

Infinite primes



INTRODUCTION

This sheet shows you why the prime numbers carry on forever, but in order to understand the proof you'll need to think a bit! The argument that you are about to see is more than 2,000 years old, and is one of the very first things that was discovered about the prime numbers.

In order to understand the next section you should make sure that you are comfortable with two facts:

Fact 1: A prime number is a number that is divisible only by itself and 1.

Fact 2: Every number can be written as a multiple of prime numbers.

This is all we need to show that the primes are infinite, so once you understand them you can move on to the next section.

THE PRIME NUMBERS ARE INFINITE – PROOF

Suppose you start a collection of prime numbers, bringing every prime that you find into a list. We'll show why you'll never finish your collection.

Suppose you have collected n primes

$$P_1, P_2, \dots P_n.$$

Example: Your collection could be the first 5 primes, so you have the numbers 2, 3, 5, 7, 11 in your list. This would mean $n = 5$ (the number of primes) with $P_1 = 2$, $P_2 = 3$, $P_3 = 5$, $P_4 = 7$ and $P_5 = 11$.



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Multiply all the primes in your collection to get a new number N

$$N = P_1 \times P_2 \times \dots \times P_n$$

Example: $N = 2 \times 3 \times 5 \times 7 \times 11$, so $N = 2,310$

Now here is the clever bit. Add 1 to your new number N to get another number X

$$X = N + 1$$

Example: $X = 2,310 + 1$, so $X = 2,311$

X is not divisible by any of the primes in your list (Why? Each of the primes in your list is a divisor of N , so dividing X by any of the primes in your list leaves a remainder of 1). So either X is a new prime or X can be divided by a new prime that's not in your list. In other words **your original list did not contain all of the primes!**

Example: 2,311 is not divisible by any of the first 5 primes. In fact it is a new prime number!

We have seen that if you have a list of primes, there is always at least one prime that is not in your list. It's important to understand that **this works for any list of prime numbers**. When you have 9 numbers in your list, you know that there is a 10th that you have not collected. When you have 11, 12, 13, or 2,090,128,390 numbers in your list, there's at least one still out there. You will never finish your collection.

So we've just proved: **There are infinitely many primes!**



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Once you have read and understood the above argument, test your mathematical skills – try to spot where we used the two facts in the introduction.

DIGGING DEEPER

The proof that there are infinitely many primes showed how we can get a new prime from any list of old ones, and we gave an example using the first 5 primes.

Have we found an easy way to get new prime numbers?

In the example that accompanied the proof we found the number 2,311 – a new prime number which wasn't on our original list. What would have happened if we took the first 6 primes? Have a think for yourself before you read the answer.

Start with a list of the first 6 primes, the numbers 2, 3, 5, 7, 11 and 13. As in the proof, we multiply them together and then add 1 to generate a new number

$$(2 \times 3 \times 5 \times 7 \times 11 \times 13) + 1 = 30,031$$

But **30,031 isn't prime!** It's actually 59×509 , and it is these two numbers that are prime. This doesn't mean there was anything wrong with the proof – we've found a new prime which is not on our original list. The proof says that either $X = 30,031$ is prime or it has a new prime number which divides it (we could pick either 59 or 509).

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It seems that if you multiply the first few prime numbers together and then add 1, sometimes you get a prime and sometimes you don't. Primes that are built in this way are called **primorial primes**. (The word 'primorial' is a mixture of the word 'prime' and the word 'factorial', which means multiplying something by all the numbers that are smaller than it.)

Apart from the first few cases, we don't know how good using this trick is for finding prime numbers, and **we don't even know whether there is a limited number of primorial primes**. If you would like to make your name as a brilliant mathematician, you could do much worse than trying to find out the answer to these questions.

So the trick doesn't give a formula to find new primes, at best it gives a new number which is a multiple of new primes. But these factors could be big or small and there is no easy way to find them. **In fact there is no mathematician in the world who has a simple formula that will always give a prime.**

It's remarkable that we've known for millennia that the primes go on forever but still have no way to find them. This is one of the reasons why a businessman in America has offered US\$1 million as a reward for understanding the pattern in the primes.

FURTHER READING

Professor Marcus du Sautoy has written a book on prime numbers and the million dollar problem. Details of the book can be found on the Music of the Primes website at <http://www.musicoftheprimes.com/index.htm>

The million dollar problem is called the Riemann Hypothesis. Full details of this Hypothesis can be found on the Clay Mathematics Institute's website at http://www.claymath.org/millennium/Riemann_Hypothesis/