

## 4.2 Applications

**Notebook:** Discrete Mathematics [CM1020]

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Cornell Notes	Topic: 4.2 Applications	Course: BSc Computer Science
		Class: Discrete Mathematics- Lecture
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Essential Question:		
What are the rules of inference and/or the rules of inference with quantifiers?		
Questions/Cues:		
<ul style="list-style-type: none"><li>• What are DeMorgan's laws for quantifiers?</li><li>• What is an argument?</li><li>• What are rules of inference?</li><li>• What are the steps to building a valid argument?</li><li>• What is fallacy and/or formal fallacy?</li><li>• What are the rules of inference with quantifiers?</li><li>• What are the steps to expressing complex statements?</li></ul>		
Notes		
<ul style="list-style-type: none"><li>• DeMorgan's laws for Quantifiers = stem from the need to consider negation of a quantified expression</li></ul> <p><b>Example</b></p> <ul style="list-style-type: none"><li>• S: "All the university's computers are connected to the network."</li><li>• P: "There is at least one computer in the university operating on Linux."</li></ul> <p><b>Intuitively</b></p> <ul style="list-style-type: none"><li>• The negation of S can be verified <b>if there is at least one</b> computer not connected to the network</li><li>• The negation of P can be verified if <b>all university computers</b> are not operating on Linux</li><li>• De Morgan's laws formalise these intuitions.</li></ul> <ul style="list-style-type: none"><li>◦ Let P be a predicate over the variable x, it follows that:</li></ul>		

## De Morgan's laws for negating quantifiers:

$$\neg \forall x P(x) \equiv \exists x \neg P(x)$$

$$\neg \exists x P(x) \equiv \forall x \neg P(x)$$

Let S: "Every student of Computer Science has taken a course in Neural Networks."

- S can be expressed as:  $\forall x P(x)$
- $U = \{\text{students in CS}\}$
- $P(x)$ : "x has taken a course in Neural Networks."

The **Negation** of S:

- "It is not the case that every student of Computer Science has taken a course in Neural Networks."  
 $\neg(\forall x P(x)) \equiv \exists x \neg P(x)$
- This implies that: "There **is** at least one student who has **not** taken a course in Neural Networks."

Let R denote: "There is a student in Computer Science who didn't take a course in Machine Learning."

- R can be expressed as:  $\exists x Q(x)$
- $U = \{\text{students in CS}\}$
- $Q(x)$ : "x didn't take a course in Machine Learning"

The **Negation** of S:

- "It is not the case that there is a student in Computer Science who didn't take a course in Machine Learning."  
 $\neg(\exists x Q(x)) \equiv \forall x \neg Q(x)$
- This implies that: "Every student in Computer Science has taken a Machine Learning course."

In the case of nested quantifiers, we apply De Morgan's laws successively from left to right.

### Example

Let  $P(x,y,z)$  denote a propositional function of variables:  $x, y$  and  $z$ .

$$\equiv \exists x \neg \exists y \forall z P(x, y, z)$$

$$\equiv \exists x \forall y \neg \forall z P(x, y, z)$$

$$\equiv \exists x \forall x \exists z \neg P(x, y, z)$$

$\neg \forall x \exists y \forall z P(x, y, z)$  is built by moving the negation to the right through all the quantifiers and replacing each  $\forall$  with  $\exists$ , and vice versa

- Argument = in propositional logic is sequence of propositions
  - final proposition is called conclusion, other propositions in argument called premises (or hypotheses)
  - Argument is valid if the truth of its premises implies truth of the conclusion

Let's consider this **argument**:

- "If you have access to the internet, you can order a book on Machine Learning."
- "You have access to the internet."

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$\therefore$  Therefore: "You can order a book on Machine Learning."

This argument is **valid** because whenever all its premises are true, the conclusion must also be true.

Let's consider this **argument** :

- "If you have access to Internet, you can order a book on Machine Learning."
- "You can order a book on Machine Learning."

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$\therefore$  Therefore: "You have access to Internet."

This argument is **not valid** because we can imagine situations where the premises are true and the conclusion is false.

- Rules of inference = seen as building block in constructing incrementally complex valid arguments
  - we can use truth table to determine whether argument is T or F, but is a long process especially with multiple variables
  - ROI, a simpler way of proving validity of argument
  - Every ROI can be proved using a tautology

## Modus ponens

- **Tautology:**  $(p \wedge (p \rightarrow q)) \rightarrow q$

- **The rule of inference:**

$$\begin{array}{l} p \rightarrow q \\ p \\ \hline \therefore q \end{array}$$

### Example

- **p:** "It is snowing."      **q:** "I will study Discrete Mathematics."
- "If it is snowing, I will study Discrete Mathematics."
- "It is snowing."
- Therefore: "I will study discrete mathematics."

- In Modus ponens, if p implies q & premise is true, then conclusion is also true

## Modus tollens

- **Tautology:**  $(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$

- **The rule of inference:**

$$\begin{array}{l} \neg q \\ p \rightarrow q \\ \hline \therefore \neg p \end{array}$$

### Example

- **p:** "It is snowing."      **q:** "I will study discrete mathematics."
- "If it is snowing, I will study Discrete Mathematics."
- "I will not study Discrete Mathematics."
- Therefore: "It is not snowing."

- In modus tollens, if premise not q is true and if conditional statement p implies q is true, conclusion not p is also true

# Conjunction

- **Tautology:**  $((p) \wedge (q)) \rightarrow (p \wedge q)$
- **The rule of inference:**

$$\begin{array}{c} p \\ q \\ \hline \therefore p \wedge q \end{array}$$

## Example

- **p:** "I will study Programming." **q:** "I will study Discrete Mathematics."
- "I will study Programming."
- "I will study Discrete Mathematics."
- Therefore: "I will study Programming and Discrete Mathematics."

- In conjunction, if premise p is true and if premise q is true, the conclusion p and q is also true

# Simplification

- **Tautology:**  $(p \wedge q) \rightarrow p$
- **The rule of inference:**

$$\begin{array}{c} p \wedge q \\ \hline \therefore p \end{array}$$

## Example

- **p:** "I will study Programming." **q:** "I will study Discrete Mathematics."
- I will study Programming and Discrete Mathematics
- Therefore: "I will study Discrete Mathematics."

- In simplification, if the premise p and q is true, the conclusion p is also true

# Addition

- **Tautology:**  $p \rightarrow (p \vee q)$
- **The rule of inference:**

$$\begin{array}{c} p \\ \hline \therefore p \vee q \end{array}$$

## Example

- **p:** "I will visit Paris."      **q:** "I will study Discrete Mathematics."
  - "I will visit Paris."
- 
- Therefore: "I will visit Paris or I will study Discrete Mathematics."

- In addition, if the premise  $p$  is true, the conclusion  $p$  or  $q$  is also true.

# Hypothetical syllogism

- **Tautology:**  $((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$
- **The rule of inference :**

$$\begin{array}{c} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$$

## Example

- **p:** "It is snowing."      **q:** "I will study Discrete Mathematics."
  - **r:** "I will pass the quizzes."
  - "If it is snowing, I will study Discrete Mathematics."
  - "If I study Discrete Mathematics, I will pass the quizzes."
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- Therefore: "If it is snowing, I will pass the quizzes."

- In hypothetical syllogism, if the premise  $p$  implies  $q$  is true and the premise  $q$  implies  $r$  is true, the conclusion  $p$  implies  $r$  is also true.



# Disjunctive syllogism

- **Tautology:**  $((p \vee q) \wedge \neg p) \rightarrow q$
- **The rule of inference:**

$$\begin{array}{l} p \vee q \\ \neg p \\ \hline \therefore q \end{array}$$

## Example

- **p:** "I will study Art."      **q:** "I will study Discrete Mathematics."
- "I will study Art or I will study Discrete Mathematics."
- "I will not study Discrete Mathematics."
- Therefore: "I will study Art."

- In Disjunctive Syllogism, if the premise  $p$  or  $q$  is true and the premise not  $p$  is true, the conclusion  $q$  is also true

# Resolution

- **Tautology:**  $((p \vee q) \wedge (\neg p \vee r)) \rightarrow (q \vee r)$
- **The rule of inference:**

$$\begin{array}{l} p \vee q \\ \neg p \vee r \\ \hline \therefore q \vee r \end{array}$$

## Example

- **p:** "It is raining."      **q:** "It is snowing."      **r:** "It is cold."
- "It is raining or it is snowing."
- "It is not raining or it is cold."
- Therefore: "It is snowing or it is cold."

- In Resolution, if the premise  $p$  or  $q$  is true and the premise not  $p$  or  $r$  is true, the conclusion  $q$  or  $r$  is also true

To build a **valid argument** we need to follow the steps below:

- If initially written in English, transform the statement into an argument form by choosing a variable for each simple proposition
- Start with the hypothesis of the argument
- Build a sequence of steps in which each step follows from the previous step by applying:
  - **rules of inference**
  - **laws of logic**
- The final step of the argument is the conclusion.

Let's build a valid argument from the following premises:

$\neg p$ : "It is not cold tonight."

$q \rightarrow p$ : "We will go to the theatre only if it is cold."

$\neg q \rightarrow r$ : "If we do not go to the theatre, we will watch a movie at home."

$r \rightarrow s$ : "If we watch a movie at home, we will need to make popcorn."

**Propositional variables:**

$p$ : "It is cold tonight."

$r$ : "We will watch a movie at home."

$q$ : "We will go to the theatre."

$s$ : "We will need to make popcorn."

	<i>Step</i>	<i>Justification</i>
1	$q \rightarrow p$	<i>Hypothesis</i>
2	$\neg p$	<i>Hypothesis</i>
3	$\therefore \neg q$	<i>Modus tollens 1, 2</i>
4	$\neg q \rightarrow r$	<i>Hypothesis</i>
5	$\therefore r$	<i>Modus ponens 3, 4</i>
6	$r \rightarrow s$	<i>Hypothesis</i>
7	$\therefore s$	<i>Modus ponens 5, 6</i>

**Conclusion:** "We will need to make popcorn."

- Fallacy = use of incorrect argument when reasoning
  - Formal fallacies can be expressed in propositional logic and proved to be incorrect
- Some of the widely used **formal fallacies** are:
  - affirming the consequent
  - a conclusion that denies premises
  - contradictory premises
  - denying the antecedent
  - existential fallacy
  - exclusive premises.



Let's consider the following argument:

- If you have internet access, you can order this book.
  - You can order this book.
  - Therefore, you have Internet access.
  - This argument can be formalised as: if  $p \rightarrow q$  and  $q$ , then  $p$
  - Where **p**: "You have Internet access."  
**q**: "You can order this book."
  - The proposition  $((p \rightarrow q) \wedge q) \rightarrow p$  is not a tautology, because it is false when  $p$  is false and  $q$  is true
  - This is an incorrect argument using the fallacy of affirming the consequent (or conclusion).
- Rules of inference with quantifiers either remove or reintroduce quantifiers within a statement.

## Universal instantiation (UI)

**The rule of inference:**

$$\frac{\forall x P(x)}{\therefore P(c)}$$

### Example

- All computer science students study discrete mathematics.

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$\therefore$  Therefore, John, who is a computer science student, studies discrete mathematics.

- Universal Instantiation is used to conclude that  $P(c)$  is true where  $c$  is a particular member of the domain, given the premise  $\forall x P(x)$ . This rule of inference removes the universal quantifier
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# Universal generalization (UG)

**The rule of inference:**

$$\frac{P(c) \text{ for an arbitrary element of the domain}}{\forall x P(x)}$$

**Example**

- **DS** = {all data science students}
- Let **c** be an arbitrary element in **DS**.
- **c** studies machine learning.

∴ Therefore,  $\forall x \in \text{DS}, x$  studies machine learning.

- Universal generalization is used to conclude that  $\forall x P(x)$  is true by taking an arbitrary element **C** from the domain and showing that **P(c)** is true. This rule of inference introduces the universal quantifier

# Existential instantiation (EI)

**The rule of inference:**

$$\frac{\exists x P(x)}{\therefore P(c) \text{ for some element of the domain}}$$

**Example**

- **DS** = {all data science students}
- There exists a student of data science who uses *Python Pandas Library*.

∴ There is a student, **c**, who is using *Python Pandas Library*.

- Existential instantiation is used to conclude that there is an element **c** in the domain for which **P(c)** is true. If we know that  $\exists x P(x)$  is true, we cannot select an arbitrary value but rather acknowledge it exists, let name it "**c**" and use in argument. This rule of inference removes the existential quantifier

# Existential generalization (EG)

**The rule of inference:**

$$\frac{P(c) \text{ for some element of the domain}}{\therefore \exists x P(x)}$$

**Example**

- **DS** = {all data science students}
- John, a student of data science, got an A in the machine learning course.

$\therefore$  Therefore, there exists someone in DS who got an A in machine learning.

- Existential generalization is used to conclude  $\exists x P(x)$  is true when  $P(c)$  is true for some elements  $c$  of the domain. This rule of inference introduces the existential quantifier

## Universal modus ponens

**The rule of inference**

$$\frac{\begin{array}{l} \forall x P(x) \rightarrow Q(x) \\ P(a) \text{ for some element of the domain} \end{array}}{Q(a)}$$

**Example**

- **DS** = {all computer science students}
- Every computer science student studying data science will study machine learning.
- John is studying data science.

$\therefore$  Therefore, John will study machine learning.

- Universal modus ponens is combination of universal instantiation and modus ponens. Universal modus ponens concludes that if for all  $x$  in the domain  $P(x)$  implies  $Q(x)$  and  $P(a)$  is true for some elements of the domain, we can conclude  $Q(a)$  is also true.

# Universal modus tollens

## The rule of inference:

$$\begin{array}{l} \forall x P(x) \rightarrow Q(x) \\ \neg Q(a) \text{ for some element of the domain} \\ \hline \neg P(a) \end{array}$$

## Example

- CS = {all computer science students}
- Every computer science student studying data science will study machine learning.
- John is not studying machine learning.

∴ Therefore, John is not studying data science.

- Universal modus tollens is a combination of universal instantiation and modus tollens. Universal modus tollens is used to conclude that if for all x in the domain P(x) implies Q(x) and if Q(a) is false for some element of the domain, we can conclude that P(a) is also false.

# Expressing complex statements

Given a statement in natural language, we can formalise it using the following steps as appropriate:

1. Determine the universe of discourse of variables.
2. Reformulate the statement by making "**for all**" and "**there exists**" explicit
3. Reformulate the statement by introducing **variables** and defining **predicates**
4. Reformulate the statement by introducing **quantifiers** and **logical operations**.

Express the statement S: "there exists a real number between any two not equal real numbers".

- The universe of discourse is: real numbers.
- Introduce variables and predicates:
  - "For all real numbers x and y, there exists z between x and y."
- Introduce quantifiers and logical operations:
  - $\forall x \forall y$  if  $x < y$  then  $\exists z$  where  $x < z < y$

Express the statement S: "every student has taken a course in machine learning".

The expression will depend on the choice of the universe of discourse

**Case 1:**  $U = \{\text{all students}\}$

- Let  $M(x)$  be: "x has taken a course in machine learning."
- S can be expressed as:  $\forall x M(x)$

**Case 2:**  $U = \{\text{all people}\}$

- Let  $S(x)$  be: "x is a student" and  $M(x)$  the same as in case 1
- S can be expressed as  $\forall x (S(x) \rightarrow M(x))$

**Note:**  $\forall x (S(x) \wedge M(x))$  is **not** correct.

Express the statement S: "some student has taken a course in machine learning".

The expression will depend on the choice of the universe of discourse.

**Case 1:**  $U = \{\text{all students}\}$

- Let  $M(x)$  be: "x has taken a course in machine learning."
- S can be expressed as:  $\exists x M(x)$

**Case 2:**  $U = \{\text{all people}\}$

- Let  $S(x)$  be: "x is a student" and  $M(x)$  the same as in case 1.
- S can be expressed as  $\exists x (S(x) \wedge M(x))$ .

**Note:**  $\exists x (S(x) \rightarrow M(x))$  is **not** correct in this case.

## Summary

In this week, we learned about DeMorgan's laws for quantifiers, rules of inference, rules of inferences with quantifiers and fallacies and/or formal fallacies. Alongside we explored the steps to building a valid argument and the steps to expressing complex statements.