Sample Exam

Part A

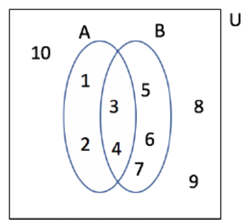
(a) Let A be a set with 5 elements. What is the number of subsets which can be formed from A?

Answer:

$$
\begin{aligned}
|A| &= 5 \\
|\mathcal{P}(A)| &= 2^{|A|} \\
2^5 &= 32 \\
&(iii)
\end{aligned}
$$

b) Given the following Venn diagram representing two sets A and B, subsets of the universal set U:

(b) Given the following Venn diagram representing two sets A and B, subsets of the universal set U:



Which one of the following sets represents $(\overline{A \cap B})$?

Choose ONE option. [4]

i. $\{1, 2, 5, 6, 7, 8, 9, 10\}$

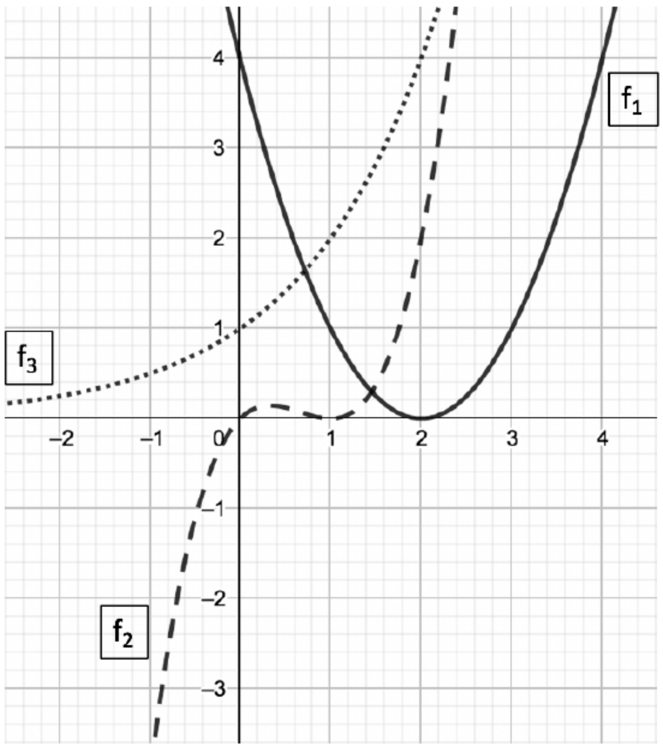
ii. $\{3, 4, 8, 9, 10\}$

iii. $\{8, 9, 10\}$

iv. none of the other options is correct

i)$S = \{1, 2, 5, 6, 7, 8, 9, 10\}$

c) The following graph shows the curves of three functions, f_1, f_2 and f_3. Which of these functions is/are not invertible?:



Answer: (iv) Functions are invertible if they are both injective and surjective. Upon inspection, we can see that both f_1 and f_2 are not injective because they have the same y value for two different x values. f_1 and f_2 are not invertible.

(d) Let p and q be the following propositions concerning a positive integer n where p means 'n has is less than or equal to 7 ' and q means 'n cannot be represented using 3 binary digits' Which one of the following logical expressions is equivalent to a correct formalisation of the sentence below?' if n is less than or equal to 7 then n can be represented using 3 binary digits'

Answer:
p: "n is less than or equal to 7"
q: n cannot be represented using 3 binary digits"

Number of digits that can be represented using n bits is $2^n - 1$. In this case it is $2^3 - 1 = 8 - 1 = 7$

"if n is less than or equal to 7 then n can be represented using 3 binary digits"

answer: iii)$p \rightarrow \neg q$

e) Given the statement $S(x) : x^2 + 1 = 5$, select the right statement from the following. Choose ONE option.

i. S can be expressed using propositional logic
ii. S is not a proposition, as its truth value is a function depending on x
iii. The truth value of $S(2)$ is False
iv. The truth value of $S(3)$ is True

Answer: $S(x)$ is a predicate. It's truth value depends on the value of x. If x is given a value, it becomes a proposition.
ii) S is not a proposition, as its truth value is a function depending on x.

f) The number of ways k objects that can be selected from n objects where the ordering of the outputs is not important can be calculated using which one of the following formulations? Choose ONE option.

Answer:
(i)When ordering is not important we can use a $C(n, k) = \frac{n!}{k!(n-k)!}$.

g) Which one of the following degree sequences cannot represent a simple graph?
i.5,3,3,2,2
ii.4,2,2,2,2
iii.2,2,2,2,2
iv.4,3,3,2,2

Answer: A simple graph is a graph that does not contain any loops or parallel edges.

1. Sum of all degree sequence / 2 should be an integer.
2. Highest degree $<$ of vertices $||$ highest degree $+ 1 <=$ of vertices
3. Draw a graph.

The answer is (i).

h) Which one the following is a correct definition of a Hamiltonian path?
i. A Hamiltonian path in a graph G is a path that uses each edge in G precisely once
ii. A Hamiltonian path in graph G is a path that visits each vertex in G exactly once
iii. A Hamiltonian path path is a trail in which neither vertices nor edges are repeated
iv. A Hamiltonian path is a walk in which no edge is repeated

Answer: ii)

i) Let $S = \{1, 2, 3\}$ and $\mathcal{R}$ be a relation on elements in S with $\mathcal{R} = \{(1, 1), (1, 2), (1, 3), (2, 2), (2, 1), (3, 3)\}$. Which one of the following statements is correct about the relation $\mathcal{R}$?

Reflexive: $\forall a \in S, (a, a) \in \mathcal{R}$ It is reflexive
Symmetric: $\forall a, b \in S, ((a, b) \in \mathcal{R} \to (b, a) \in \mathcal{R})$ Not symmetric because (3,1) is missing
Anti-symmetric: $\forall a, b \in S((a, b) \in \mathcal{R} \wedge (b, a) \in \mathcal{R}) \to a = b)$ It is not anti-symmetric
Transitive: $\forall a, b, c \in S((a, b) \in \mathcal{R} \wedge (b, c) \in \mathcal{R} \to (a, c) \in \mathcal{R})$ Not transitive, because (2,3) is missing.
No correct answer on sample paper.

j) Let $S = a, b, c$ and R be a relation on elements in Swith $R = \{(c, b), (a, a), (b, c)\}$ Which one of the following statements is correct about the relation R?

Reflexive: $\forall a \in S, (a, a) \in \mathcal{R}$ Not reflexive, no (b, b) and no (c, c)
Symmetric:$\forall (a, b) \in S((a, b) \in \mathcal{R} \to (b, a) \in \mathcal{R})$ Is symmetric, (c,b), and (b,c)
Anti-symmetry:$\forall a, b \in S((a, b) \in \mathcal{R} \wedge (b, a) \in \mathcal{R}) \to a = b)$ Not anti-symmetric, the implication doesn't hold
Transitive: $\forall a, b, c \in S((a, b) \in \mathcal{R} \wedge (b, c) \in \mathcal{R}) \to (a, c) \in \mathcal{R})$ Not transitive, because (c,c) and (b,b) are missing.
The correct answer is iii)

Part B question 2

a) Rewrite the following three sets using the listing method:

$A = \{n^2 + (-1)^n : n \in \mathbb{Z} \text{ and } 0 \le n < 5\}$
$0 \to 0^2 + (-1)^0 = 1$
$1 \to 1^2 + (-1)^1 = 0$
$2 \to 2^2 + (-1)^2 = 5$
$3 \to 3^2 + (-1)^3 = 8$
$4 \to 4^2 + (-1)^4 = 17$

Hence $A = \{1, 0, 5, 8, 17\}$

$B = \{n + \frac{1}{n} : n \in \mathbb{Z}^+ \text{ and } n < 6\}$

$1 \to 1 + \frac{1}{1} = 2$

$2 \to 2 + \frac{1}{2} = \frac{2 \cdot 2}{2} + \frac{1}{2} = \frac{5}{2}$

$3 \to 3 + \frac{1}{3} = \frac{10}{3}$

$4 \to 4 + \frac{1}{4} = \frac{17}{4}$

$5 \to 5 + \frac{1}{5} = \frac{26}{5}$

$C = \{(-2)^{-n} : n \in \mathbb{Z}^+ \text{ and } n < 5\}$

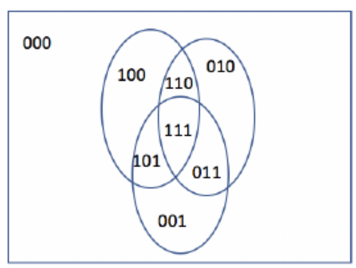
$$1 \to (-2)^{-1} = \frac{1}{(-2)^1} = -\frac{1}{2}$$

$$1 \to (-2)^{-2} = \frac{1}{(-2)^2} = \frac{1}{4}$$

$$1 \to (-2)^{-3} = \frac{1}{(-2)^3} = -\frac{1}{8}$$

$$1 \to (-2)^{-4} = \frac{1}{(-2)^4} = \frac{1}{16}$$

ii. Given the following Venn diagram representing three sets A, B and C intersecting in the most general way. Three binary digits are used to refer to each one of the 8 region in this diagram. In terms of A, B and C, Write the set representing the area comprising the regions 011, 101 and 111. The answer must be written in its simplest



[3]

The bottom set is A, the top right set is C, top left B. **Answer**: $A \cap (B \cup C)$

iii) Given three sets A, B and C. Using set identities, prove that the expression $(A - B) - (B - C)$ is equivalent to $A - B$.

$(A - B) - (B - C)$

$$A - B = A \cap \overline{B}$$

$$\begin{aligned}
(A - B) - (B - C) &= (A \cap \overline{B}) - (B \cap \overline{C}) && \text{Set Difference law, expressions in parentheses} \\
&= (A \cap \overline{B}) \cap \overline{(B \cap \overline{C})} && \text{Set Differnce between both expressions} \\
&= (A \cap \overline{B}) \cap (\overline{B} \cup \overline{\overline{C}}) && \text{De Morgan's law} \\
&= (A \cap \overline{B}) \cap (\overline{B} \cup C) && \text{Double Complement} \\
&= ((A \cap \overline{B}) \cap \overline{B}) \cup ((A \cap \overline{B}) \cap C) && \text{Distributive Law} \\
&= (A \cap (\overline{B} \cap \overline{B})) \cup ((A \cap \overline{B}) \cap C) && \text{Associativity} \\
&= (A \cap \overline{B}) \cup ((A \cap \overline{B}) \cap C) && \text{Idempotent Law} \\
&= D \cup (D \cap C) && \text{Substitution: } A \cap \overline{B} = D \\
&= D && \text{Absorption Law} \\
&= A \cap \overline{B} && \text{from substitution} \\
&= A - B && \text{Set difference law}
\end{aligned}$$

$\square$

(b) i. Let p and q be two propositions. Assume that p is false and q is true.Determine the truth value of each of the following:
p : false
q : true

$p \to q = \textit{false} \to \textit{true} = \textit{true}$
$\neg p \lor q = \textit{true} \lor \textit{true} = \textit{true}$
$q \oplus \neg p = \textit{true} \oplus \textit{true} = false$
$\neg q \to p = \textit{false} \to \textit{false} = \textit{true}$

ii. Let p,q and r denote the following statements:
p:"I finish my home work"
q:"I will go to the gym"
r:"It's raining"

1. Write the following the following statements into their corresponding symbolic forms:

- if I finish my homework and it is raining then I will go to the gym.

  $(p \land r) \to q$
  $p \land r \to q$

  Both are correct because of the order of operations of logical operators.

- I will go to the gym or finish my homework but not both

  $q \oplus p$

2. Write in words the contrapositive of the the following statement:

"if I finish my homework and it is raining then I will go to the gym."

$(p \land r) \to q$
$\neg q \to \neg(p \land r)$
$\neg q \to \neg p \lor \neg r$ by De Morgan's law:

If I will not go to the gym, then I won't finished my homework or it's not raining. (captures $\neg q \to \neg p \lor \neg r$)

If I will not go to the gym, then it is not the case that I finish my homework and it's raining. (capture the statement $\neg q \to \neg(p \wedge r)$)

I will not go to the gym only if I don't finish my homework or it's not raining (captures $\neg q \to \neg p \vee \neg r$)

c) Let A be the set of all students,

$p(x)$ be the proposition: 'x is enrolled on the Discrete Mathematics module' and
$S(x)$ is the set of all students in the same year as x' where x is an element of A.
Use rules of inference with quantifiers to formalise the three following statements:
i. None of the students is enrolled in Discrete Mathematics module.
$\neg\exists x P(x)$

$\forall x \neg P(x)$ DeMorgan's law

ii. Not every student is enrolled in Discrete Mathematics module.

$\neg\forall x, P(x)$

$\exists x \neg P(x)$ Demorgan's law

iii. There exists a student, none of whose classmates play football. "Classmates of a student x" means 'students in the same class as x'

Let $F(x)$ be the proposition "x plays football"

Let $C(x, y)$ be the proposition "y is in the same class as x"

$\exists x \in A, \forall y \in S, x \neq y, (\neg F(y) \wedge C(x, y))$

(d) Suppose that a sales person has to visit eight different cities including London. They must begin their trip in London, but can visit the other seven cities in any order they wish. How many possible orders can the sales person use when visiting these cities?

In this example order matters because we need to visit all of the cities, we can only visit one city once, but we can do so in any order, the only thing that matters is that we have visited each city once during the trip.

From London, we can go to any of the 7 cities.

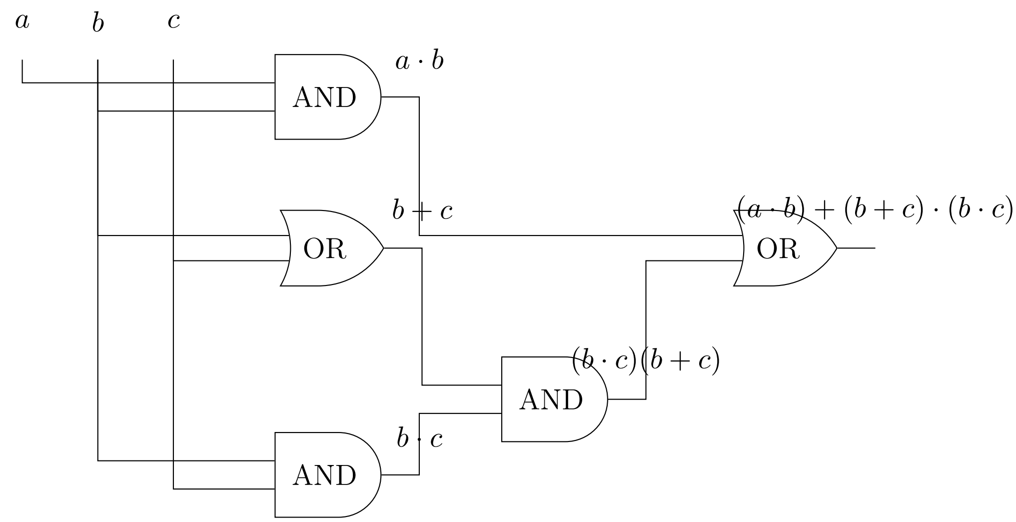
$P(7,7) = 7!$ This is called k permutations, we must show the full formula.

$P(7,7) = \frac{n!}{(n-r)!} = \frac{7!}{(7-7)!} = \frac{7!}{0!} = \frac{7!}{1} = 7!$

$P(7) = 7!$

1 Question 3

(a) Given the following logical circuit with three inputs a, b and c:



i. Identify the logical gates used in this circuit.

Answer:

There are 3 AND gates and 2 OR gates.

ii. What is the the logical expression of the output of this circuit?

Answer:

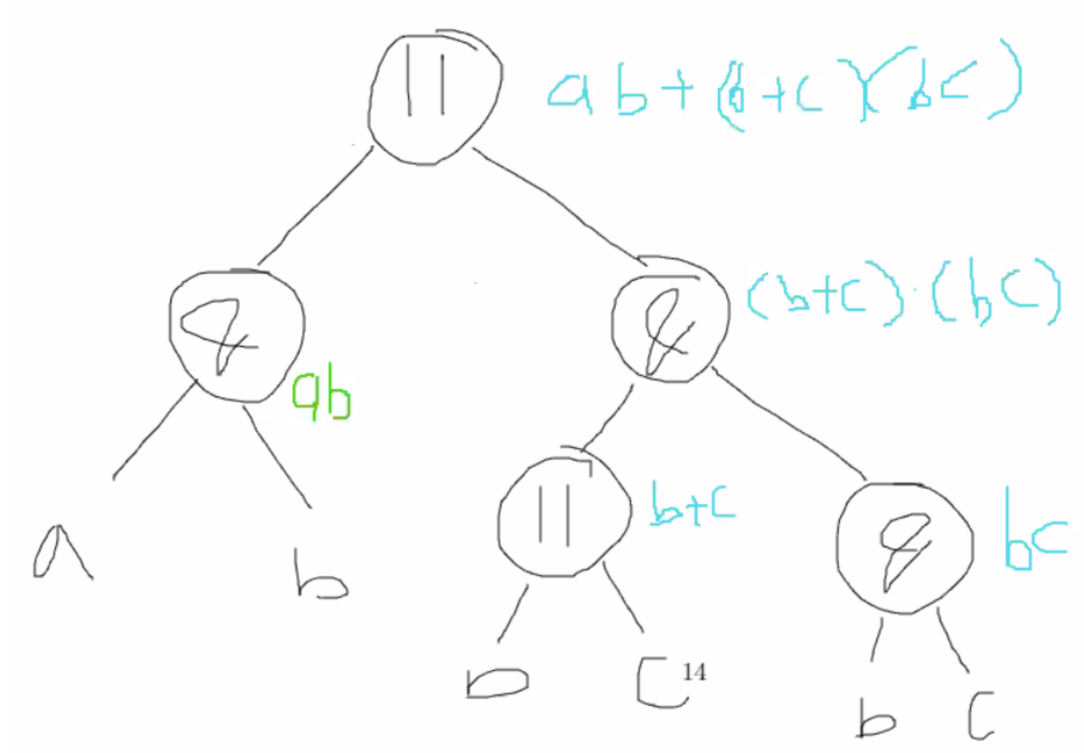
$a \cdot b$

$b + c$

$b \cdot c$

$(b + c)(b \cdot c)$

$(a \cdot b) + ((b + c) \cdot (b \cdot c))$

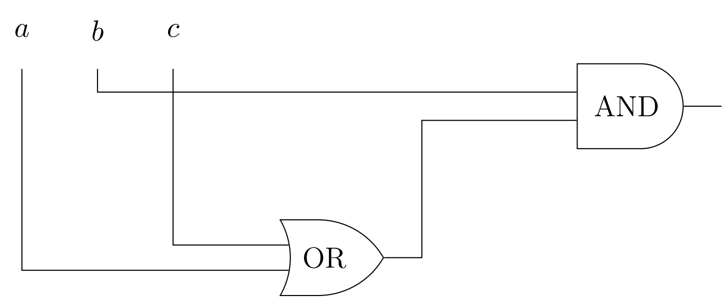

ab+(b+c)(bc)
ab
(b+c)·(bc)
b+c
bc
a
b
b
c
b
c
14

iii. Simplify the logical expression in (ii). Explain your answer.

$$\begin{aligned} a(b+c) &= ab + ac \\ a + bc &= (a+b)(a+c) \qquad \textit{Only works in Boolean algebra} \end{aligned}$$

$$\begin{aligned}
(a \cdot b) + ((b+c) \cdot (b \cdot c)) &= ab + (b \cdot (b \cdot c)) + (c \cdot (b \cdot c)) && \text{Distributive law} \\
&= ab + ((b \cdot b) \cdot c) + (c \cdot (b \cdot c)) && \text{Associative law} \\
&= ab + ((b \cdot b) \cdot c) + (c \cdot c \cdot b) && \text{Commutativity law} \\
&= ab + ((b \cdot b) \cdot c) + ((c \cdot c) \cdot b) && \text{Associativity law} \\
&= ab + (b \cdot c) + (c \cdot b) && \text{Idempotent law} \\
&= ab + (b \cdot c) + (b \cdot c) && \text{Commutativity law} \\
&= (a \cdot b) + (b \cdot c) && \text{Idempotent law} \\
&= b \cdot (a+c) && \text{Distributive law used in reverse}
\end{aligned}$$

iv. Draw the resulting simplified circuit.



(b) i. Name two properties a function has to satisfy to be an invertible function.

For a function to be invertible it must be injective and surjective (having both properties in bijective).

ii. Show that the function $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = 3x - 5$ is a bijection.

f is injective if for all $a, b \in \mathbb{R}$, if $a \neq b$, then $f(a) \neq f(b)$.

$a \neq b \to 3a \neq 3b \to 3a - 5 \neq 3b - 5 \to f(a) \neq f(b) \to$ f is injective .

f is surjective is for all $y \in \mathbb{R}$, there exists $x \in \mathbb{R}$, such that $f(x) = y$:

$f(x) = y \to 3x - 5 = y \to 3x = y + 5 \to x = \dfrac{y+5}{3} \in \mathbb{R} \to$ f is surjective

Since f is both injective and surjective (bijective), it is an invertible function.

$\square$

iii. Consider the following function invertible $f : \mathbb{R} \to \mathbb{R}^+$ with $f(x) = 2^{x+1}$. Find its inverse function, f^{-1}.

$$
\begin{aligned}
y &= 2^{x+1} \\
x &= 2^{y+1} \text{swap x and y and solve for y} \\
log_2 x &= log_2 2^{y+1} \\
log_2 x &= (y+1) \cdot log_2 2 \\
log_2 x &= (y+1) \\
log_2 x - 1 &= y \\
y &= log_2 x - 1 \in \mathbb{R} \\
x &\in \mathbb{R}^+
\end{aligned}
$$

$f^{-1} : \mathbb{R}^+ \to \mathbb{R}$ with $f^{-1}(x) = log_2 x - 1$

iv. Plot the curve of f and f^{-1} in the graph.

$f(x)$ and $f^{-1}(x)$ is symmetric with respect to $y = x$. $f(x)$ crosses at $(0, 2)$ and $f^{-1}(x)$ crosses at $(2, 0)$.

Alternate form:

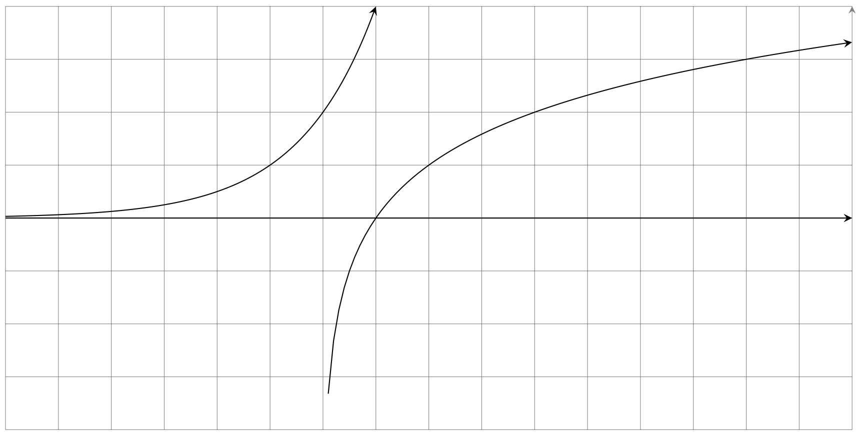
Given $f: \mathbb{R} \to \mathbb{R}^+ \; with \; f(x) = 2^{x+1}$

Let $y \in \mathbb{R}^+, where \; y = 2^{x+1}$

$\therefore \log_2 y = x + 1$

$\Rightarrow x = \log_2 y - 1 \in \mathcal{R}$

$\therefore f^{-1}: \mathbb{R}^+ \to \mathbb{R} \; with \; f^{-1}(x) = \log_2 x - 1$



(c) i. Consider the recursive relation:

What are the values of f_2 and f_3?

$f_n = n \cdot f_{n-1}$ for all integer $n \geq 1$ with $f_1 = 1$.

$f_2 = 2 \cdot f_{n-1} = 2 \cdot f_1 = 2 \cdot 1 = 2$

$f_3 = 3 \cdot f_{n-1} = 3 \cdot f_2 = 3 \cdot 2 = 6$

ii. Let $S_n = \sum_{i=1}^{i=n}(4i-1)$ for all $n \in \mathbb{Z}^+$.

1. Find S_1 and S_2.

$$S_1 = \sum_{i=1}^{i=1}(4i-1) = S_1 = (4 \cdot 1) - 1 = 3$$

$$S_2 = \sum_{i=1}^{i=2}(4i-1) = S_1 + S_2 = 3 + (4 \cdot 2) - 1 = 10$$

2. Prove by induction that $S_n = 2n^2 + n$ for all $n \in \mathbb{Z}^+$

Proof. Let $P(n) : S_n = 2n^2 + n$

Basis step Show $S(1)$ is true.

$$S_1 = 2 \cdot 1^2 + 1 = 3 \qquad \text{Which is correct as per 1.}$$

Inductive step Assuming $P(n)$ we need to show $P(n+1)$

$$\text{Our inductive hypothesis: } S_n = \sum_{i=1}^{i=n}(4i - 1) = 2n^2 + n$$

$$\begin{aligned}
S_{n+1} &= S_n + (4 \cdot (n+1) - 1) \quad \text{adding n+1th term} \\
&= 2n^2 + n + 4n + 4 - 1 \\
&= 2n^2 + 5n + 3 \\
2(n+1)^2 + (n+1) &= 2n^2 + 5n + 3 \\
2(n+1)(n+1) + (n+1) &= 2n^2 + 5n + 3 \\
2(n^2 + n + n + 1) + (n+1) &= 2n^2 + 5n + 3 \\
2(n^2 + 2n + 1) + (n+1) &= 2n^2 + 5n + 3 \\
2n^2 + 4n + 2 + (n+1) &= 2n^2 + 5n + 3 \\
2n^2 + 5n + 3 &= 2n^2 + 5n + 3
\end{aligned}$$

$\square$

$(a+b)^2 = a^2 + 2ab + b^2$

Inductive step

Let S_n $be\ true\ for\ some\ n = k \ \in \mathbb{Z}^+$

$$\therefore S_k = \sum_{i=1}^{i=k}(4i-1) = 2k^2 + k$$

$$S_{k+1} = \sum_{i=1}^{i=k}(4i-1) + [4(k+1)-1] = 2k^2 + k + [4(k+1)-1]$$

$$= 2k^2 + 5k + 3$$

$$= 2(k^2 + 2k + 1) + (k+1)$$

$$= 2(k+1)^2 + (k+1)$$

which is the same as $S_n = 2n^2 + n\ for\ n = k+1$

$\therefore S_k \Rightarrow S_{k+1}$

2 Question 4Graph Theory, Trees Relations

(a) i. Give a definition of a simple graph.

A simple graph contains no loops, and no parallel edges, and is undirected.

ii. Is it possible to draw a simple graph with a degree sequence, 4, 3, 3,2? If yes draw the graph and if no, explain why.

No it's not possible because there are 4 vertices and the max degree of each vertex must be $n - 1$. This means the graph represented by the degree sequence may contain a parallel edge or loop.

iii. Draw the two graphs with adjacency lists

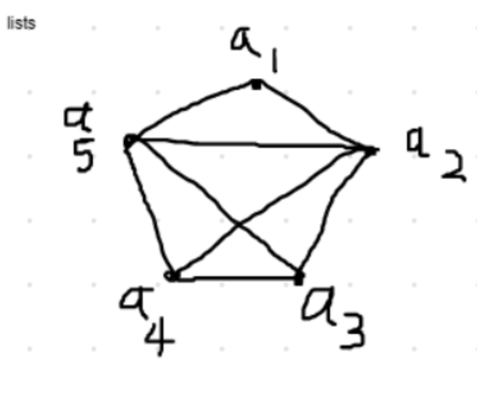
•a1:a2, a5

•a2:a1, a3, a4, a5

•a3:a2, a4, a5

•a4:a2, a3, a5

•a5:a1, a2, a3, a4



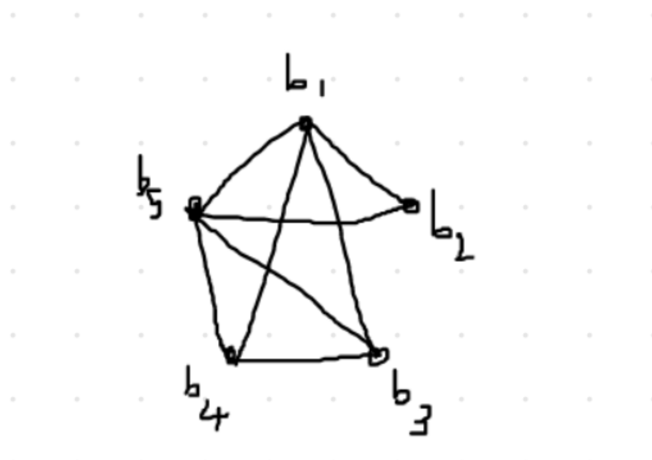
•b1:b2, b3, b4, b5

•b2:b1, b5

•b3:b1, b4, b5

•b4:b1, b3, b5

•b5:b1, b2, b3, b4



1. Write down the degree sequence for each graph above.

$G_a = 4, 4, 3, 3, 2$ $G_b = 4, 4, 3, 3, 2$

2. Are these graphs isomorphic? If so, show the correspondence between them. Mapping 1:

$f(a_1) = b_2$

$f(a_2) = b_1$

$f(a_3) = b_3$

$f(a_4) = b_4$

$f(a_5) = b_5$

Mapping 2: $f(a_3) = b_4$

$f(a_4) = b_3$

$f(a_1) = b_2$

$f(a_5) = b_5$

$f(a_2) = b_1$

Mapping 3:

$f(a_3) = b_3$

$f(a_4) = b_4$

$f(a_1) = b_2$

$f(a_5) = b_5$

$f(a_2) = b_1$

They are isomorphic, we've shown multiple mappings.

b) i. What is the number of vertices in a tree with n edges?

A tree with n vertices has $n - 1$ edges. Then a tree with n edges has $n + 1$ vertices.

ii) Explain how to find the minimum spanning tree in a weighted graph using Prim's algorithm.

Start with any node in the spanning tree. Add the cheapest edge, and add the node it leads to (without creating a cycle) that's not already in the tree.

Prim's algorithm from wiki:

The Prim's algorithm may informally be described as performing the following steps:

1. Initialize a tree with a single vertex, chosen arbitrarily from the graph. 2. Grow the tree by one edge: of the edges that connect the tree to vertices not

yet in the tree, find the minimum-weight edge, and transfer it to the tree. 3. Repeat step 2 (until all vertices are in the tree).

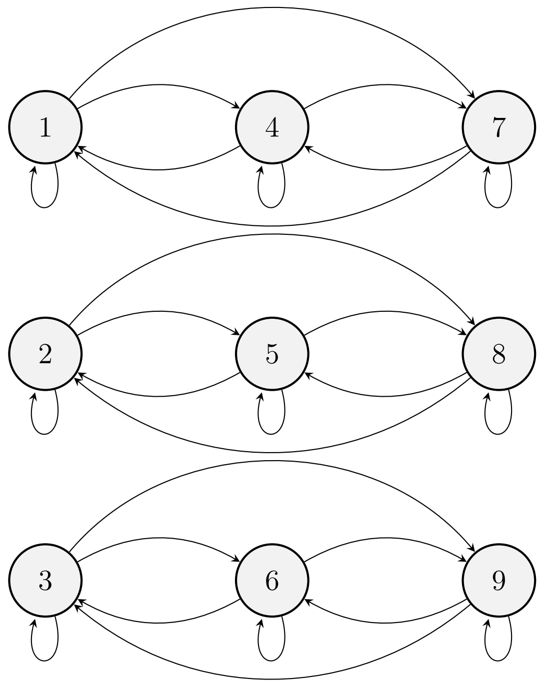
Kruskal's algorithm from wiki:

1. create a forest F (a set of trees), where each vertex in the graph is a separate tree 2. create a set S containing all the edges in the graph 3. while S is nonempty and F is not yet spanning 3.1 remove an edge with minimum weight from S 3.2 if the removed edge connects two different trees then add it to the forest F, combining two trees into a single tree At the termination of the algorithm, the forest forms a minimum spanning forest of the graph. If the graph is connected, the forest has a single component and forms a minimum spanning tree

At the termination of the algorithm, the forest forms a minimum spanning forest of the graph. If the graph is connected, the forest has a single component and forms a minimum spanning tree

The reader should note the difference between Prim's and Kruskal's algorithms. In Prim's algorithm edges of minimum weight that are incident to a vertex already in the tree, and not forming a circuit, are chosen; whereas in Kruskal's algorithm edges of minimum weight that are not necessarily incident to a vertex already in the tree, and that do not form a circuit, are chosen

The reader should note the difference between Prim's and Kruskal's algorithms. In Prim's algorithm edges of minimum weight that are incident to a vertex already in the tree, and not forming a circuit, are chosen; whereas in Kruskal's algorithm edges of minimum weight that are not necessarily incident to a vertex already in the tree, and that do not form a circuit, are chosen



(c) Let S be the set of integers $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ Let $\mathcal{R}$ be a relation defined on S by the following condition such that,
for all $x, y \in S, x\mathcal{R}y$ if x mod 3 $= y$ mod 3.

i. Draw the digraph of R. :

1 mod 3 = 1

2 mod 3 = 2

3 mod 3 = 0

4 mod 3 = 1

5 mod 3 = 2

6 mod 3 = 0

7 mod 3 = 1

8 mod 3 = 2

9 mod 3 = 0

Equivalence classes:

$[1] = \{1, 4, 7\}$

$[2] = \{2, 5, 8\}$

$[0] = \{3, 6, 9\}$

i. Say with reason whether or not R is

•reflexive;

•symmetric;

•anti-symmetric;

•transitive.

In the cases where the given property does not hold provide a counter example to justify this.

Reflexive: $\forall a \in S, (a, a) \in \mathcal{R}$ It is reflexive because $a\mathcal{R}a$, example (3,3)

Symmetric: $\forall a, b \in S, ((a, b) \in \mathcal{R} \rightarrow (b, a) \in \mathcal{R})$ It's symmetric, example (3,6) and (6,3) or (1, 4) and (4,1) . Anti-symmetric: $\forall a, b \in S((a, b) \in \mathcal{R} \wedge (b, a) \in \mathcal{R}) \rightarrow a = b)$ Not anti-symmtric, counter (1,4) and (4,1), but x does not = y

Transitive: $\forall a, b, c \in S((a, b) \in \mathcal{R} \wedge (b, c) \in \mathcal{R} \rightarrow (a, c) \in \mathcal{R})$ Transitive, example $(1, 4) \wedge (4, 7) \rightarrow (1, 7)$.

iii) is R a partial order? Explain your answer.

A partial order must be reflexive, anti-symmetric, and transitive. This relation is not a Partial order

iv. is R an equivalence relation? If the answer is yes, write down the equivalence classes for this relation and if the answer is no, explain why.

R is an equivalence relation, because it is reflexive, symmetric, and transitive.

$[1] = \{1, 4, 7\}$

$[2] = \{2, 5, 8\}$

$[0] = \{3, 6, 9\}$