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Higher-Order Partial Differential Equations > Navier–Stokes Equations

$$2. \quad \frac{\partial w}{\partial y} \frac{\partial}{\partial x} (\Delta w) - \frac{\partial w}{\partial x} \frac{\partial}{\partial y} (\Delta w) = \nu \Delta \Delta w, \quad \Delta w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}.$$

There is a **two-dimensional stationary equation of motion of a viscous incompressible fluid** (it is obtained from the Navier–Stokes equations by eliminating pressure and introducing the stream function w , see Remark).

1°. Suppose $w(x, y)$ is a solution of the equation in question. Then the functions

$$\begin{aligned} w_1 &= -w(y, x), \\ w_2 &= w(C_1 x + C_2, C_1 y + C_3) + C_4, \\ w_3 &= w(x \cos \alpha + y \sin \alpha, -x \sin \alpha + y \cos \alpha), \end{aligned}$$

where $C_1, \dots, C_4$ and α are arbitrary constants, are also solutions of the equation.

2°. Any solution of the Poisson equation $\Delta w = C$ is also a solution of the original equation (these are “inviscid” solutions).

3°. Solutions in the form of a one-variable function or the sum of functions with different arguments:

$$\begin{aligned} w(y) &= C_1 y^3 + C_2 y^2 + C_3 y + C_4, \\ w(x, y) &= C_1 x^2 + C_2 x + C_3 y^2 + C_4 y + C_5, \\ w(x, y) &= C_1 \exp(-\lambda y) + C_2 y^2 + C_3 y + C_4 + \nu \lambda x, \\ w(x, y) &= C_1 \exp(\lambda x) - \nu \lambda x + C_2 \exp(\lambda y) + \nu \lambda y + C_3, \\ w(x, y) &= C_1 \exp(\lambda x) + \nu \lambda x + C_2 \exp(-\lambda y) + \nu \lambda y + C_3, \end{aligned}$$

where $C_1, \dots, C_5$ and λ are arbitrary constants.

4°. Generalized separable solutions:

$$\begin{aligned} w(x, y) &= A(kx + \lambda y)^3 + B(kx + \lambda y)^2 + C(kx + \lambda y) + D, \\ w(x, y) &= A e^{-\lambda(y+kx)} + B(y+kx)^2 + C(y+kx) + \nu \lambda(k^2 + 1)x + D, \\ w(x, y) &= 6\nu x(y + \lambda)^{-1} + A(y + \lambda)^3 + B(y + \lambda)^{-1} + C(y + \lambda)^{-2} + D, \\ w(x, y) &= (Ax + B)e^{-\lambda y} + \nu \lambda x + C, \\ w(x, y) &= [A \sinh(\beta x) + B \cosh(\beta x)] e^{-\lambda y} + \frac{\nu}{\lambda}(\beta^2 + \lambda^2)x + C, \\ w(x, y) &= [A \sin(\beta x) + B \cos(\beta x)] e^{-\lambda y} + \frac{\nu}{\lambda}(\lambda^2 - \beta^2)x + C, \\ w(x, y) &= A e^{\lambda y + \beta x} + B e^{\gamma x} + \nu \gamma y + \frac{\nu}{\lambda} \gamma(\beta - \gamma)x + C, \quad \gamma = \pm \sqrt{\lambda^2 + \beta^2}, \end{aligned}$$

where A, B, C, D, k, β , and λ are arbitrary constants.

5°. Generalized separable solution linear in x :

$$w(x, y) = F(y)x + G(y), \quad (1)$$

where the functions $F = F(y)$ and $G = G(y)$ are determined by the autonomous system of fourth-order ordinary differential equations

$$F'_y F''_{yy} - F F'''_{yyy} = \nu F''''_{yyyy}, \quad (2)$$

$$G'_y F''_{yy} - F G'''_{yyy} = \nu G''''_{yyyy}. \quad (3)$$

Equation (2) has the following particular solutions:

$$\begin{aligned} F &= ay + b, \\ F &= 6\nu(y + a)^{-1}, \\ F &= ae^{-\lambda y} + \lambda\nu, \end{aligned}$$

where a , b , and λ are arbitrary constants.

Let $F = F(y)$ is a solution of equation (2) ($F \neq \text{const}$). Then, the corresponding general solution of equation (3) can be written out in the form

$$G = \int U dy + C_4, \quad U = C_1 U_1 + C_2 U_2 + C_3 \left(U_2 \int \frac{U_1}{\Phi} dy - U_1 \int \frac{U_2}{\Phi} dy \right),$$

where C_1 , C_2 , C_3 , and C_4 are arbitrary constants, and

$$U_1 = \begin{cases} F''_{yy} & \text{if } F''_{yy} \neq 0, \\ F & \text{if } F''_{yy} = 0, \end{cases} \quad U_2 = U_1 \int \frac{\Phi dy}{U_1^2}, \quad \Phi = \exp\left(-\frac{1}{\nu} \int F dy\right).$$

6°. There is an exact solution of the form (generalizes the solution of Item 5°):

$$w(x, y) = F(z)x + G(z), \quad z = y + kx, \quad k \text{ is any.}$$

7°. Self-similar solution:

$$w = \int F(z) dz + C_1, \quad z = \arctan\left(\frac{x}{y}\right),$$

where the function F is determined by the first-order autonomous ordinary differential equation $3\nu(F'_z)^2 - 2F^3 + 12\nu F^2 + C_2 F + C_3 = 0$ (C_1 , C_2 , and C_3 are arbitrary constants).

8°. There is an exact solution of the form (generalizes the solution of Item 7°):

$$w = C_1 \ln|x| + \int V(z) dz + C_2, \quad z = \arctan\left(\frac{x}{y}\right).$$

Remark. The two-dimensional steady-state equations of a viscous incompressible fluid (the Navier–Stokes equations)

$$\begin{aligned} u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_1}{\partial y} &= -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \Delta u_1, \\ u_1 \frac{\partial u_2}{\partial x} + u_2 \frac{\partial u_2}{\partial y} &= -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \Delta u_2, \\ \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} &= 0 \end{aligned}$$

are reduced to the equation in question by the introduction of a stream function w such that $u_1 = \frac{\partial w}{\partial y}$ and $u_2 = -\frac{\partial w}{\partial x}$ followed by the elimination of the pressure p (with cross differentiation) from the first two equations.

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