Exponential Object in Geb

Artem Gureev

Theorem. `Geb` *category is equivalent to* ***FinSet*** *via a functor* GF : `Geb` $\to$ ***FinSet*** *defined on objects by induction:*

$$\begin{aligned} GF(T) &:= \boldsymbol{1} \\ GF(I) &:= \boldsymbol{0} \\ GF(X \oplus Y) &:= GF(X) + GF(Y) \\ GF(X \otimes Y) &:= GF(X) \times GF(Y) \end{aligned}$$

and preserving all categorical structure on the nose, as well as the distribution morphism.

Definition. *For* $X, Y \in Obj(\texttt{Geb})$ *define* $X \Rightarrow Y \in Obj(\texttt{Geb})$ *by induction:*

$$\begin{aligned} T \Rightarrow Y &:= Y \\ I \Rightarrow Y &:= T \\ (A \oplus B) \Rightarrow Y &:= (A \Rightarrow Y) \otimes (B \Rightarrow Y) \\ (A \otimes B) \Rightarrow Y &:= (A \Rightarrow (B \Rightarrow Y)) \end{aligned}$$

Proposition. *The above object is an exponential, i.e.* $GF(A \Rightarrow B) \cong GF(A)^{GF(B)}$

Proof. By induction:

$$GF(T \Rightarrow B) := GF(B) \cong GF(B)^{\mathbf{1}} := GF(B)^{GF(T)}$$

$$GF(I \Rightarrow B) := GF(T) := \mathbf{1} \cong GF(B)^{0} := GF(B)^{GF(I)}$$

$$\begin{aligned} GF(A \oplus B \Rightarrow Y) &:= GF((A \Rightarrow Y) \otimes (B \Rightarrow Y)) \\ &:= GF(A \Rightarrow Y) \times GF(B \Rightarrow Y) \\ &\cong GF(Y)^{GF(A)} \times GF(Y)^{GF(B)} \qquad \text{By induction} \\ &\cong GF(Y)^{GF(A) \times GF(B)} \end{aligned}$$

$$\begin{aligned} GF(A \otimes B \Rightarrow Y) &:= GF(A \Rightarrow (B \Rightarrow Y)) \\ &\cong GF(B \Rightarrow Y)^{GF(A)} \\ &\cong (GF(Y)^{GF(B)})^{GF(A)} \\ &\cong GF(Y)^{GF(B) \times GF(A)} \\ &:= GF(Y)^{GF(B \otimes A)} \end{aligned}$$

□

For computational details, look in `geb.agda`

Definition. *Define* $evalG : (Y \Rightarrow X) \otimes Y \to X$ *by induction:*

$$\begin{tikzcd} (T \Rightarrow X) \otimes T \arrow[r, "evalG"] \arrow[d, equal] & X \\ X \times T \arrow[ur, "\pi_1"'] & \end{tikzcd}$$

that is, $eval(G) := \lambda(x, *).x$, *arguing in* ***FinSet***

$$\begin{tikzcd} X^I \otimes I \arrow[rr, "evalG"] \arrow[dr, "\pi_2"'] & & X \\ & I \arrow[ur, "!"'] & \end{tikzcd}$$

that is, $evalG$ *is the unique morphism from the initial object:* $evalG(x, i) :=!(i)$

$$\begin{tikzcd} (A \Rightarrow X) \otimes (B \Rightarrow X) \otimes (A \oplus B) \arrow[rr, "evalG"] \arrow[dr, "D"'] & & X \\ & [((A \Rightarrow X) \otimes (B \Rightarrow X)) \otimes A] \oplus [((A \Rightarrow X) \otimes (B \Rightarrow X)) \otimes B] \arrow[ur, "{[c1,c2]}"'] & \end{tikzcd}$$

where D *is the distributivity iso, in* ***FinSet*** *presented as:*

$$\lambda(f, g, inl(a)) = inl(f, g, a)$$

$$\lambda(f, g, inr(b)) = inr(f, g, b)$$

and

$$\begin{tikzcd} ((A \Rightarrow X) \otimes (B \Rightarrow X)) \otimes A \arrow[r, "fM"] \arrow[rr, bend right, "c1"'] & (A \Rightarrow X) \otimes A \arrow[r, "evalG"] & X \end{tikzcd}$$

$\lambda(f, g, a).GF(evalG(f, a))$ *in* ***FinSet***

$$\begin{tikzcd} ((A \Rightarrow X) \otimes (B \Rightarrow X)) \otimes A \arrow[r, "fL"] \arrow[rr, bend right, "c2"'] & (B \Rightarrow X) \otimes B \arrow[r, "evalG"] & X \end{tikzcd}$$

$\lambda(f, g, b).(eval(f, b))$ *in* ***FinSet***

In other words, $GF(evalG)$ *here is equivalently*

$$\lambda(f, g, (inla)) = eval(f, a)$$

$$\lambda(f, g, (inrb)) = eval(g, b)$$

$$\begin{tikzcd} ((A \Rightarrow (B \Rightarrow X)) \otimes (A \otimes B) \arrow[rr, "evalG"] \arrow[dr, "\langle p1{,}p2 \rangle"'] & & X \\ & (B \Rightarrow X) \otimes B \arrow[ur, "evalG"'] & \end{tikzcd}$$

where

$$\begin{tikzcd}
((A \Rightarrow (B \Rightarrow X)) \otimes (A \otimes B) \arrow[r, "\lambda(f,a,b).(f,a)"] \arrow[rr, bend right, "p1"'] & (A \Rightarrow (B \Rightarrow X)) \otimes A \arrow[r, "evalG"] & (B \Rightarrow X)
\end{tikzcd}$$

$$\begin{tikzcd}
((A \Rightarrow (B \Rightarrow X)) \otimes (A \otimes B) \arrow[r, "\lambda(f,a,b).b"] \arrow[rr, bend right, "p1"'] & B \arrow[r, "evalG"] & (B \Rightarrow X)
\end{tikzcd}$$

*working in **FinSet** (in* `Geb` *this is of course replaced by projections)*

*so in **FinSet** this is* $\lambda(f,a,b).evalG((eval(f,a)),b)$

Proposition. *$GF(evalG)$ is isomorphic to eval, i.e.*

$$\begin{tikzcd}
GF((A \Rightarrow B) \otimes A) \arrow[rr, "\cong"] \arrow[dr, "GF(evalG)"'] & & GF(A)^{GF(B)} \times GF(A) \arrow[dl, "eval"] \\
& GF(B) &
\end{tikzcd}$$

Proof. By induction. See `exp.agda` ☐

Definition. *We define $curryG : Hom(X \otimes Y, Z) \to Hom(X, (Y \Rightarrow Z))$ by induction*

Let $f \in Hom(X \otimes Y, Z)$

for $Y := T$, $curryG : Hom(X \otimes T, Z) \to Hom(X, Z)$ is defined as

$$curryG(f) = f \circ \langle 1_X, !_X \rangle$$

$\lambda(x).f(x,*)$ *in **FinSet***

for $Y := I$, $curryG : Hom(X \otimes I, Z) \to Hom(X, T)$ is defined by

$$curryG(f) :=!_X$$

for $Y := A \oplus B$ then $curryG : Hom(X \otimes (A \oplus B), Z) \to Hom(X, Z^A \otimes Z^B)$ is defined as

$$\begin{tikzcd}
& Z^A & \\
X \arrow[ur, "curryG(f \circ d \circ inl)"] \arrow[rr, "!"] \arrow[dr, "curryG(f \circ d \circ inr)"'] & & Z^A \otimes Z^B \arrow[ul, two heads] \arrow[dl, two heads] \\
& Z^B &
\end{tikzcd}$$

*where d is the distributivity morphism $d : (X \otimes Y) \oplus (X \otimes Z) \to X \otimes (Y \oplus Z)$. In **FinSet** it becomes the usual:*

$$\lambda(inl(x,y)).(x, inl(y))$$
$$\lambda(inr(x,z)).(x, inr(z))$$

for $Y := A \otimes B$ $curryG : Hom(X \otimes (A \otimes B)) \to Hom(X, Z^{B^A})$ is defined as

Proposition. *$GF(curryG(f))$ for each f is ismorphic to $curry(f)$:*

$$\begin{tikzcd}
& GF(X) \arrow[dl, "GF(curryG(f))"'] \arrow[dr, "curry(GF(f))"] & \\
GF(Z^Y) \arrow[rr, "\cong"] & & GF(Z)^{GF(Y)}
\end{tikzcd}$$

Proof. By induction. See `exp.agda` $\square$

Since equivalences reflect equalities and are fully faithful, and GF preserves product structure on the nose, it suffices to show that $GF(curryG)$ has the universal property (in terms of making the needed triangle commute) of currying in order to confirm that it is the correct currying function for `Geb`

Yet this follows as the statement corresponds to the right wall of the following tetrahedron:

$$\begin{tikzcd}
& & GF(Z) \times GF(Y) \arrow[dll, bend right, "curry(GF(e))\times 1"'] \arrow[d, "GF(curryG(e))\times 1"] \arrow[drr, bend left, "GF(e)"] & & \\
& & GF(X)^{GF(Y)} \times GF(Y) \arrow[dll, dash, "\cong"] \arrow[drr, "GF(evalG)"] & & \\
GF(Y \Rightarrow X) \times GF(Y) \arrow[rrrr, "eval"'] & & & & GF(X)
\end{tikzcd}$$

as we know the commutativity of all the other sides of the tetrahedron, we are done