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Higher-Order Partial Differential Equations > Boussinesq Equation

$$1. \quad \frac{\partial^2 w}{\partial t^2} + \frac{\partial}{\partial x} \left(w \frac{\partial w}{\partial x} \right) + \frac{\partial^4 w}{\partial x^4} = 0.$$

Boussinesq equation. This equation arises in hydrodynamics and some physical applications.

1°. Suppose $w(x, t)$ is a solution of the Boussinesq equation in question. Then the function

$$w_1 = C_1^2 w(C_1 x + C_2, \pm C_1^2 t + C_3),$$

where C_1 , C_2 , and C_3 are arbitrary constants, is also a solution of the equation.

2°. Solutions:

$$w(x, t) = 2C_1 x - 2C_1^2 t^2 + C_2 t + C_3,$$

$$w(x, t) = (C_1 t + C_2)x - \frac{1}{12C_1^2} (C_1 t + C_2)^4 + C_3 t + C_4,$$

$$w(x, t) = -\frac{(x + C_1)^2}{(t + C_2)^2} + \frac{C_3}{t + C_2} + C_4(t + C_2)^2,$$

$$w(x, t) = -\frac{x^2}{t^2} + C_1 t^3 x - \frac{C_1^2}{54} t^8 + C_2 t^2 + \frac{C_4}{t},$$

$$w(x, t) = -\frac{(x + C_1)^2}{(t + C_2)^2} - \frac{12}{(x + C_1)^2},$$

$$w(x, t) = -3\lambda^2 \cos^{-2} \left[\frac{1}{2} \lambda (x \pm \lambda t) + C_1 \right],$$

where $C_1, \dots, C_4$ and λ are arbitrary constants.

3°. Traveling-wave solution (generalizes the last solution of Item 1°):

$$w(x, t) = w(\zeta), \quad \zeta = x + \lambda t,$$

where the function $w(\zeta)$ is determined by the second-order ordinary differential equation $w''_{\zeta\zeta} + w^2 + 2\lambda^2 w + C_1 \zeta + C_2 = 0$.

4°. Self-similar solution:

$$w(x, t) = t^{-1} u(z), \quad z = xt^{-1/2},$$

where the function $u = u(z)$ is determined by the ordinary differential equation $u''''_{zzzz} + (uu'_z)'_z + \frac{1}{4} z^2 u''_{zz} + \frac{7}{4} z u'_z + 2u = 0$.

5°. There are exact solutions of the following forms:

$$w(x, t) = (x + C)^2 F(t) - 12(x + C)^{-2};$$

$$w(x, t) = G(\xi) - 4C_1^2 t^2 - 4C_1 C_2 t, \quad \xi = x - C_1 t^2 - C_2 t;$$

$$w(x, t) = \frac{1}{t} H(\eta) - \frac{1}{4} \left(\frac{x}{t} + Ct \right)^2, \quad \eta = \frac{x}{\sqrt{t}} - \frac{1}{3} C t^{3/2};$$

$$w(x, t) = (a_1 t + a_0)^2 U(\zeta) - \left(\frac{a_1 x + b_1}{a_1 t + a_0} \right)^2, \quad \zeta = x(a_1 t + a_0) + b_1 t + b_0,$$

where $C, C_1, C_2, a_1, a_0, b_1$, and b_0 are arbitrary constants,

6°. The Boussinesq equation is solved by the inverse scattering method. Any rapidly decaying function $F = F(x, y; t)$ as $x \rightarrow +\infty$ and satisfying simultaneously the two linear equations

$$\frac{1}{\sqrt{3}} \frac{\partial F}{\partial t} + \frac{\partial^2 F}{\partial x^2} - \frac{\partial^2 F}{\partial y^2} = 0, \quad \frac{\partial^3 F}{\partial x^3} + \frac{\partial^3 F}{\partial y^3} = 0$$

generates a solution of the Boussinesq equation in the form

$$w = 12 \frac{d}{dx} K(x, x; t),$$

where $K(x, y; t)$ is a solution of the linear Gel'fand–Levitan–Marchenko integral equation

$$K(x, y; t) + F(x, y; t) + \int_x^\infty K(x, s; t) F(s, y; t) ds = 0.$$

Time t appears here as a parameter.

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