10.2 Further techniques

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Essential Question:

What are the rules/strategies used when counting objects when they are sampled with or without replacement?

Questions/Cues:

- What is a binomial expression?
- What is the Binomial Theorem?
- What is Pascal's Identity?
- What is Pascal's Triangle?
- What are permutations with repetition?
- What are permutations without repetition?
- What are combinations with repetition?
- What are combinations without repetition?
- How do we choose which formula to use when selecting k-objects from a set with n-elements?
- How do we distribute objects into boxes?
- What is meant by distinguishable/indistinguishable?
- What is meant by with/without exclusion?

Notes

Binomial expression

An expression consisting of two terms, connected by a + or − sign is called a binomial expression.

Examples of binomial expressions:

$x + a;\ 2x - y;\ x^2 - y^2;\ 2x - 3y, \ldots$

Binomial theorem

$(x+y)^1 = x + y$
$(x+y)^2 = x^2 + 2xy + y^2$
$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$
...
$(x+y)^{30}$

Let x and y be variables, and n a non-negative integer.
The expansion of $(x+y)^n$ can be formalised as:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Example

What is the coefficient of x^8y^7 in the expansion of $(3x - y)^{15}$?

Solution:

- We can view the expression as $(3x + (-y))^{15}$
- By the binomial theorem:

  $(3x + (-y))^{15} = \sum_{k=0}^{15} \binom{15}{k} (3x)^k (-y)^{n-k}$

- Consequently, the coefficient of x^8y^7 in the expansion is obtained when k = 8:
  - $\binom{15}{8}(3)^8(-1)^7 = -3^8 \frac{15!}{8!7!}$

Application of the binomial theorem

Let's prove the identity $2^n = \sum_{k=0}^{n} \binom{n}{k}$

Using binomial theorem:

- With x = 1 and y = 1, from the binomial theorem we see that the identity is verified.

Using Sets:

- Consider the subsets of a set with n elements
- There are subsets with zero elements, with one element, with two elements and so on ... with n elements
- Therefore the total number of subsets is: $\sum_{k=0}^{n} \binom{n}{k}$
- Also, since we know that a set with n elements has 2^n subsets, we can conclude that: $2^n = \sum_{k=0}^{n} \binom{n}{k}$

Pascal's identity

If n and k are integers with $n \geq k \geq 1$, then:

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$$

Proof :

- Let T be a set where $|T| = n + 1$, $a \in T$, and $S = T - \{a\}$
- There are $\binom{n+1}{k}$ subsets of T containing k elements. Each of these subsets either:
  - contains a with $k - 1$ other elements, or
  - contains k elements of S and not a
- There are:
  - $\binom{n}{k-1}$ subsets of k elements that contain a
  - $\binom{n}{k}$ subsets of k elements of T that don't contain a
- Hence $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$.

Pascal's triangle

Pascal's triangle is a number triangle with numbers arranged in staggered rows such that $a_{n,r}$ **is the binomial coefficient** $\binom{n}{r}$:

$$\begin{array}{c}
\binom{0}{0} \\
\binom{1}{0}\ \binom{1}{1} \\
\binom{2}{0}\ \binom{2}{1}\ \binom{2}{2} \\
\binom{3}{0}\ \binom{3}{1}\ \binom{3}{2}\ \binom{3}{3} \\
\binom{4}{0}\ \binom{4}{1}\ \binom{4}{2}\ \binom{4}{3}\ \binom{4}{4}
\end{array}
\qquad
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1
\end{array}$$

$$\binom{4}{3} = \binom{3}{2} + \binom{3}{3} = 4 + 1 = 5$$

Using Pascal's identity, we can show that the result of adding two **adjacent** coefficients in this triangle is **equal** to the binomial coefficient in the **next row between these two coefficients**.

Permutations with repetition

The number of **r-permutations** of a set of n objects with repetition allowed is $\mathbf{n^r}$.

Proof:

- Since we have n choices each time, there are n possibilities for the 1^{st} choice, n possibilities for the 2^{nd} choice, ..., and n possibilities when choosing the last number
- By the product rule, multiplying each time:

$$\underbrace{\boxed{n} \times \boxed{n} \times \boxed{n} \times \cdots \times \boxed{n}}_{\text{r times}} = \mathbf{n^r}$$

Example

How many strings of length **r** can be formed if we are using only uppercase letters in the English alphabet?

Solution:

The number of such strings is 26^r, which is the number of r-permutations with repetition of a set with 26 elements.

Permutations without repetition

In the case of permutations without repetition, we reduce the number of available choices each time by 1. The number of **r-permutations** of a set with n objects without repetition is:

$$\underbrace{\boxed{n} \times \boxed{n-1} \times \boxed{n-2} \times \cdots \times \boxed{n-r+1}}_{\text{r times}}$$

$$P(n,r) = P_n^r = n(n-1)(n-2)\ldots(n-r+1) = \frac{n!}{(n-r)!}$$

Example

During a running competition how many different ways can the first and the second place be awarded if 10 runners are taking part in the race?

Solution:

$$P(10,2) = P_{10}^2 = \frac{10!}{(10-2)!} = \frac{10!}{8!} = 90$$

Combination with repetition

The number of ways in which k objects can be selected from n categories of objects, with repetition permitted, can be calculated as: $\binom{k+n-1}{k} = \frac{(k+n-1)!}{k!(n-1)!}$

- It is also the total **number** of ways to put **k identical balls** into n distinct boxes.

- It is also the total **number of functions** from a set of k identical elements to a set of n distinct elements.

Example

Let's find all multisets of size 3 from the set {1, 2, 3, 4}.

Solution:

- Using bars and crosses, think of the values 1, 2, 3, 4 as four categories
- We will denote each multiset of size 3 by placing three crosses in the various categories
- For instance, the multiset {1, 1, 3} is represented by × × | | × |
- This counting problem can be modelled as distributing the 3 crosses among the 3+4−1 positions, the remaining positions being occupied by bars
- Thus the number of multisets of size 3 is: $C(6,3) = \frac{6!}{3!3!} = 20.$

Combination without repetition

- The number of ways in which r objects can be selected from n categories of objects with repetition not permitted can be calculated as: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

- This counting problem is the same as the **number** of ways of **putting** k **identical balls** into n distinct boxes, where each box receives at most one ball

- It is also the **number of one-to-one functions** from a set of k identical elements into a set of n distinct elements

- It is also the number of **k-element subsets** of an n-element set.

Choice of formulas

- We have discussed four different ways of selecting k objects from a set with n elements:
  - the order in which the choices are made may or may not matter,
  - repetition may or may not be allowed.
- The following table summarises the formula in each case:

	Order matters	Order does not matter
Repetition is not permitted	$\frac{n!}{(n-k)!}$	$\frac{n!}{k!\,(n-k)!}$
Repetition is permitted	n^k	$\frac{(k+n-1)!}{k!\,(n-1)!}$

Example

John is the chair of a committee. In how many ways can a committee of 3 be chosen from 10 people, given that John must be one of the people selected?

Solution:

- Since John is already chosen, we need to choose another 2 out of 9 people.
- In choosing a committee, the order doesn't matter, so we need to apply the combination without repetition formula: $C(9,2) = \frac{9!}{2!(9-2)!} = 36$ ways.

Distributing objects into boxes

Counting problems can be phrased in terms of distributing k **objects** into n **boxes** under various conditions:

- The **objects** can be either distinguishable or indistinguishable
- The **boxes** can be either distinguishable or indistinguishable
- The distribution can be done either with exclusion or without exclusion.

Distinguishable = refers to objects or boxes that are marked in some way that makes each one distinguishable from the other
Indistinguishable = refers to objects or boxes that are identical, so that there is no way to tell them apart

Note When placing indistinguishable objects into distinguishable boxes, it makes no difference which object is placed into which box
With exclusion = means that no box can contain more than one object
Without exclusion = means that a box may contain more than one object

Distinguishable objects and distinguishable boxes with exclusion

In this case, we want to **distribute** k **balls, numbered** from 1 to k, into n **boxes**, **numbered** from 1 to n, in such a way that **no box** receives **more than one ball**.

This is equivalent to making an **ordered selection of k boxes** from **n boxes**, where the **balls** do the selecting for us:

- the ball labelled 1 chooses the first box
- the ball labelled 2 chooses the second box
- and so on ...

Distinguishable objects and distinguishable boxes with exclusion

Theorem:

Distributing **k** distinguishable **balls** into n distinguishable **boxes**, with exclusion, is equivalent to forming a permutation of size k from a set of size n.

Therefore, the number of ways of placing k distinguishable balls into n distinguishable boxes is as follows:

$$P(n,k) = n(n-1)(n-2)\ldots(n-k+1) = \frac{n!}{(n-k)!}$$

Distinguishable objects and distinguishable boxes without exclusion

In this case, we want to **distribute** k **balls**, **numbered** from 1 to k, into n **boxes**, **numbered** from 1 through n, **without restrictions** on the number of **balls** in each box.

This is equivalent to making an **ordered selection of k boxes from n**, **with repetition**, where the balls do the selecting for us:

- the ball labelled 1 chooses the first box
- the ball labelled 2 chooses the second box
- and so on ...

Distinguishable objects and distinguishable boxes without exclusion

Theorem:

Distributing k distinguishable balls into n distinguishable boxes, without exclusion, is equivalent to forming a permutation of size *k* from a set of size *n,* with repetition.

Therefore, there are:

n^k different ways.

Indistinguishable objects and distinguishable boxes with exclusion

In this case, we want to **distribute** k **balls**, into n **boxes**, **numbered** from 1 through n, in such a way that **no box** receives more **than one ball**.

Theorem:

Distributing k indistinguishable balls into n distinguishable boxes, with exclusion, is equivalent to forming a combination of size *k* from a set of size *n.*

Therefore, there are

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$ different ways

Indistinguishable objects and distinguishable boxes without exclusion

In this case, we want to **distribute** k **balls**, into n **boxes**, **numbered** from 1 through n, **without restrictions** on the number of balls in each box.

Theorem :

Distributing k indistinguishable balls into n distinguishable boxes, without exclusion, is equivalent to forming a combination of size *k* from a set of size *n,* with repetition.

Therefore, there are

$\binom{n+k-1}{k} = \frac{(n+k-1)!}{k!(n-1)!}$ different ways

Example

How many ways are there of placing 8 indistinguishable balls into 6 distinguishable boxes?

$$\binom{8+6-1}{8} = \binom{13}{8} = \frac{13!}{8!5!} = 1{,}287$$

Summary

In this week, we learned about distinguishable & indistinguishable objects/boxes and the different permutation/combination formulas to apply in various such scenarios.