p.316, icon at Example 1
#2. Use the Principle of Mathematical Induction to prove the "generalized" distributive law

$$a(b_1 + b_2 + \cdots + b_n) = ab_1 + ab_2 + \cdots + ab_n$$

for all integers $n \geq 2$.

Prove the above proposition is true.

BASIS STEP

P(2) (n=2):

$a(b_1 + b_2) = ab_1 + ab_2$ is true because of the distributive property

INDUCTIVE STEP

1. Assume P(k) is true.

$a(b_1 + b_k) = ab_1 + ab_k$

P(k): $a(b_1 + ... + b_k) = ab_1 + ... + ab_k$

2. Show P(k) -> P(k+1).

P(k+1): $a(b_1 + ... + b_k + b_{k+1}) = ab_1 + ... + ab_k + ab_{k+1}$

The $b_1 + ... + b_k + b_{k+1}$ inside of the left expression can be expressed as combined or separate terms according to the associative property:

$b + ... + b_k + b_{k+1} = (b + ... + b_k) + (b_{k+1})$

As in basis step, use the distributive property to distribute *a* to both terms in the expression:

$a((b_1 + ... + b_k) + (b_{k+1})) = a(b_1 + ... + b_k) + a(b_{k+1})$

Distribute the *a* to $a(b_{k+1})$:

$a((b_1 + ... + b_k) + a(b_{k+1})) = a(b_1 + ... + b_k) + ab_{k+1}$

Replace $a(b_1 + ... + b_k)$, which is P(k) on the right side with $ab_1 + ... + ab_k + ab_{k+1}$ according to step 2

$a((b_1 + ... + b_k) + a(b_{k+1})) = ab_1 + ... + ab_k + ab_{k+1}$

$a(b_1 + ... + b_k + b_{k+1}) = ab_1 + ... + ab_k + ab_{k+1}$ is P(k+1)

p.316, icon at Example 1
#3. Use the Principle of Mathematical Induction to prove that

$$\sum_{i=1}^{n} (2i + 3) = n(n + 4) \quad \text{for all } n \geq 1.$$

$\sum_{i=1}^{n} (2i + 3) = n(n + 4)$ for all $n \geq 1$

1) Let P(n) be the proposition $\sum_{i=1}^{n}(2i+3) = n(n+4)$ for all $n \geq 1$

2) Basis step: Show that P(1) is true:
   2(1) + 3 = 1(1+4)
   5 = 5

3) Inductive step: P(k) -> P(k+1). Let's assume that P(k) is true.

   P(k): $\sum_{i=1}^{k}(2i+3) = k(k+4)$ for n = k.

   Under this assumption, we must show that P(k+1) is true.

   Show P(k+1): $\sum_{i=1}^{k+1}(2i+3) = (k+1)(k+1+4)$

   $$\sum_{i=1}^{k+1}(2i+3) = (k+1)(k+5)$$

   $$\sum_{i=1}^{k+1}(2i+3) = \sum_{i=1}^{k}(2i+3) + (2k+5) = (k+1)(k+5)$$

   ASIDE: (2k+5), comes from the summation formula where the pattern of sums would have been (2(1) +3) + (2(2)+3) + (2k+3), the last value would have k+1 substituted into the equation. (2(k+1)+3) = (2k+2+3) = 2k+5.

   $\sum_{i=1}^{k}(2i+3) + (2k+5)$ = k(k+4) + (2k+5)

   $$\sum_{i=1}^{k}(2i+3) + (2k+5) = k^2 + 4k + 2k + 5$$

   $$\sum_{i=1}^{k}(2i+3) + (2k+5) = k^2 + 6k + 5$$

   $$\sum_{i=1}^{k}(2i+3) + (2k+5) = (k+1)(k+5)$$

   Therefore we have shown that P(n) for all n>= 1.

p.319, icon at Example 5

#1. Use the Principle of Mathematical Induction to show that the following inequality is true for all integers $n \geq 2$:

$$\sum_{i=1}^{n} \frac{1}{\sqrt{i}} > \sqrt{n}.$$

1) Let P(n) be the proposition $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} > \sqrt{n}$

2) Basis step: Show that P(2) is true: $\sum_{i=1}^{n} \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} > \sqrt{2}$

   1 + 0.7 > 1.4 We have shown that P(2) is true.

3) Inductive step: P(k) -> P(k+1). Let's assume that P(k) is true.

   P(k): $\sum_{i=1}^{k} \frac{1}{\sqrt{i}} > \sqrt{k}$ for n = k.

   Under this assumption we must show that P(k+1) is true.

   P(k+1) : $\sum_{i=1}^{k+1} \frac{1}{\sqrt{i}} > \sqrt{k+1}$ = $\sum_{i=1}^{k} \frac{1}{\sqrt{i}} + \frac{1}{\sqrt{k+1}} > \sqrt{k+1}$

Start on the right side of the equation:

1. $\sum_{i=1}^{k} \frac{1}{\sqrt{i}} + \frac{1}{\sqrt{k+1}} > \sqrt{k+1}$

We substitute $\sum_{i=1}^{k} \frac{1}{\sqrt{i}}$ with $\sqrt{k}$ because we know it's even smaller as P(k): $\sum_{i=1}^{k} \frac{1}{\sqrt{i}} > \sqrt{k}$:

2. $\sqrt{k} + \frac{1}{\sqrt{k+1}} > \sqrt{k+1}$

We multiply both sides by $\sqrt{k+1}$:

3. $(\sqrt{k+1})(\sqrt{k} + \frac{1}{\sqrt{k+1}}) > (\sqrt{k+1})(\sqrt{k+1})$

   $(\sqrt{k+1})(\sqrt{k} + \frac{1}{\sqrt{k+1}}) > k+1$ (just showing each side separately for clarity)

   $\sqrt{k+1}\ \sqrt{k} + 1 > k+1$

Subtract 1 from both sides:

4. $\sqrt{k+1}\ \sqrt{k} > k$

Multiply $\sqrt{k+1}$ by $\sqrt{k}$:

5. $\sqrt{k^2+k} > k$

Represent the right side as $\sqrt{k^2}$:

6. $\sqrt{k^2+k} > \sqrt{k^2}$

We can see that this is true.

Explaining this procedure: we did a series of “safe” transformations to the expression we were originally trying to prove. The final expression is true, so we know the original expression is true.

Need to prove: Y > A

Known X < Y

If we prove X > A, then, given Y > X, Y > X > A => Y > A

p.321, icon at Example 8

#1. Prove that 6 is a divisor of $4^n + 7^n + 1$ for all positive integers n.

1) Let P(n) be the proposition that 6 is a divisor of 4^n+7^n+1 for all positive integers *n.*
2) Basis Step: Show that P(1) is true: 4^1+7^1+1 = 12, we know that 12 is divisible by 6.
3) Inductive step: P(k)->P(k+1). We assume that P(k) is true.
   Inductive Hypothesis: P(k): 4^k+7^k+1 is divisible by 6 for n = k.
   4^k+7^k+1 = 6m $m \in \mathbb{N}$
   Under this assumption, we must show that P(k+1) is also true.
   P(k+1): $4^{k+1}+7^{k+1}+1$ = 6m where $m \in \mathbb{N}$

4)Use the exponent laws to remove the +1 from the 4^{k+1} and 7^{k+1} (eg: $2^3=2(2^2)$):

$4^{k+1}+7^{k+1}+1$ = $4 \cdot 4^k + 7 \cdot 7^k + 1$ = …

$= 4 \cdot 4^k + 7 \cdot 7^k + 1$

$= (3+1) \cdot 4^k + (6+1) \cdot 7^k + 1$

On the right side, distribute the 4^k to the (3+1) and the 7^k to the (6+1). Remove the 1s.

$= 3 \cdot 4^k + 1 \cdot 4^k + 6 \cdot 7^k + 1 \cdot 7^k + 1$

5) Group terms:

$= (3 \cdot 4^k + 6 \cdot 7^k) + (4^k + 7^k + 1)$ <- this is P(k) which = 6m

$= (3 \cdot 4^k + 6 \cdot 7^k) + 6m$

$= (3 \cdot 4 \cdot 4^{k-1} + 6 \cdot 7^k) + 6m$

$= (12 \cdot 4^{k-1} + 6 \cdot 7^k) + 6m$

$= 6(2 \cdot 4^{k-1} + 7^k + m)$ which is divisible by 6.

Generalization of step done in 4:

$(a+b)4^k = a4^k + b4^k$

$(a+b)4^k = a4^k + b4^k$

4 = 3 + 1

$4(4^k) = (1+3)(4^k) = 4^k + 3(4^k)$

p.316, icon at Example 1
#4. Find a formula for

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{n^2}\right)$$

for $n \geq 2$, and use the Principle of Mathematical Induction to prove that the formula is correct.

1) Let P(n) be the proposition: $\prod_{i=1}^{n}(1-\frac{1}{i^2}) = \frac{n+1}{2n} when\ n \geq 2$

   Note: The character Π is used for multiplication in the same way we use Sigma for summation.

   n=2 =¾ n=3 = 8/9 n=4=15/16 n=5=24/25 n=6=35/36 n=7=48/49 n=8=63/64
   n=9=80/81 n=10=99/100 n=11=120/121
   Multiplication sets:
   2=¾ ||| 2,3=⅔ ||| 2,3,4=⅝ ||| 2...5=3/5 ||| 2...6=7/12 ||| 2...7=4/7
   2...8=9/16 ||| 2...9=5/9 ||| 2...10=11/20

2) Basis Step: Show that P(2) is true = $(1-\frac{1}{2^2})$ = ¾ ...3 is 2+1 and 4 is 2(2)
3) Induction Step: P(k) -> P(k+1) . Assume that P(k) is true.

   P(k): $\prod_{i=1}^{k}(1-\frac{1}{i^2}) = \frac{k+1}{2k}$ (our inductive hypothesis)

   Under this assumption, we must show that P(k+1) is also true.

   P(k+1): $\prod_{i=1}^{k+1}(1-\frac{1}{i^2}) = \frac{k+2}{2(k+1)} = \frac{k+2}{2k+2}$

Replace the left side with

$\frac{k+1}{2k}(1-\frac{1}{k+1^2}) = \frac{k+2}{2k+2}$ (the left most expression is P(k))

$\frac{k+1}{2k}(1-\frac{1}{k+1^2}) = \frac{k+2}{2k+2}$

$\frac{k+1}{2k}(\frac{k+1^2}{k+1^2}-\frac{1}{k+1^2}) = \frac{k+2}{2k+2}$

$\frac{k+1}{2k}(\frac{k+1^2-1}{k+1^2}) = \frac{k+2}{2k+2}$

$\frac{k+1}{2k}(\frac{k+1^2-1}{k+1^2}) = \frac{k+2}{2k+2}$

$\frac{k+1}{2k}(\frac{k+1^2-1}{k+1^2}) = \frac{k+2}{2k+2}$

(k+1)^2 - 1 = (k+1 + 1)(k+1-1) = (k+2)k

$\frac{k+1}{2k}(\frac{(k+1-1)(k+1+1)}{(k+1)(k+1)}) = \frac{k+2}{2k+2}$

$\frac{k+1}{2k}(\frac{k(k+2)}{(k+1)(k+1)}) = \frac{k+2}{2k+2}$

$\frac{1}{2}\frac{(k+2)}{(k+1)} = \frac{k+2}{2k+2}$

\

$\frac{k^2-1}{k^2}$

(k-1)/k = 1 - 1/k != 1 - 1/(k+1)^2 !

$\prod_{i=1}^{k}(1-\frac{1}{i^2})(1-\frac{1}{k+1^2}) = \frac{k+1}{2k}\ (1-\frac{1}{k+1^2}\ )$

9) a)Find a formula for the sum of the first n even positive Integers.

N	2n	= F(n)
1	2	= 2
2	4	= 6
3	6	= 12
4	8	= 20
5	10	= 30

F(n) = Sum(i=1; i<=n){2i}

F(n) = 2 + 4 + 6 + …+ 2n = ??

$\sum_{i=1}^{n}$ 2i $= n^2 + n$ = n(n+1)

1) Let P(n) be the proposition: $\sum_{i=1}^{n}$ 2i = n(n+1).
2) Basis step: Show P(1):

2(1) = 1(1+1)
2 = 2
We have shown that P(1) is true.

3) Inductive step: P(k) -> P(k+1). Let's assume that P(k) is true.
P(k): $\sum_{i=1}^{n}$ 2i= $k(k+1)$ where n=k
Under this assumption we must show that P(k+1) is also true.

P(k+1): $\sum_{i=1}^{k+1}$ 2i = $(k+1)(k+1+1)$
$\sum_{i=1}^{k+1}$ 2i = $(k+1)(k+2)$

$\sum_{i=1}^{k+1}$ 2i = $\sum_{i=1}^{k}$ 2i + 2(k+1) = k(k+1) + 2(k+1)
= $k^2 + k + 2k + 2$
= $k^2 + 3k + 2$
= (k+1)(k+2)
Therefore, we have shown that P(n) holds for all n>=1.