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Second-Order Hyperbolic Partial Differential Equations > Sine-Gordon Equation

$$6. \quad \frac{\partial^2 w}{\partial t^2} = a \frac{\partial^2 w}{\partial x^2} + b \sin(\lambda w).$$

Sine-Gordon equation. It arises in differential geometry and various areas of physics.

1°. Traveling-wave solutions:

$$w(x, t) = \frac{4}{\lambda} \arctan \left\{ \exp \left[\pm \frac{b\lambda(kx + \mu t + \theta_0)}{\sqrt{b\lambda(\mu^2 - ak^2)}} \right] \right\} \quad \text{if } b\lambda(\mu^2 - ak^2) > 0,$$

$$w(x, t) = -\frac{\pi}{\lambda} + \frac{4}{\lambda} \arctan \left\{ \exp \left[\pm \frac{b\lambda(kx + \mu t + \theta_0)}{\sqrt{b\lambda(ak^2 - \mu^2)}} \right] \right\} \quad \text{if } b\lambda(\mu^2 - ak^2) < 0,$$

where k , μ , and θ_0 are arbitrary constants. The first expression corresponds to a single-soliton solution.

2°. Functional separable solutions:

$$w(x, t) = \frac{4}{\lambda} \arctan \left[\frac{\mu \sinh(kx + A)}{k\sqrt{a} \cosh(\mu t + B)} \right], \quad \mu^2 = ak^2 + b\lambda > 0;$$

$$w(x, t) = \frac{4}{\lambda} \arctan \left[\frac{\mu \sin(kx + A)}{k\sqrt{a} \cosh(\mu t + B)} \right], \quad \mu^2 = b\lambda - ak^2 > 0;$$

$$w(x, t) = \frac{4}{\lambda} \arctan \left[\frac{\gamma}{\mu} \frac{e^{\mu(t+A)} + ak^2 e^{-\mu(t+A)}}{e^{k\gamma(x+B)} + e^{-k\gamma(x+B)}} \right], \quad \mu^2 = ak^2\gamma^2 + b\lambda > 0,$$

where A , B , k , and γ are arbitrary constants.

3°. An N -soliton solution is given by ($a = 1$, $b = -1$, and $\lambda = 1$)

$$w(x, t) = \arccos \left[1 - 2 \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2} \right) (\ln F) \right],$$

$$F = \det [M_{ij}], \quad M_{ij} = \frac{2}{a_i + a_j} \cosh \left(\frac{z_i + z_j}{2} \right), \quad z_i = \pm \frac{x - \mu_i t + C_i}{\sqrt{1 - \mu_i^2}}, \quad a_i = \pm \sqrt{\frac{1 - \mu_i}{1 + \mu_i}},$$

where μ_i and C_i are arbitrary constants.

4°. For other exact solutions of the sine-Gordon equation, see the [nonlinear Klein–Gordon equation](#) with $f(w) = b \sin(\lambda w)$.

5°. The sine-Gordon equation is integrated by the inverse scattering method.

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