4.1 The Basics

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Cornell Notes	**Topic:** 4.1 The Basics	Course: BSc Computer Science Class: Discrete Mathematics-Lecture Date: November 13, 2019

Essential Question:

What is Predicate Logic and Quantification?

Questions/Cues:

- Why is propositional logic insufficient?
- What is a predicate?
- What is Quantification?
- What is the Universal Quantifier?
- What is the Existential Quantifier?
- What is the Uniqueness Quantifier?
- What are Nested Quantifiers?
- What is meant by Binding variables in terms of Predicate logic?
- What the logical operators in terms of Predicate Logic?
- What is the Order of Precedence in Predicate Logic?

Notes

- Insufficiency of Propositional logic = Cannot express precisely the meaning of complex statements in Math & only studies propositions; statements with known truth values
- Predicate = Predicates more general form of proposition
  - Predicates behave as functions whose values are T or F depending on their variables
  - Predicates become propositions when their variables are given actual values.

The statement “x squared is equal to 4” has two parts:

- The **variable** x, which is the subject of the statement
- The **predicate** “squared is equal to 4”, which is the property that the subject of the statement can have

- This statement can be formalised as **P(x)** where P is the predicate “squared is equal to 4” and x is the variable
- **P** is said to be the **propositional function**
- Once a value is assigned to the variable x, the statement P(x) becomes a proposition and has a truth value.

A **predicate** can depend on **more than one** variable.

Examples

Let P(x, y) denote “$x^2 > y$”;

- $P(-2, 3) \equiv (4 > 3)$ is True
- $P(2, 4) \equiv (2^2 > 4)$ is False

Let Q(x, y, z) denote “$x+y< z$”;

- $Q(2, 4, 5) \equiv (6 < 5)$ is F
- $Q(1, 3, 7) \equiv (4 < 7)$ is T
- $Q(1, 3, z) \equiv (4 < z)$ is **not** a proposition

Logical operations from propositional logic carry over to predicate logic.

Example

If P(x) denotes “ $x^2 < 16$ ”, then:

- $P(1) \vee P(-5) \equiv (1 < 16) \vee (25 < 16) \equiv T \vee F \equiv T$
- $P(1) \wedge P(-5) \equiv T \wedge F \equiv F$
- $P(3) \wedge P(y)$ is **not** a proposition. It **becomes** a proposition when y is assigned a **value**.

- Quantification = expresses the extent to which a predicate is true over a range of elements
  - express meaning of words all and some
  - Two most important quantifiers are: universal and existential quantifier

Example

- “All men are mortal.”
- “Some computers are not connected to the network.”

Universal quantifier

Definition

- The universal quantification of a predicate $P(x)$ is the proposition:
  - "$P(x)$ is true for all values of x in the universe of discourse."
- We use the notation: $\forall x\, P(x)$, which is read "for all x".

If the universe of discourse is finite, say $\{n_1, n_2, \ldots, n_k\}$, then the universal quantifier is simply the **conjunction** of the propositions over all the elements:

$$\forall x\, P(x) \Leftrightarrow P(n_1) \wedge P(n_2) \wedge \ldots \wedge P(n_k)$$

Let P, Q denote the following propositional functions of x:

- P(x): "x must take a discrete mathematics course."
- $Q(x)$: "x is a Computer Science student."
- Where, the universe of discourse for both $P(x)$ and $Q(x)$ is all university students.

Let's express the following statements:

- "Every CS student must take a discrete mathematics course."

$$\forall x\, Q(x) \rightarrow P(x)$$

- "Everybody must take a discrete mathematics course or be a CS student."

$$\forall x\, (P(x) \vee \mathrm{Q}(\mathrm{x}))$$

- "Everybody must take a discrete mathematics course and be a CS student."

$$\forall x\, (P(x) \wedge \mathrm{Q}(\mathrm{x}))$$

Let's formalise the following statement S:

S: "For every x and every y, $x+y > 10$."

- Let $P(x,y)$ be the statement $x+y > 10$, where the universe of discourse for x, y is the set of integers

- The statement S is: $\forall x \forall y\, P(x,y)$
- Which can also be written as: $\forall x,y\, P(x,y)$

Existential quantifier

Definition
The existential quantification of a predicate $P(x)$ is the proposition:

- "There exists a value x in the universe of discourse such that $P(x)$ is true."

- We use the notation: $\exists x\, P(x)$, which is read "there exists x".

If the universe of discourse is finite, say $\{n_1, n_2, \dots, n_k\}$, then the existential quantifier is simply the **disjunction** of the propositions over all the elements:

$$\exists x\, P(x) \Leftrightarrow P(n_1) \vee P(n_2) \vee \dots \vee P(n_k)$$

- Let $P(x,y)$ denote the statement "x+y = 5"

The expression **E**: $\exists x \exists y\, P(x,y)$ means:

- "There exists a value x and a value y in the universe of discourse such that x+y = 5 is true."

For instance

- If the universe of discourse is **positive integers, E** is **True**
- If the universe of discourse is **negative integers, E** is **False**.

- Let a, b, c denote fixed real numbers.
- And S be the statement: "**There exists** a **real** solution to $\boldsymbol{ax^2+bx-c=0}$"
- S can be expressed as $\exists x\ P(x)$, where:
  - $P(x)$ is $ax^2+bx-c=0$ and the universe of discourse for x is the set of real numbers.
- Let's evaluate the truth value of S:
  - When $b^2>=4ac$, S *is* ***true***, *as* $P(-b\pm\sqrt{(b^2-4ac)})/2a) = 0$
  - When $b^2<4ac$, S *is* ***false***, as there is no real number x that can satisfy the predicate.

Uniqueness quantifier

Definition

- The uniqueness quantification of a predicate $P(x)$ is the proposition:
  - "There exists a unique value x in the universe of discourse such that $P(x)$ is true."

- We use the notation: $\exists ! x \ \ P(x)$, which is read "There exists a unique *x.*"

  - special case of the existential quantifier

- Let $P(x)$ denote the statement "$x^2 = 4$"

- The expression **E**: $\exists ! x \ \ P(x)$ means:
  - "There exists **a unique** value x in the universe of discourse such that $x^2 = 4$ is true."

For instance

- If the universe of discourse is **positive integers**, **E** is **True** (as x = 2 is the unique solution)
- If the universe of discourse is **integers**, **E** is **False** (as x = 2 and x = -2 are both solutions).

Nested quantifiers

To express statements with multiple variables we use nested quantifiers.

Nested quantifiers	Meaning
$\forall x \ \forall y \ P(x, y)$	P(x, y) is true for every pair x, y.
$\exists x \ \exists y \ P(x, y)$	There is a pair x, y for which P(x, y) is true.
$\forall x \ \exists y \ P(x, y)$	For every x there is a y for which P(x, y) is true.
$\exists x \ \forall y \ P(x, y)$	There is an x for which P(x, y) is true for every y.

Binding variables

- A variable is said to be **bound** if it is within the scope of a quantifier
- A variable is **free** if it is not bound by a quantifier or particular values.

Example

- Let P be a propositional function.
- And S the statement: $\exists$ x P(x,y)
- We can say that:
  - x is bound
  - y is free.

- **Logical operations**, which were discussed in the topic on propositional logic ($\neg \wedge \vee \rightarrow \leftrightarrow$), can also be applied to quantified statements.

Example

- If P(x) denotes "x > 3" and Q(x) denotes "x squared is even" then:
- $\exists$ x (P(x) $\wedge$ Q(x)) $\equiv$ T (ex. x = 4)
- $\forall$ x (P(x) $\rightarrow$ Q(x)) $\equiv$ F (ex. x = 5)

- When nested quantifiers are of the **same** type, the order does **not** matter
- With quantifiers of **different** types, the order **does** matter.

Example

- $\forall$ x $\forall$ y P(x,y) $\equiv$ $\forall$ y $\forall$ x P(x,y)
- $\exists$ x $\exists$ y P(x,y) $\equiv$ $\exists$ y $\exists$ x P(x,y)
- $\forall$ x $\exists$ y P(x,y) is different from $\exists$ y $\forall$ x P(x,y)

- The quantifiers ∀ and ∃ have a **higher precedence priority** than all **logical** operators.

Example

- Let **P(x)** and **Q(x)** denote two propositional functions.
- $\forall x\, P(x) \lor Q(x)$ is the **disjunction** of $\forall x\, P(x)$ and $Q(x)$ rather than $\forall x\, (P(x) \lor Q(x))$.
- $\forall x\, P(x) \rightarrow Q(x)$ is the **implication** of $\forall x\, P(x)$ and $Q(x)$ rather than $\forall x\, (P(x) \rightarrow Q(x))$.

Summary

In this week, we learned what Predicate logic is and what predicates are. Also, we looked at quantification and the different quantifiers in predicate logic. Finally, we explored the order of precedence and logical operators in terms of predicate logic.