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Third-Order Partial Differential Equations > Korteweg–de Vries Equation

$$1. \quad \frac{\partial w}{\partial t} + \frac{\partial^3 w}{\partial x^3} - 6w \frac{\partial w}{\partial x} = 0.$$

Korteweg–de Vries equation. It is used in many sections of nonlinear mechanics and physics.

1°. Suppose $w(x, t)$ is a solution of the Korteweg–de Vries equation. Then the function

$$w_1 = C_1^2 w(C_1 x + 6C_1 C_2 t + C_3, C_1^3 t + C_4) + C_2,$$

where $C_1, \dots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. One-soliton solution:

$$w(x, t) = -\frac{a}{2 \cosh^2 \left[\frac{1}{2} \sqrt{a} (x - at - b) \right]},$$

where a and b are arbitrary constants.

3°. Two-soliton solution:

$$w(x, t) = -2 \frac{\partial^2}{\partial x^2} \ln(1 + B_1 e^{\theta_1} + B_2 e^{\theta_2} + AB_1 B_2 e^{\theta_1 + \theta_2}),$$

$$\theta_1 = a_1 x - a_1^3 t, \quad \theta_2 = a_2 x - a_2^3 t, \quad A = \left(\frac{a_1 - a_2}{a_1 + a_2} \right)^2,$$

where B_1, B_2, a_1 , and a_2 are arbitrary constants.

4°. N -soliton solution:

$$w(x, t) = -2 \frac{\partial^2}{\partial x^2} \left\{ \ln \det [\mathbf{I} + \mathbf{C}(\mathbf{x}, \mathbf{t})] \right\}.$$

Here, $\mathbf{I}$ is the $N \times N$ identity matrix and $\mathbf{C}(x, t)$ the $N \times N$ symmetric matrix with entries

$$C_{mn}(x, t) = \frac{\sqrt{\rho_m(t)\rho_n(t)}}{p_m + p_n} \exp[-(p_m + p_n)x],$$

where the normalizing factors $\rho_n(t)$ are given by

$$\rho_n(t) = \rho_n(0) \exp(8p_n^3 t), \quad n = 1, 2, \dots, N.$$

The solution involves $2N$ arbitrary constants p_n and $\rho_n(0)$.

The above solution can be represented, for $t \rightarrow \pm\infty$, as the sum of N single-soliton solutions.

5°. “One soliton + one pole” solution:

$$w(x, t) = -2p^2 [\cosh^{-2}(pz) - (1 + px)^{-2} \tanh^2(pz)] [1 - (1 + px)^{-1} \tanh(pz)]^{-2}, \quad z = x - 4p^2 t - c,$$

where p and c are arbitrary constants.

6°. Rational solutions (algebraic solitons):

$$w(x, t) = \frac{6x(x^3 - 24t)}{(x^3 + 12t)^2},$$

$$w(x, t) = -2 \frac{\partial^2}{\partial x^2} \ln(x^6 + 60x^3 t - 720t^2).$$

7°. There is a self-similar solution of the form $w = t^{-2/3} U(z)$, where $z = t^{-1/3} x$.

8°. Solution:

$$w(x, t) = 2\varphi(z) + 2C_1t, \quad z = x + 6C_1t^2 + C_2t,$$

where C_1 and C_2 are arbitrary constants, and the function $\varphi(z)$ is determined by the second-order ordinary differential equation $\varphi''_{zz} = 6\varphi^2 - C_2\varphi - C_1z + C_3$.

9°. The Korteweg–de Vries equation is solved by the inverse scattering method. Any rapidly decreasing function $F = F(x, y; t)$ as $x \rightarrow +\infty$ that satisfies simultaneously the two linear equations

$$\frac{\partial^2 F}{\partial x^2} - \frac{\partial^2 F}{\partial y^2} = 0, \quad \frac{\partial F}{\partial t} + \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right)^3 F = 0$$

generates a solution of the Korteweg–de Vries equation in the form

$$w = -2 \frac{d}{dx} K(x, x; t),$$

where $K(x, y; t)$ is a solution of the linear Gel'fand–Levitan–Marchenko integral equation

$$K(x, y; t) + F(x, y; t) + \int_x^\infty K(x, z; t)F(z, y; t) dz = 0.$$

Time t appears in this equation as a parameter.

See also:

- [cylindrical Korteweg–de Vries equation](#) ,
- [modified Korteweg–de Vries equation](#) ,
- [generalized Korteweg–de Vries equation](#) .

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