

9.2 Equivalence, and partial and total order relations

Notebook: Discrete Mathematics [CM1020]

Created: 2019-10-07 2:31 PM

Updated: 2020-01-14 2:01 PM

Author: SUKHJIT MANN

Cornell Notes

Topic:
9.2 Equivalence, and partial
and total order relations

Course: BSc Computer Science

Class: Discrete Mathematics-
Lecture

Date: January 14, 2020

Essential Question:

What is the difference between a equivalence class & relation? Also how is order demonstrated in relations?

Questions/Cues:

- What is an equivalence relation?
- What is an equivalence class?
- What is a partial order in relations?
- What is a total order in relations?

Notes

Definition of equivalence relation

Let R be a relation of elements on a set S . R is
an **equivalence relation**

if and only if

R is **reflexive**, **symmetric** and **transitive**.

Example 1

- Let **R** be **relation** of elements in **Z**:
$$R = \{ (a, b) \in \mathbb{Z}^2 \mid a \bmod 2 = b \bmod 2 \}$$
- We have already proved that this relation is:
 - **reflexive** as $a R a, \forall a \in \mathbb{Z}$
 - **symmetric** as if $a R b$ then $b R a, \forall a, b \in \mathbb{Z}$
 - **transitive** as if $a R b$ and $b R c$ then $a R c, \forall a, b, c \in \mathbb{Z}$
- **R** is an **equivalence relation**.

Example 2

- Let **R** be a **relation** of elements in **Z**:
$$R = \{ (a, b) \in \mathbb{Z}^2 \mid a \leq b \}$$
- We have already proved that this relation is:
 - **reflexive** as $a R a$ for all a in \mathbb{Z}
 - **transitive** as if $a R b$ and $b R c$ then $a R c, \forall a, b, c \in \mathbb{Z}$
 - **not Symmetric** as $2 \leq 3$ but $3 \not\leq 2, \forall a, b \in \mathbb{Z}$
- **R** is not an **equivalence relation**.

Definition of equivalence class

Let **R** be an **equivalence relation** on a set **S**. Then, the **equivalence class** of $a \in S$ is:

the **subset** of **S** containing all the **elements** **related** to **a** through '**R**'.

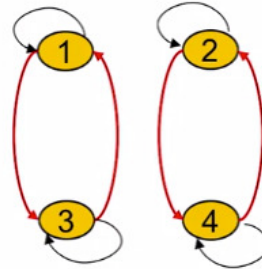
$$[a] = \{x: x \in S \text{ and } x R a\}$$

Example 1

- Let $S = \{1, 2, 3, 4\}$ and R be a relation on elements in S :
 $R = \{ (a, b) \in S^2 \mid a \bmod 2 = b \bmod 2 \}$

- R is an **equivalence relation** with 2 equivalence classes:

- $[1] = [3] = \{1, 3\}$
- $[2] = [4] = \{2, 4\}$



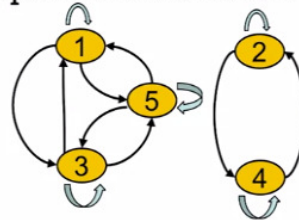
Example 2

- Let $Z = \{1, 2, 3, 4, 5\}$ and R be relation of elements in Z :
 $R = \{ (a, b) \in Z^2 \mid a - b \text{ is an even number} \}$

$$R = \{ (1,1), (2,2), (3,3), (4,4), (5,5), (2,4), (4,2), (1,3), (3,1), (1,5), (5,1), (3,5), (5,3) \}$$

- R is an **equivalence relation** with 2 equivalence classes:

- $[1] = [3] = [5] = \{1, 3, 5\}$
- $[2] = [4] = \{2, 4\}$



Definition of partial order

Let R be a **relation** on elements in a set S . R is a partial order

if and only if

R is **reflexive**, **anti-symmetric** and **transitive**.

Example 1

- Let **R** be a **relation** of elements in **Z**:
$$R = \{ (a, b) \in \mathbb{Z}^2 \mid a \leq b \}$$
- It can easily be proved that **R** is:
 - **reflexive** as $a \leq a, \forall a \in \mathbb{Z}$
 - **transitive** as if $a \leq b$ and $b \leq c$ then $a \leq c, \forall a, b \in \mathbb{Z}$
 - **anti-symmetric** as if $a \leq b$ and $b \leq a$ then $a = b, \forall a, b \in \mathbb{Z}$
- **R** is a **partial order**.

Example 2

- Let **R** be a **relation** of elements in **Z⁺**:
$$R = \{ (a, b) \in \mathbb{Z}^+ \mid a \text{ divides } b \}$$
- It can easily be proved that **R** is:
 - **reflexive** as $a \text{ divides } a, \forall a \in \mathbb{Z}^+$
 - **transitive** as if $a \text{ divides } b$ and $b \text{ divides } c$ then $a \text{ divides } c, \forall a, b, c \in \mathbb{Z}^+$
 - **anti-symmetric** as if $a \text{ divides } b$ and $b \text{ divides } a$ then $a = b, \forall a, b \in \mathbb{Z}^+$
- **R** is a **partial order**.

Definition of Total Order

Let R be a relation on elements in a set S . R is a total order

if and only if

R is a partial order & $\{\forall a, b \in S \mid aRb \text{ or } bRa\}$

This means that R has to be a partial order & every two elements of the set S can be comparable with respect to the relation R

Example 1

- Let **R** be a **relation** of elements in **Z**:
$$\mathbf{R} = \{ (a, b) \in \mathbb{Z}^2 \mid a \leq b \}$$
- It has been previously shown that **R** is a partial order
- Also, $\forall a, b \in \mathbb{Z}, a \leq b$ or $b \leq a$ is true
- **R** is a **total order**.

Example 2

- Let **R** be a **relation** on elements in **Z⁺**:
$$\mathbf{R} = \{ (a, b) \in \mathbb{Z}^+ \mid a \text{ divides } b \}$$
- It has been proved that **R** is a **partial** order
- **Z⁺** contains elements that are **incomparable**, such as 5 and 7
- **R** is **not totally** ordered.

Summary

In this week, we learned what an equivalence relation & class are. Finally we explored the partial & total ordering of a relation.