The Geb Category

Artem Gureev

The document provides the outline of the intended semantics of the Geb programming language without dependent types, i.e. of the initial model of Geb.

We have two ways of describing the Geb category, one via the universal property and one via explicit construction.

Recall that category theory is an essentially algebraic theory and, moreover, the theory of finite products and coproducts in a category is essentially algebraic. Finally, adding the axiom of distributivity is also an equationally specified operation. Hence we have the universal characterization:

Definition. *The* `Geb` *category is the initial model of an essentially algebraic theory of categories with finite products, coproducts, and distributivity. In other words, it is a free model of said theory on the empty set.*

This provides the characterization via the universal property of the free$\dashv$forgetful adjunction.

We can also specify the category explicitly.

Definition. *Define* $G0_n$ *for* $n \in \mathbb{N}$ *inductively:*

$G0_0 := \{I, T\}$

$G0_{n+1} := G0_n \cup \{a + b \mid a, b \in G0_n\} \cup \{a \times b \mid a, b \in G0_n\}$

Define G_{obj} *as a colimit of*

$$G0_0 \hookrightarrow G0_1 \hookrightarrow \dots \hookrightarrow G0_n \hookrightarrow G0_{n+1} \hookrightarrow \dots$$

or equivalently $\bigcup_{n \in \mathbb{N}} G0_n$, *or the initial algebra in* Set *of the endofunctor* $2X^2 + 2$

Definition. *Define* $G1_n$ *and* $dom \times codom : G1_n \times G1_n \to G_{obj} \times G_{obj}$ *for* $n \in \mathbb{N}$ *by induction:*

$$\begin{aligned} G1_0 := &\{1_x \mid x \in G_{obj}\} \\ &\cup \{!_{I,x} \mid x \in G_{obj}\} \\ &\cup \{!_{x,T} \mid x \in G_{obj}\} \\ &\cup \{\pi_{k,x,y} \mid k \in 2, x, y \in G_{obj}\} \\ &\cup \{i_{k,x,y} \mid k \in 2, x, y \in G_{obj}\} \\ &\cup \{d_{x,y,z} \mid x, y, z \in G_{obj}\} \end{aligned}$$

with the evident (co)domain functions where d is supposed to be the distributivity map, i.e. $dom(d_{x,y,z}) =$

$x \times (y+z)$ and $codom(d_{x,y,z}) = (x \times y) + (x \times z)$

$$\begin{aligned} G1_{n+1} := G1_n &\cup \{f \circ g \mid f, g \in G1_n, codom(g) = dom(f)\} \\ &\cup \{\langle f, g\rangle \mid f, g \in G1_n\} \\ &\cup \{[f, g] \mid f, g \in G1_n\} \end{aligned}$$

similarly with evident (co)domain functions.

Define $G_{mor} := \bigcup_{n \in \mathbb{N}} G1_n / \sim$ where $\sim$ is the smallest equivalence relation spanned by $(f \circ g) \circ k \sim f \circ (g \circ k)$, $\pi_{1,x,y} \circ \langle f, g\rangle \sim f$, $\pi_{2,x,y} \circ \langle f, g\rangle \sim g$, $[f, g] \circ i_{1,x,y} \sim f$, $[f, g] \circ i_{2,x,y} \sim g$, $1_x \circ f \sim f$, $f \circ 1_x \sim f$, and making $d_{x,y,z}$ the inverse of the evident morphism.

Definition. *Let* `Geb` *designate the category whose class of objects is G_{obj} and set of morphisms G_{mor} with the evident (co)dom functions defined inductively, composition given by $comp([f], [g]) := [f \circ g]$, and $\{[1_x] \mid x \in G_{obj}\}$ being the set of identities.*