7.1 Introduction to graph theory: basic concepts

Notebook:	Discrete Mathematics [CM1020]		
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Cornell Notes	**Topic:** 7.1 Introduction to graph theory: basic concepts	Course: BSc Computer Science Class: Discrete Mathematics-Lecture Date: December 11, 2019

Essential Question:

What is a graph and how it is represented with edges, vertices, loops and paths?

Questions/Cues:

- What is a graph?
- What are the origins of graph theory?
- What are some real-world applications of graph theory?
- What is a more formal definition of a graph?
- What is a vertex?
- What is an edge?
- What is meant by adjacency in graph theory?
- What are loops and parallel edges?
- What is a directed graph or Digraph?
- What is a walk?
- What is a trail?
- What is a circuit?
- What is a path?
- What is a cycle?
- What is an Euler path?
- What is an Hamiltonian path?
- What is an Hamiltonian cycle?
- What is an Hamiltonian graph?
- What is connectivity in terms of graphs?
- What is strong connectivity?
- What is Transitive Closure?
- What is the degree of a vertex in terms of an undirected graph?
- What is the in-degree/out-degree of a vertex?
- What is the degree sequence of a graph?
- What are the properties of a degree sequence?
- What is a simple graph?
- What are the properties of simple graphs?
- What is a regular graph?
- What are the properties of a regular graph?
- What are some special regular graphs?
- What is a complete graph?

- What are the properties of a complete graph?

Notes

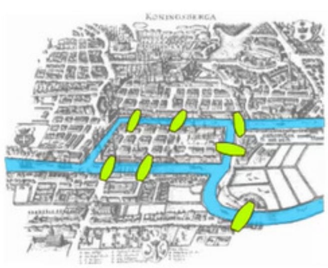
What is a graph?

Graphs are **discrete** structures consisting of **vertices (nodes)** and **edges** connecting them

Graph theory is an area in discrete mathematics which studies these type of discrete structures.

Origins of graph theory

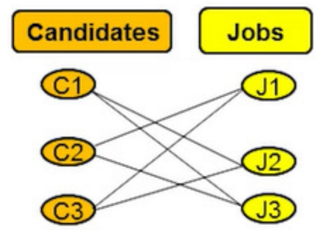
The first problem in graph theory is the **Seven Bridges** of Konigsberg problem solved by Leonhard Euler in 1735



Application of graphs

In a variety of disciplines, problems can be solved using graph models:

- Modelling computer networks
- Modelling road maps
- Solving shortest path problems between cities
- Assigning jobs to employees in an organisation.



Definition: Graph

G is an ordered triple G:=(V, E).

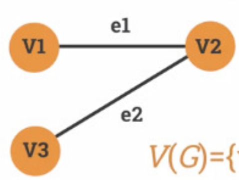
V is a set of nodes, or vertices.

E is a set of edges, lines or connections.

Definition: Vertex

Vertex

- Basic Element of a graph
- Drawn as a *node* or a *dot*
- Set of vertices of *G* is usually denoted by *V*(*G*) or *V*.

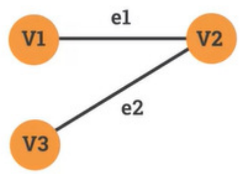


$V(G)=\{v_1, v_2, v_3\}$

Definition: Edges

Edge

- A is a link between 2 vertices
- Drawn as a line connecting two vertices
- The set of edges in a graph G is usually denoted by E(G), or E.



$E(G)=\{e_1, e_2\} = \{\{v_1, v_2\}, \{v_2, v_3\}\}$

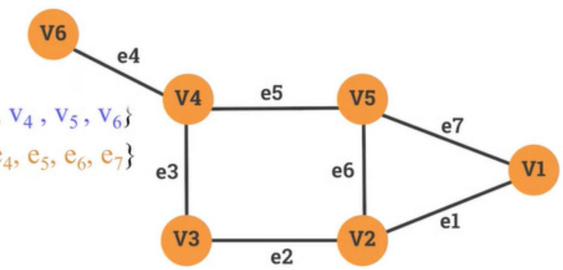
Definition: Adjacency

Adjacency

- Two **vertices** are said to be **adjacent** if they are endpoints of the same edge
- Two **edges** are said **adjacent** if they share the same vertex
- If a vertex **v** is an **endpoint** of an edge **e**, then we say that **e** and **v** are **incident**.

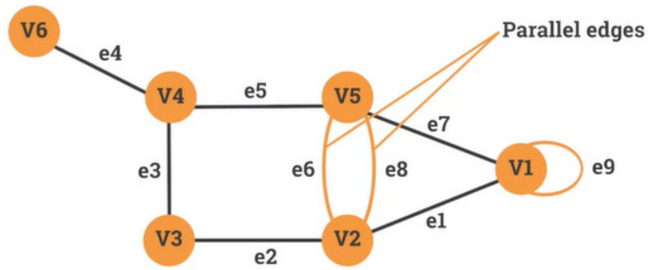
Example

- $V:=\{v_1, v_2, v_3, v_4, v_5, v_6\}$
- $E=\{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$



v_1 and v_2 are endpoints of the edge e_1. We say that v_1 and v_2 are **adjacent**.
The edges e_1 and e_7 share the same vertex v_1. We say that e_1 and e_7 are **adjacent**.
The vertex v2 is an endpoint of the edge e_1. We say that e_1 and v_2 are **incident**.

Loops and parallel edges



v_2 and v_5 are are linked with two edges (e_6 and e_8).
e_6 and e_8 are called parallel edges.

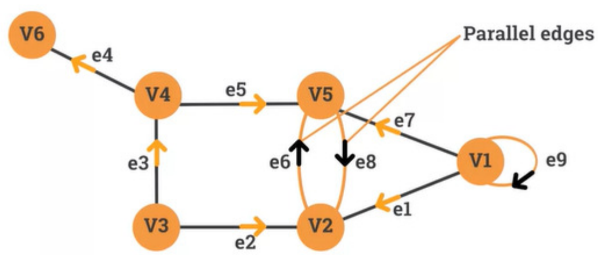
v_1 is linked to itself by e_9. The edge e_9 is called a loop.

Directed graphs — Digraph

A **directed graph**, also called a **digraph**, is a graph in which the edges have a direction.

This is usually indicated with an arrow on the edge.

Directed graphs



e_1 is a connection from v_1 to v_2 but not from v_2 to v_1

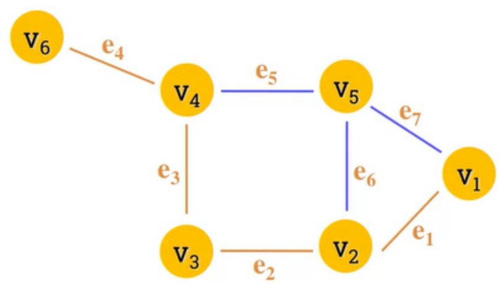
e_6 is a connection from v_2 to v_5 whereas e_8 is a connection from v_5 to v_2

Definition of a walk

A walk is a sequence of vertices and edges of a graph were vertices and edges can be repeated.

A **walk of length k** in a graph is a succession of k (not necessarily different) edges of the form
uv, vw, wx,..., yz.

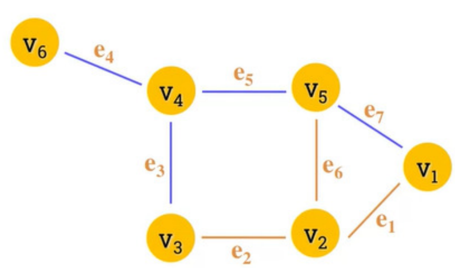
Example 1



$v_1v_2,\ v_2v_3,\ v_3v_4\ ,\ v_4v_6 = e_1\ ,e_2\ ,e_3\ ,e_{4.} = v_1v_2v_3v_4v_6$

A walk of **length 4** from **v_1 to v_{6t}**

Example 2



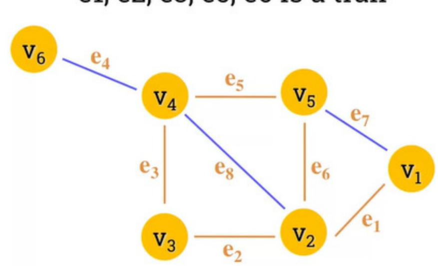
$v_1v_2,\ v_2v_3,\ v_3v_2\ ,\ v_2v_5 = e_1\ ,e_2\ ,e_2\ ,e_{6.}\ =.\ v_1v_2v_3v_2v_5$

A walk of **length 4** from **v_1 to v_5** (passes twice through the edge e_2)

Trail

A **trail** is a walk in which no edge is repeated. In a trail, vertices can be repeated but no edge is ever repeated.

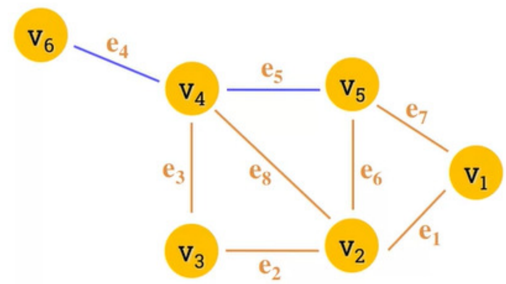
e1, e2, e3, e5, e6 is a trail



Circuit

A **circuit** is a is a closed trail. Circuits can have repeated vertices only.

e7, e6, e8, e3, e2, e1 is a circuit

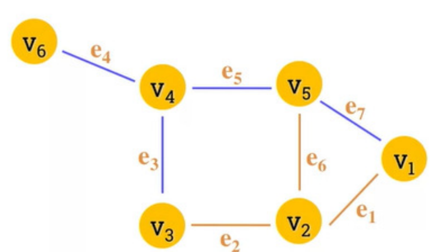


Definition of a path

A **path** is a trail in which neither vertices nor edges are repeated.

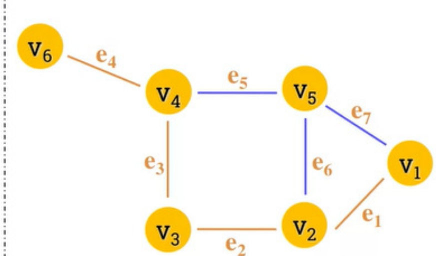
- The length of a path is given by the number of edges it contains

Example



v_1v_2, v_2v_3, v_3v_2, v_2v_5 = e_1, e_2, e_2, e_6. . $v_1v_2v_3v_2v_5$

A walk of **length 4** from **v_1. to v_5** but not a path

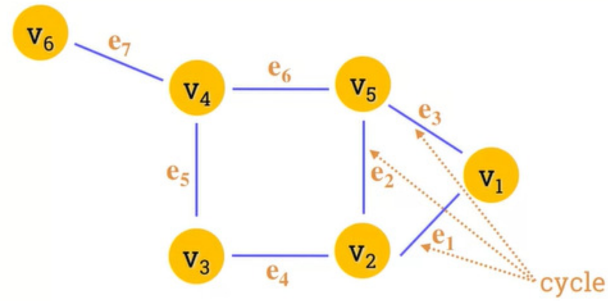


v_1v_2, v_2v_3, v_3v_4 , v_4v_6 = e_1 ,e_2 ,e_3 ,e_4. = $v_1v_2v_3v_4v_6$

A path of **length 4** from **v_1 to v_6**

Cycle

A cycle is a closed path, consisting of edges and vertices where a vertex is reachable from itself.



$v_1v_2,\ v_2v_5,\ v_5v_1 = e_1,\ e_2,\ e_3 = v_1v_2v_5v_1$

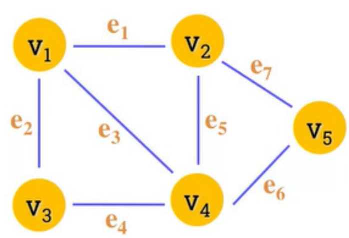
A walk of **length 3** from **v_1 to v_1** = closed path = cycle

- Note** In example above for cycle, it is from $V_1 \,\text{to}\, V_5$, instead of $V_1 \,\text{to}\, V_1$.

Euler path

Definition: A **Eulerian path** in a graph is a path that uses each edge precisely once. If such a path exists, the graph is called **traversable**.

Example:



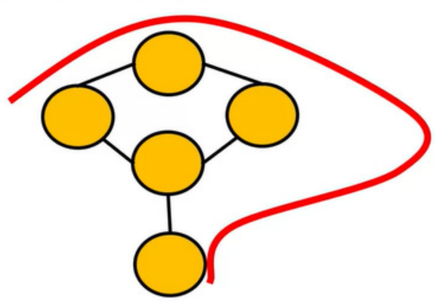
$e_2,\ e_4,\ e_6,\ e_7,\ e_5,\ e_3,\ e_1 = v_1v_3v_4v_5v_2v_4v_1v_2$

is a Euler path

Hamiltonian path

A **Hamiltonian path** (also called a *traceable path*) is a path that visits each vertex exactly once.

Example:

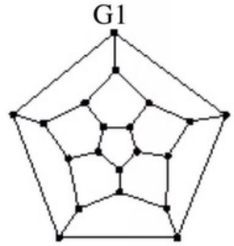


A graph that contains a Hamiltonian path is called a **traceable graph**.

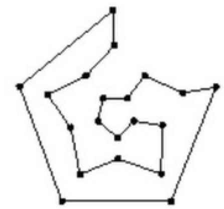
Hamiltonian cycle

A **Hamiltonian cycle** is a cycle that visits each vertex exactly once (except for the starting vertex, which is visited once at the start and once again at the end).

G1



G2



Hamiltonian cycle

Hamiltonian graph

A graph that contains a Hamiltonian cycle is called a **Hamiltonian graph**.

Any Hamiltonian cycle can be converted to a Hamiltonian path by removing one of its edges.

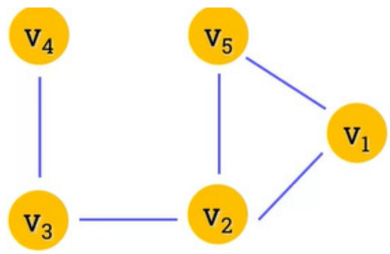
Connectivity

An ***undirected*** graph is ***connected*** if

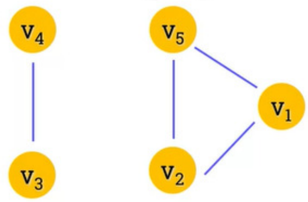
you can get from ***any node to any other*** by following a ***sequence of edges***

OR

any two nodes are **connected** by a path.



Connected graph

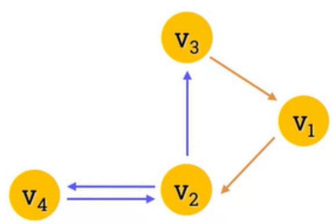


Not connected graph

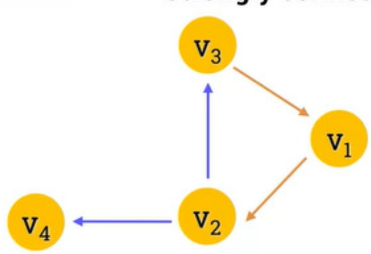
there is no path from (v_1, v_2 or v_6) to (v_3 or v_4)

Strong Connectivity

A directed graph is **strongly connected** if there is a **directed path** from any node to any other node.



Strongly connected directed graph

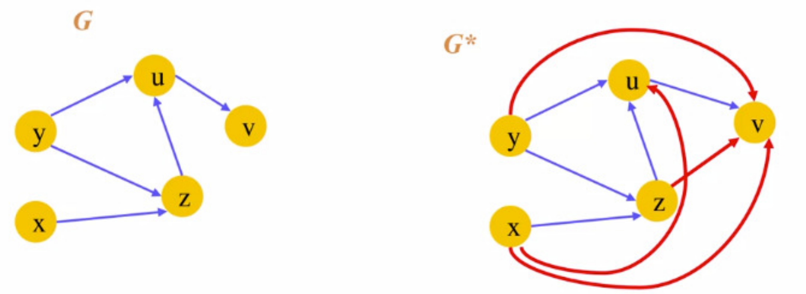


Not Strongly connected directed graph

No directed path from v_4 to any of the other 3 vertices

Transitive Closure

Given a digraph G, the transitive closure of G is the digraph G^* such that:
G^* has the same vertices as G
if G has a directed path from u to v ($u \neq v$), G^* has a directed edge from u to v



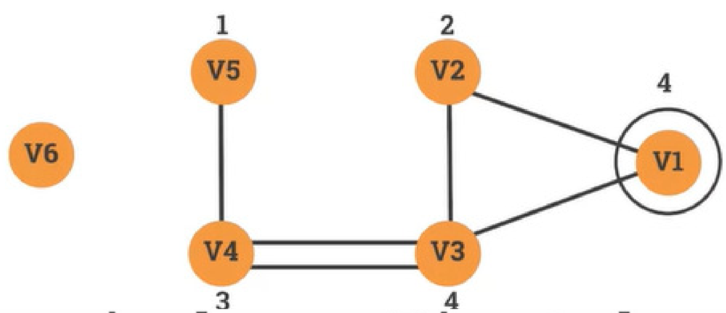
The transitive closure provides reachability information about a digraph.

Terminology – Undirected graphs

Degree of a vertex (deg (v)): the number of edges incident on v

A loop contributes **twice** to the degree

An isolated vertex has a degree : 0



Terminology – Directed graphs

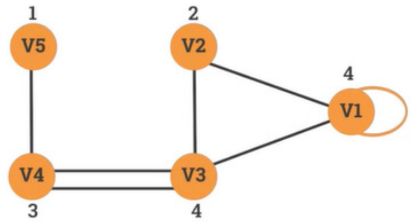
In-deg (v): number of edges for which v is the terminal vertex

Out-deg (v): number of edges for which v the is initial vertex

deg (v) = Out-deg (v) + IN-deg (v)

A loop contributes **twice** to the degree as it contributes 1 to both in-degree and out-degree.

Example



$\deg(v_1) = \text{in-deg}(v_1) + \text{out-deg}(v_1) = 2 + 2 = 4$
$\deg(v_2) = \text{in-deg}(v_2) + \text{out-deg}(v_2) = 1 + 1 = 2$
$\deg(v_3) = \text{in-deg}(v_3) + \text{out-deg}(v_3) = 2 + 2 = 4$
$\deg(v_4) = \text{in-deg}(v_4) + \text{out-deg}(v_4) = 1 + 2 = 3$
$\deg(v_5) = \text{in-deg}(v_5) + \text{out-deg}(v_5) = 1 + 0 = 1$
$\deg(v_6) = \text{in-deg}(v_6) + \text{out-deg}(v_6) = 0 + 0 = 0$

Degree sequence of a graph

Given an undirected graph G, a **degree sequence** is a **monotonic nonincreasing** sequence of the vertex degrees of all the vertices of G.

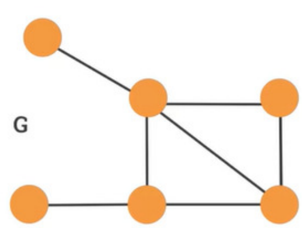
- Written in descending order separated by commas

Degree sequence property 1

The **sum of the degree sequence** of a graph is always **even**.

Therefore, it is impossible to construct a graph where the sum of the degree sequence is odd.

Example



The degree sequence of G is: 4,3,3,2,1,1

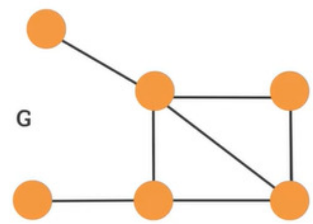
Sum of the degree sequence = 1+1+2+3+3+4 = 14

Degree sequence property 2

Given a graph G, the sum of the degree sequence of G is **twice** the number of edges in G.

Number of edges(G) = (sum of degree sequences of G) / 2

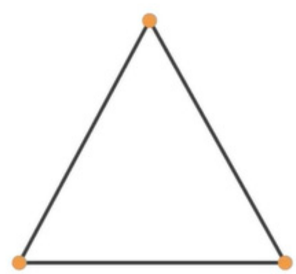
Example 1



The degree degree sequence of G is: 4,3,3,2,1,1

Number of edges = (1+1+2+3+3+4)/2 = 14/2 = 7

Example 2

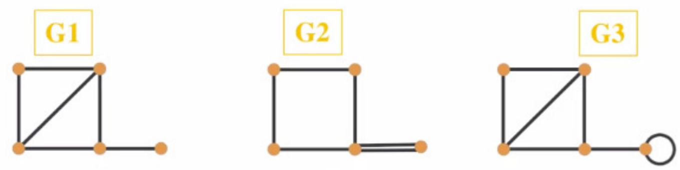


The degree sequence of G is: 2,2,2

Number of edges = (2+2+2)/2 = 6/2 = 3

Simple graphs

A *simple graph* is a graph without loops and parallel edges.



Simple

Not Simple
(parallel edges)

Not Simple
(loop)

Properties of simple graphs

Given a simple graph G with n vertices, then the degree of each vertex of G is at most equal to n-1.

Proof :

Let v be a vertex of G such that deg(v) > n-1

However, we have only n-1 other vertices for v to be connected to

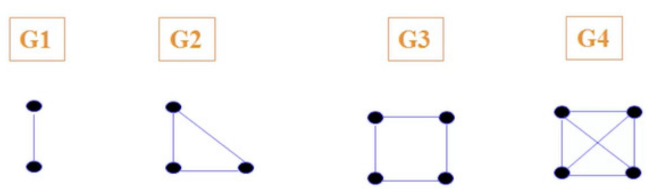
Hence, the other connections can only be a result of parallel edges of loops.

Regular graphs

A **graph** is said to be **regular** of degree if all local degrees are the same number.

A graph G where all the vertices the same degree, r, is called an r-regular graph.

Examples



1-regular graph with 2 vertices

1, 1

2-regular graph with 3 vertices

2, 2, 2

2-regular graph with 4 vertices

2, 2, 2, 2

3-regular graph with 4 vertices

3, 3, 3, 3

Properties of regular graphs

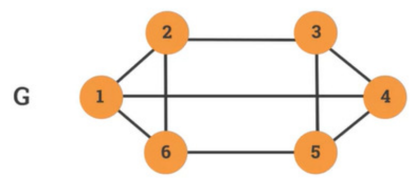
Given an r-regular G with n vertices, then the following is true:

Degree sequence of G = r,r,r,...,r (n times)

Sum of degree sequence of G = r x n

Number of edges in G = r x n/2

Example: 3-regular with 6 vertices

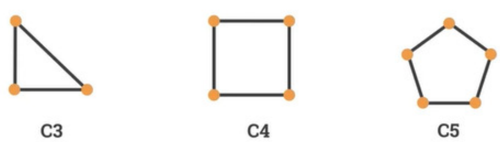


Degree Sequence = 3,3,3,3,3,3

Sum of degree sequence = 3x6=18

Number of edges = 18/2=9

Special regular graphs: cycles



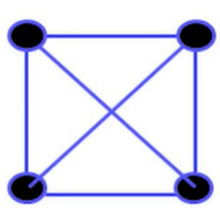
C_3 is 2-regular graph with 3 vertices
C_4 is 2-regular graph with 4 vertices
C_5 is 2-regular graph with 5 vertices

deg seq. of C_3 = 2,2,2
deg seq. of C_4 = 2,2,2,2
deg seq. of C_5 = 2,2,2,2,2

3-regular with 4 vertices

Sum of degree sequence = 3x4 = 12

The sum is even, hence it is possible to construct 3-regular graph with 4 vertices.



3-regular graph with 5 vertices

Sum of degree sequence = 3x5= 15

The sum if odd, hence it is impossible to construct a 3-regular graph with 5 vertices.

Complete graphs

A **complete graph** is a *simple* graph where *every pair of vertices* are *adjacent* (linked with an edge).

We represent a complete graph with n vertices using the symbol K_n.

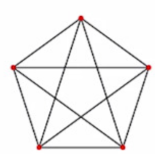
Complete graph properties

A complete graph with n vertices, $\mathbf{k_n}$, has the following properties:

- Every vertex has a degree **(n-1)**
- Sum of degree sequence = **n(n-1)**
- Number of edges = **n(n-1)/2**

Another example of a complete graph

k_5



There are 5 vertices
Degree of each vertex = (5-1) = 4
Sum of deg. Seq. = 5(5-1) = 20
Number of edges = 5(5-1)/2=20/2=10

Summary

In this week, we learned what graph is, how it is defined by its edges, vertices & direction. Alongside this, we explored the different configuration of vertices & edges that result in a path, circuit, cycle and etc. Also we looked at the meaning of a degree sequences and how to count the degree of each vertex with in/out degree for vertices. Finally, we examined special graphs like simple, r-regular and complete graphs.