4.2 Applications-Reading

Notebook:	Discrete Mathematics [CM1020]		
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Cornell Notes	**Topic:** 4.2 Applications-Reading	Course: BSc Computer Science Class: Discrete Mathematics-Reading Date: November 17, 2019

Essential Question:

What are the rules of inference and/or the rules of inference with quantifiers?

Questions/Cues:

- What is an argument?
- What are rules of inference?
- What are the steps to building a valid argument?
- What are fallacies?
- What are the rules of inference?
- What is Universal instantiation?
- What is Universal generalization?
- What is Existential instantiation?
- What is Existential generalization?
- What is Universal modus ponens?
- What is is Universal modus tollens?

Notes

- Argument = sequence of statements that with a conclusion
- Valid = meaning the conclusion of the argument must follow from the truth of the preceding premises of argument. That is argument is valid if and only if it is impossible for all premises to be true and conclusion to be false
- Fallacies = incorrect reasoning which leads to invalid arguments

An *argument* in propositional logic is a sequence of propositions. All but the final proposition in the argument are called *premises* and the final proposition is called the *conclusion*. An argument is *valid* if the truth of all its premises implies that the conclusion is true.

An *argument form* in propositional logic is a sequence of compound propositions involving propositional variables. An argument form is *valid* no matter which particular propositions are substituted for the propositional variables in its premises, the conclusion is true if the premises are all true.

The tautology $(p \land (p \to q)) \to q$ is the basis of the rule of inference called **modus ponens**, or the **law of detachment**. (Modus ponens is Latin for *mode that affirms*.) This tautology leads to the following valid argument form, which we have already seen in our initial discussion about arguments (where, as before, the symbol $\therefore$ denotes "therefore"):

$$\begin{array}{l} \quad p \\ \quad p \to q \\ \hline \therefore q \end{array}$$

Using this notation, the hypotheses are written in a column, followed by a horizontal bar, followed by a line that begins with the therefore symbol and ends with the conclusion. In particular, modus ponens tells us that if a conditional statement and the hypothesis of this conditional statement are both true, then the conclusion must also be true. Example 1 illustrates the use of modus ponens.

TABLE 1 Rules of Inference.

Rule of Inference	*Tautology*	*Name*
p $p \to q$ $\therefore q$	$(p \land (p \to q)) \to q$	Modus ponens
$\neg q$ $p \to q$ $\therefore \neg p$	$(\neg q \land (p \to q)) \to \neg p$	Modus tollens
$p \to q$ $q \to r$ $\therefore p \to r$	$((p \to q) \land (q \to r)) \to (p \to r)$	Hypothetical syllogism
$p \lor q$ $\neg p$ $\therefore q$	$((p \lor q) \land \neg p) \to q$	Disjunctive syllogism
p $\therefore p \lor q$	$p \to (p \lor q)$	Addition
$p \land q$ $\therefore p$	$(p \land q) \to p$	Simplification
p q $\therefore p \land q$	$((p) \land (q)) \to (p \land q)$	Conjunction
$p \lor q$ $\neg p \lor r$ $\therefore q \lor r$	$((p \lor q) \land (\neg p \lor r)) \to (q \lor r)$	Resolution

Show that the premises "It is not sunny this afternoon and it is colder than yesterday," "We will go swimming only if it is sunny," "If we do not go swimming, then we will take a canoe trip," and "If we take a canoe trip, then we will be home by sunset" lead to the conclusion "We will be home by sunset."

Solution: Let p be the proposition "It is sunny this afternoon," q the proposition "It is colder than yesterday," r the proposition "We will go swimming," s the proposition "We will take a canoe trip," and t the proposition "We will be home by sunset." Then the premises become $\neg p \wedge q$, $r \rightarrow p$, $\neg r \rightarrow s$, and $s \rightarrow t$. The conclusion is simply t. We need to give a valid argument with premises $\neg p \wedge q$, $r \rightarrow p$, $\neg r \rightarrow s$, and $s \rightarrow t$ and conclusion t.

We construct an argument to show that our premises lead to the desired conclusion as follows.

Step	Reason
1. $\neg p \wedge q$	Premise
2. $\neg p$	Simplification using (1)
3. $r \rightarrow p$	Premise
4. $\neg r$	Modus tollens using (2) and (3)
5. $\neg r \rightarrow s$	Premise
6. s	Modus ponens using (4) and (5)
7. $s \rightarrow t$	Premise
8. t	Modus ponens using (6) and (7)

Show that the premises "If you send me an e-mail message, then I will finish writing the program," "If you do not send me an e-mail message, then I will go to sleep early," and "If I go to sleep early, then I will wake up feeling refreshed" lead to the conclusion "If I do not finish writing the program, then I will wake up feeling refreshed."

Solution: Let p be the proposition "You send me an e-mail message," q the proposition "I will finish writing the program," r the proposition "I will go to sleep early," and s the proposition "I will wake up feeling refreshed." Then the premises are $p \rightarrow q$, $\neg p \rightarrow r$, and $r \rightarrow s$. The desired conclusion is $\neg q \rightarrow s$. We need to give a valid argument with premises $p \rightarrow q$, $\neg p \rightarrow r$, and $r \rightarrow s$ and conclusion $\neg q \rightarrow s$.

This argument form shows that the premises lead to the desired conclusion.

Step	Reason
1. $p \rightarrow q$	Premise
2. $\neg q \rightarrow \neg p$	Contrapositive of (1)
3. $\neg p \rightarrow r$	Premise
4. $\neg q \rightarrow r$	Hypothetical syllogism using (2) and (3)
5. $r \rightarrow s$	Premise
6. $\neg q \rightarrow s$	Hypothetical syllogism using (4) and (5)

The proposition $((p \to q) \wedge q) \to p$ is not a tautology, because it is false when p is false and q is true. However, there are many incorrect arguments that treat this as a tautology. In other words, they treat the argument with premises $p \to q$ and q and conclusion p as a valid argument form, which it is not. This type of incorrect reasoning is called the **fallacy of affirming the conclusion**.

Is the following argument valid?

> If you do every problem in this book, then you will learn discrete mathematics. You learned discrete mathematics.
>
> Therefore, you did every problem in this book.

Solution: Let p be the proposition "You did every problem in this book." Let q be the proposition "You learned discrete mathematics." Then this argument is of the form: if $p \to q$ and q, then p. This is an example of an incorrect argument using the fallacy of affirming the conclusion. Indeed, it is possible for you to learn discrete mathematics in some way other than by doing every problem in this book. (You may learn discrete mathematics by reading, listening to lectures, doing some, but not all, the problems in this book, and so on.) ◀

The proposition $((p \to q) \wedge \neg p) \to \neg q$ is not a tautology, because it is false when p is false and q is true. Many incorrect arguments use this incorrectly as a rule of inference. This type of incorrect reasoning is called the **fallacy of denying the hypothesis**.

Let p and q be as in Example 10. If the conditional statement $p \to q$ is true, and $\neg p$ is true, is it correct to conclude that $\neg q$ is true? In other words, is it correct to assume that you did not learn discrete mathematics if you did not do every problem in the book, assuming that if you do every problem in this book, then you will learn discrete mathematics?

Solution: It is possible that you learned discrete mathematics even if you did not do every problem in this book. This incorrect argument is of the form $p \to q$ and $\neg p$ imply $\neg q$, which is an example of the fallacy of denying the hypothesis. ◀

Universal instantiation is the rule of inference used to conclude that $P(c)$ is true, where c is a particular member of the domain, given the premise $\forall x\, P(x)$. Universal instantiation is used when we conclude from the statement "All women are wise" that "Lisa is wise," where Lisa is a member of the domain of all women.

Universal generalization is the rule of inference that states that $\forall x\, P(x)$ is true, given the premise that $P(c)$ is true for all elements c in the domain. Universal generalization is used when we show that $\forall x\, P(x)$ is true by taking an arbitrary element c from the domain and showing that $P(c)$ is true. The element c that we select must be an arbitrary, and not a specific, element of the domain. That is, when we assert from $\forall x\, P(x)$ the existence of an element c in the domain, we have no control over c and cannot make any other assumptions about c other than it comes from the domain. Universal generalization is used implicitly in many proofs in mathematics and is seldom mentioned explicitly. However, the error of adding unwarranted assumptions about the arbitrary element c when universal generalization is used is all too common in incorrect reasoning.

Existential instantiation is the rule that allows us to conclude that there is an element c in the domain for which $P(c)$ is true if we know that $\exists x\, P(x)$ is true. We cannot select an arbitrary value of c here, but rather it must be a c for which $P(c)$ is true. Usually we have no knowledge of what c is, only that it exists. Because it exists, we may give it a name (c) and continue our argument.

Existential generalization is the rule of inference that is used to conclude that $\exists x\, P(x)$ is true when a particular element c with $P(c)$ true is known. That is, if we know one element c in the domain for which $P(c)$ is true, then we know that $\exists x\, P(x)$ is true.

TABLE 2 Rules of Inference for Quantified Statements.

Rule of Inference	*Name*
$\dfrac{\forall x P(x)}{\therefore\ P(c)}$	Universal instantiation
$\dfrac{P(c) \text{ for an arbitrary } c}{\therefore\ \forall x P(x)}$	Universal generalization
$\dfrac{\exists x P(x)}{\therefore\ P(c) \text{ for some element } c}$	Existential instantiation
$\dfrac{P(c) \text{ for some element } c}{\therefore\ \exists x P(x)}$	Existential generalization

Show that the premises "Everyone in this discrete mathematics class has taken a course in computer science" and "Marla is a student in this class" imply the conclusion "Marla has taken a course in computer science."

Solution: Let $D(x)$ denote "x is in this discrete mathematics class," and let $C(x)$ denote "x has taken a course in computer science." Then the premises are $\forall x(D(x) \to C(x))$ and $D(\text{Marla})$. The conclusion is $C(\text{Marla})$.

The following steps can be used to establish the conclusion from the premises.

Step	**Reason**
1. $\forall x(D(x) \to C(x))$	Premise
2. $D(\text{Marla}) \to C(\text{Marla})$	Universal instantiation from (1)
3. $D(\text{Marla})$	Premise
4. $C(\text{Marla})$	Modus ponens from (2) and (3)

◀

Show that the premises "A student in this class has not read the book," and "Everyone in this class passed the first exam" imply the conclusion "Someone who passed the first exam has not read the book."

Solution: Let $C(x)$ be "x is in this class," $B(x)$ be "x has read the book," and $P(x)$ be "x passed the first exam." The premises are $\exists x(C(x) \wedge \neg B(x))$ and $\forall x(C(x) \to P(x))$. The conclusion is $\exists x(P(x) \wedge \neg B(x))$. These steps can be used to establish the conclusion from the premises.

Step	**Reason**
1. $\exists x(C(x) \wedge \neg B(x))$	Premise
2. $C(a) \wedge \neg B(a)$	Existential instantiation from (1)
3. $C(a)$	Simplification from (2)
4. $\forall x(C(x) \to P(x))$	Premise
5. $C(a) \to P(a)$	Universal instantiation from (4)
6. $P(a)$	Modus ponens from (3) and (5)
7. $\neg B(a)$	Simplification from (2)
8. $P(a) \wedge \neg B(a)$	Conjunction from (6) and (7)
9. $\exists x(P(x) \wedge \neg B(x))$	Existential generalization from (8)

◀

universal modus ponens. This rule tells us that if $\forall x(P(x) \rightarrow Q(x))$ is true, and if $P(a)$ is true for a particular element a in the domain of the universal quantifier, then $Q(a)$ must also be true. To see this, note that by universal instantiation, $P(a) \rightarrow Q(a)$ is true. Then, by modus ponens, $Q(a)$ must also be true. We can describe universal modus ponens as follows:

$$\begin{array}{rl} & \forall x(P(x) \rightarrow Q(x)) \\ & \underline{P(a), \text{ where } a \text{ is a particular element in the domain}} \\ \therefore & Q(a) \end{array}$$

Assume that "For all positive integers n, if n is greater than 4, then n^2 is less than 2^n" is true. Use universal modus ponens to show that $100^2 < 2^{100}$.

Solution: Let $P(n)$ denote "$n > 4$" and $Q(n)$ denote "$n^2 < 2^n$." The statement "For all positive integers n, if n is greater than 4, then n^2 is less than 2^n" can be represented by $\forall n(P(n) \rightarrow Q(n))$, where the domain consists of all positive integers. We are assuming that $\forall n(P(n) \rightarrow Q(n))$ is true. Note that $P(100)$ is true because $100 > 4$. It follows by universal modus ponens that $Q(100)$ is true, namely that $100^2 < 2^{100}$. ◀

universal modus tollens. Universal modus tollens combines universal instantiation and modus tollens and can be expressed in the following way:

$$\begin{array}{rl} & \forall x(P(x) \rightarrow Q(x)) \\ & \underline{\neg Q(a), \text{ where } a \text{ is a particular element in the domain}} \\ \therefore & \neg P(a) \end{array}$$

Summary

In this week, we learned about rules of inference, rules of inferences with quantifiers and fallacies. Alongside we explored the steps to building a valid argument.