Lecture Notes for

EE 261

The Fourier Transform and its Applications

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Contents

1 Fourier Series **1**

1.1 Introduction and Choices to Make 1

1.2 Periodic Phenomena 2

1.3 Periodicity: Definitions, Examples, and Things to Come 4

1.4 It All Adds Up 9

1.5 Lost at $\boldsymbol{c}$ 10

1.6 Period, Frequencies, and Spectrum 13

1.7 Two Examples and a Warning 16

1.8 The Math, the Majesty, the End 21

1.9 Orthogonality 26

1.10 Appendix: The Cauchy-Schwarz Inequality and its Consequences 33

1.11 Appendix: More on the Complex Inner Product 36

1.12 Appendix: Best L^2 Approximation by Finite Fourier Series 38

1.13 Fourier Series in Action 39

1.14 Notes on Convergence of Fourier Series 50

1.15 Appendix: Pointwise Convergence vs. Uniform Convergence 58

1.16 Appendix: Studying Partial Sums via the Dirichlet Kernel: The Buzz Is Back 59

1.17 Appendix: The Complex Exponentials Are a Basis for $L^2([0,1])$ 61

1.18 Appendix: More on the Gibbs Phenomenon 62

2 Fourier Transform **65**

2.1 A First Look at the Fourier Transform 65

2.2 Getting to Know Your Fourier Transform 75

3 Convolution **95**

3.1 A $*$ is Born 95

3.2 What is Convolution, Really? 99

3.3 Properties of Convolution: It's a Lot like Multiplication 101

3.4 Convolution in Action I: A Little Bit on Filtering . . . 102
3.5 Convolution in Action II: Differential Equations . . . 106
3.6 Convolution in Action III: The Central Limit Theorem . . . 116
3.7 The Central Limit Theorem: The Bell Curve Tolls for Thee . . . 128
3.8 Fourier transform formulas under different normalizations . . . 130
3.9 Appendix: The Mean and Standard Deviation for the Sum of Random Variables . . . 131
3.10 More Details on the Central Limit Theorem . . . 132
3.11 Appendix: Heisenberg's Inequality . . . 133

4 Distributions and Their Fourier Transforms 137

4.1 The Day of Reckoning . . . 137
4.2 The Right Functions for Fourier Transforms: Rapidly Decreasing Functions . . . 142
4.3 A Very Little on Integrals . . . 148
4.4 Distributions . . . 152
4.5 A Physical Analogy for Distributions . . . 164
4.6 Limits of Distributions . . . 165
4.7 The Fourier Transform of a Tempered Distribution . . . 168
4.8 Fluxions Finis: The End of Differential Calculus . . . 174
4.9 Approximations of Distributions . . . 177
4.10 The Generalized Fourier Transform Includes the Classical Fourier Transform . . . 178
4.11 Operations on Distributions and Fourier Transforms . . . 179
4.12 Duality, Changing Signs, Evenness and Oddness . . . 179
4.13 A Function Times a Distribution Makes Sense . . . 182
4.14 The Derivative Theorem . . . 185
4.15 Shifts and the Shift Theorem . . . 186
4.16 Scaling and the Stretch Theorem . . . 188
4.17 Convolutions and the Convolution Theorem . . . 190
4.18 δ Hard at Work . . . 195
4.19 Appendix: The Riemann-Lebesgue lemma . . . 205
4.20 Appendix: Smooth Windows . . . 206
4.21 Appendix: $\mathbf{1/x}$ as a Principal Value Distribution . . . 209

5 Ш, Sampling, and Interpolation 211

5.1 X-Ray Diffraction: Through a Glass Darkly[1] . . . 211
5.2 The Ш Distribution . . . 212
5.3 The Fourier Transform of Ш . . . 216

5.4 Periodic Distributions and Fourier series . . . 219
5.5 Sampling Signals . . . 223
5.6 Sampling and Interpolation for Bandlimited Signals . . . 225
5.7 Interpolation a Little More Generally . . . 229
5.8 Finite Sampling for a Bandlimited Periodic Signal . . . 231
5.9 Troubles with Sampling . . . 236
5.10 Appendix: How Special is Ш? . . . 246
5.11 Appendix: Timelimited vs. Bandlimited Signals . . . 248

6 Discrete Fourier Transform 251

6.1 From Continuous to Discrete . . . 251
6.2 The Discrete Fourier Transform (DFT) . . . 254
6.3 Two Grids, Reciprocally Related . . . 259
6.4 Appendix: Gauss's Problem . . . 260
6.5 Getting to Know Your Discrete Fourier Transform . . . 261
6.6 Periodicity, Indexing, and Reindexing . . . 262
6.7 Inverting the DFT and Many Other Things Along the Way . . . 264
6.8 Properties of the DFT . . . 273
6.9 Different Definitions for the DFT . . . 277
6.10 The FFT Algorithm . . . 279
6.11 Zero Padding . . . 292

7 Linear Time-Invariant Systems 295

7.1 Linear Systems . . . 295
7.2 Examples . . . 296
7.3 Cascading Linear Systems . . . 301
7.4 The Impulse Response . . . 302
7.5 Linear Time-Invariant (LTI) Systems . . . 304
7.6 Appendix: The Linear Millennium . . . 307
7.7 Appendix: Translating in Time and Plugging into $\boldsymbol{L}$ . . . 308
7.8 The Fourier Transform and LTI Systems . . . 309
7.9 Matched Filters . . . 311
7.10 Causality . . . 313
7.11 The Hilbert Transform . . . 314
7.12 Appendix: The Hilbert Transform of **sinc** . . . 320
7.13 Filters Finis . . . 321

7.14 Appendix: Geometric Series of the Vector Complex Exponentials 330
7.15 Appendix: The Discrete Rect and its DFT 332

8 *n*-dimensional Fourier Transform 335
8.1 Space, the Final Frontier 335
8.2 Getting to Know Your Higher Dimensional Fourier Transform 347
8.3 Higher Dimensional Fourier Series 361
8.4 Ш, Lattices, Crystals, and Sampling 371
8.5 Crystals 381
8.6 Bandlimited Functions on $\mathbf{R}^2$ and Sampling on a Lattice 383
8.7 Naked to the Bone 386
8.8 The Radon Transform 388
8.9 Getting to Know Your Radon Transform 391
8.10 Appendix: Clarity of Glass 395
8.11 Medical Imaging: Inverting the Radon Transform 396

A Mathematical Background 403
A.1 Complex Numbers 403
A.2 The Complex Exponential and Euler's Formula 406
A.3 Algebra and Geometry 409
A.4 Further Applications of Euler's Formula 409

B Some References 413

Index 415

Chapter 1

Fourier Series

1.1 Introduction and Choices to Make

Methods based on the Fourier transform are used in virtually all areas of engineering and science and by virtually all engineers and scientists. For starters:

- Circuit designers
- Spectroscopists
- Crystallographers
- Anyone working in signal processing and communications
- Anyone working in imaging

I'm expecting that many fields and many interests will be represented in the class, and this brings up an important issue for all of us to be aware of. With the diversity of interests and backgrounds present not all examples and applications will be familiar and of relevance to all people. We'll all have to cut each other some slack, and it's a chance for all of us to branch out. Along the same lines, it's also important for you to realize that this is one course on the Fourier transform among many possible courses. The richness of the subject, both mathematically and in the range of applications, means that we'll be making choices almost constantly. Books on the subject do not look alike, nor do they look like these notes — even the notation used for basic objects and operations can vary from book to book. I'll try to point out when a certain choice takes us along a certain path, and I'll try to say something of what the alternate paths may be.

The very first choice is where to start, and my choice is a brief treatment of Fourier series.[1] Fourier analysis was originally concerned with representing and analyzing periodic phenomena, via Fourier *series*, and later with extending those insights to nonperiodic phenomena, via the Fourier *transform*. In fact, one way of getting from Fourier series to the Fourier transform is to consider nonperiodic phenomena (and thus just about any general function) as a limiting case of periodic phenomena as the period tends to infinity. A discrete set of frequencies in the periodic case becomes a continuum of frequencies in the nonperiodic case, the *spectrum* is born, and with it comes the most important principle of the subject:

> Every signal has a spectrum and is determined by its spectrum. You can analyze the signal either in the time (or spatial) domain or in the frequency domain.

[1] Bracewell, for example, starts right off with the Fourier transform and picks up a little on Fourier series later.

I think this qualifies as a Major Secret of the Universe.

All of this was thoroughly grounded in physical applications. Most often the phenomena to be studied were modeled by the fundamental differential equations of physics (heat equation, wave equation, Laplace's equation), and the solutions were usually constrained by boundary conditions. At first the idea was to use Fourier series to find explicit solutions.

This work raised hard and far reaching questions that led in different directions. It was gradually realized that setting up Fourier series (in sines and cosines) could be recast in the more general framework of orthogonality, linear operators, and eigenfunctions. That led to the general idea of working with *eigenfunction expansions* of solutions of differential equations, a ubiquitous line of attack in many areas and applications. In the modern formulation of partial differential equations, the Fourier transform has become the basis for *defining* the objects of study, while still remaining a tool for solving specific equations. Much of this development depends on the remarkable relation between Fourier transforms and convolution, something which was also seen earlier in the Fourier series days. In an effort to apply the methods with increasing generality, mathematicians were pushed (by engineers and physicists) to reconsider how general the notion of "function" can be, and what kinds of functions can be — and should be — admitted into the operating theater of calculus. Differentiation and integration were both generalized in the service of Fourier analysis.

Other directions combine tools from Fourier analysis with symmetries of the objects being analyzed. This might make you think of crystals and crystallography, and you'd be right, while mathematicians think of number theory and Fourier analysis on groups. Finally, I have to mention that in the purely mathematical realm the question of convergence of Fourier series, believe it or not, led G. Cantor near the turn of the 20th century to investigate and invent the theory of infinite sets, and to distinguish different sizes of infinite sets, all of which led to Cantor going insane.

1.2 Periodic Phenomena

To begin the course with Fourier series is to begin with periodic functions, those functions which exhibit a regularly repeating pattern. It shouldn't be necessary to try to sell periodicity as an important physical (and mathematical) phenomenon — you've seen examples and applications of periodic behavior in probably (almost) every class you've taken. I would only remind you that periodicity often shows up in two varieties, sometimes related, sometimes not. Generally speaking we think about periodic phenomena according to whether they are *periodic in time* or *periodic in space*.

1.2.1 Time and space

In the case of time the phenomenon comes to you. For example, you stand at a fixed point in the ocean (or on an electrical circuit) and the waves (or the electrical current) wash over you with a regular, recurring pattern of crests and troughs. The height of the wave is a periodic function of time. Sound is another example: "sound" reaches your ear as a longitudinal pressure wave, a periodic compression and rarefaction of the air. In the case of space, you come to the phenomenon. You take a picture and you observe repeating patterns.

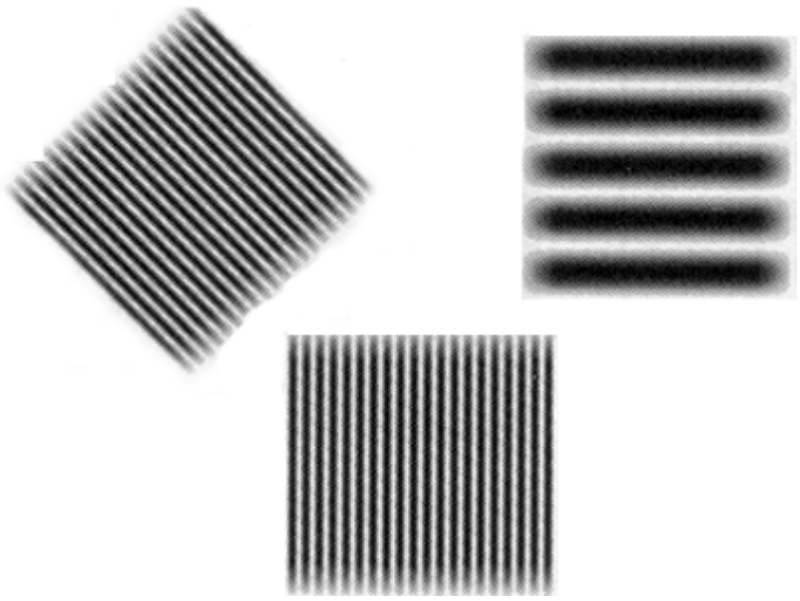
Temporal and spatial periodicity come together most naturally in wave motion. Take the case of one spatial dimension, and consider a single sinusoidal wave traveling along a string (for example). For such a wave the periodicity in time is measured by the frequency ν, with dimension 1/sec and units Hz (Hertz = cycles per second), and the periodicity in space is measured by the wavelength λ, with dimension length and units whatever is convenient for the particular setting. If we fix a point in space and let the time vary (take a video of the wave motion at that point) then successive crests of the wave come past that

point ν times per second, and so do successive troughs. If we fix the time and examine how the wave is spread out in space (take a snapshot instead of a video) we see that the distance between successive crests is a constant λ, as is the distance between successive troughs. The frequency and wavelength are related through the equation $v = \lambda\nu$, where v is the speed of propagation — this is nothing but the wave version of speed = distance/time. Thus the higher the frequency the shorter the wavelength, and the lower the frequency the longer the wavelength. If the speed is fixed, like the speed of electromagnetic waves in a vacuum, then the frequency determines the wavelength and vice versa; if you can measure one you can find the other. For sound we identify the physical property of frequency with the perceptual property of pitch, for light frequency is perceived as color. Simple sinusoids are the building blocks of the most complicated wave forms — that's what Fourier analysis is about.

1.2.2 More on spatial periodicity

Another way spatial periodicity occurs is when there is a repeating pattern or some kind of symmetry in a spatial region and physically observable quantities associated with that region have a repeating pattern that reflects this. For example, a crystal has a regular, repeating pattern of atoms in space; the arrangement of atoms is called a *lattice*. The electron density distribution is then a periodic function of the spatial variable (in $\mathbf{R}^3$) that describes the crystal. I mention this example because, in contrast to the usual one-dimensional examples you might think of, here the function, in this case the electron density distribution, has three independent periods corresponding to the three directions that describe the crystal lattice.

Here's another example — this time in two dimensions — that is very much a natural subject for Fourier analysis. Consider these stripes of dark and light:

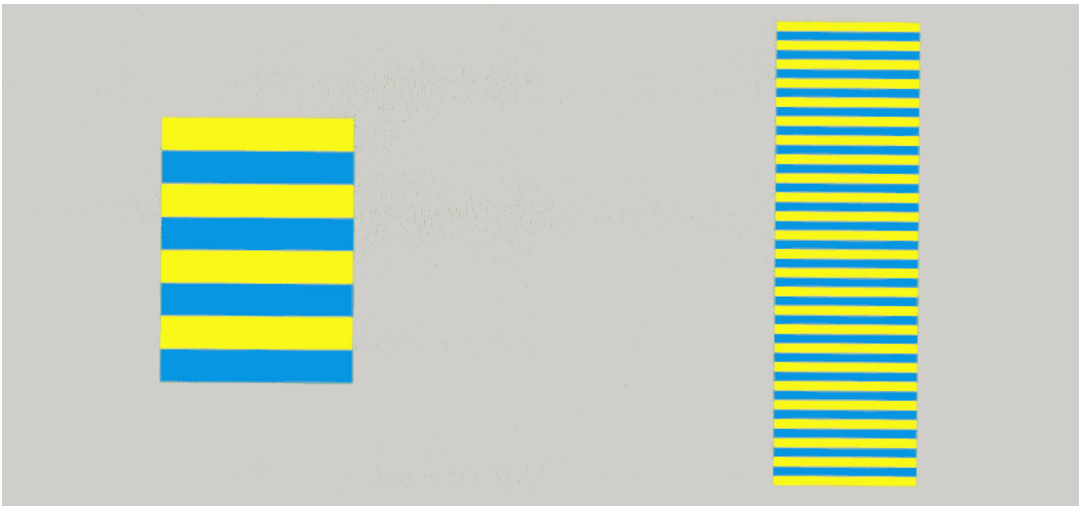


No doubt there's some kind of spatially periodic behavior going on in the respective images. Furthermore, even without stating a precise definition, it's reasonable to say that one of the patterns is "low frequency" and that the others are "high frequency", meaning roughly that there are fewer stripes per unit length in the one than in the others. In two dimensions there's an extra subtlety that we see in these pictures: "spatial frequency", however we ultimately define it, must be a vector quantity, not a number. We have to say that the stripes occur with a certain spacing in a certain direction.

Such periodic stripes are the building blocks of general two-dimensional images. When there's no color, an image is a two-dimensional array of varying shades of gray, and this can be realized as a synthesis — a

Fourier synthesis — of just such alternating stripes.

There are interesting perceptual questions in constructing images this way, and color is more complicated still. Here's a picture I got from *Foundations of Vision* by Brian Wandell, who is in the Psychology Department here at Stanford.



The shades of blue and yellow are the same in the two pictures —the only a change is in the frequency. The closer spacing "mixes" the blue and yellow to give a greenish cast. Here's a question that I know has been investigated but I don't know the answer. Show someone blue and yellow stripes of a low frequency and increase the frequency till they just start to see green. You get a number for that. Next, start with blue and yellow stripes at a high frequency so a person sees a lot of green and then lower the frequency till they see only blue and yellow. You get a number for that. Are the two numbers the same? Does the orientation of the stripes make a difference?

1.3 Periodicity: Definitions, Examples, and Things to Come

To be certain we all know what we're talking about, a function $f(t)$ is *periodic* of period T if there is a number $T > 0$ such that

$$f(t+T) = f(t)$$

for all t. If there is such a T then the smallest one for which the equation holds is called the *fundamental period* of the function f.[2] Every *integer* multiple of the fundamental period is also a period:

$$f(t+nT) = f(t)\,, \quad n = 0, \pm 1, \pm 2, \ldots\,.^{3}$$

I'm calling the variable t here because I have to call it something, but the definition is general and is not meant to imply periodic functions of time.

[2] Sometimes when people say simply "period" they mean the smallest or fundamental period. (I usually do, for example.) Sometimes they don't. Ask them what they mean.

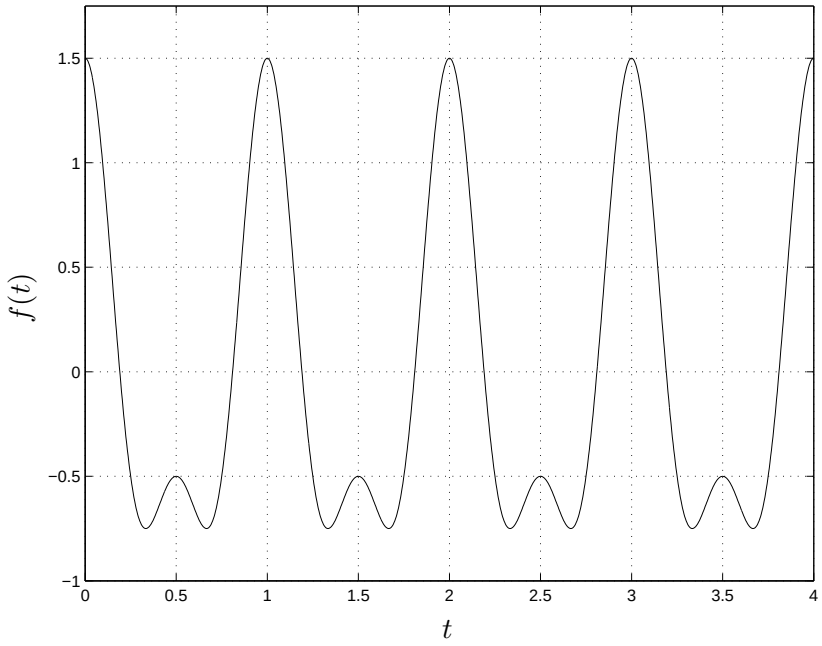
[3] It's clear from the geometric picture of a repeating graph that this is true. To show it algebraically, if $n \geq 1$ then we see inductively that $f(t+nT) = f(t+(n-1)T+T) = f(t+(n-1)T) = f(t)$. Then to see algebraically why negative multiples of T are also periods we have, for $n \geq 1$, $f(t-nT) = f(t-nT+nT) = f(t)$.

The graph of f over *any* interval of length T is one *cycle*. Geometrically, the periodicity condition means that the shape of one cycle (any cycle) determines the graph everywhere; the shape is repeated over and over. A homework problem asks you to turn this idea into a formula.

This is all old news to everyone, but, by way of example, there are a few more points I'd like to make. Consider the function

$$f(t) = \cos 2\pi t + \tfrac{1}{2}\cos 4\pi t\,,$$

whose graph is shown below.



The individual terms are periodic with periods 1 and 1/2 respectively, but the sum is periodic with period 1:

$$\begin{aligned} f(t+1) &= \cos 2\pi(t+1) + \tfrac{1}{2}\cos 4\pi(t+1) \\ &= \cos(2\pi t + 2\pi) + \tfrac{1}{2}\cos(4\pi t + 4\pi) = \cos 2\pi t + \tfrac{1}{2}\cos 4\pi t = f(t)\,. \end{aligned}$$

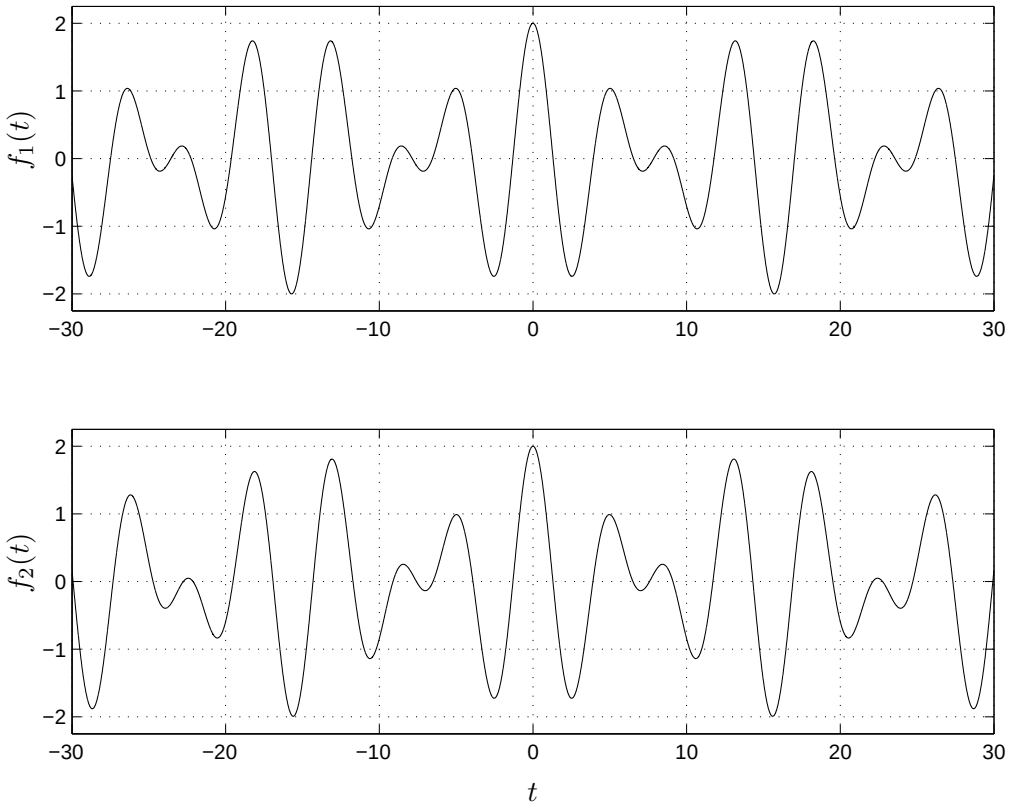
There is no smaller value of T for which $f(t+T) = f(t)$. The overall pattern repeats every 1 second, but if this function represented some kind of wave would you say it had frequency 1 Hz? Somehow I don't think so. It has one *period* but you'd probably say that it has, or contains, *two* frequencies, one cosine of frequency 1 Hz and one of frequency 2 Hz.

The subject of adding up periodic functions is worth a general question:

- Is the sum of two periodic functions periodic?

I guess the answer is no if you're a mathematician, yes if you're an engineer, i.e., no if you believe in irrational numbers and leave it at that, and yes if you compute things and hence work with approximations. For example, $\cos t$ and $\cos(\sqrt{2}t)$ are each periodic, with periods 2π and $2\pi/\sqrt{2}$ respectively, but the sum $\cos t + \cos(\sqrt{2}t)$ is not periodic.

Here are plots of $f_1(t) = \cos t + \cos 1.4t$ and of $f_2(t) = \cos t + \cos(\sqrt{2}t)$.



(I'm aware of the irony in making a big show of computer plots depending on an irrational number when the computer has to take a rational approximation to draw the picture.) How artificial an example is this? Not artificial at all. We'll see why, below.

1.3.1 The view from above

After *years* (centuries) of work, there are, in the end, relatively few mathematical ideas that underlie the study of periodic phenomena. There are many details and subtle points, certainly, but these are of less concern to us than keeping a focus on the bigger picture and using that as a guide in applications. We'll need the following.

1. The functions that model the simplest periodic behavior, i.e., sines and cosines. In practice, both in calculations and theory, we'll use the complex exponential instead of the sine and cosine separately.
2. The "geometry" of square integrable functions on a finite interval, i.e., functions for which
$$\int_a^b |f(t)|^2\,dt < \infty\,.$$
3. Eigenfunctions of linear operators (especially differential operators).

The first point has been familiar to you since you were a kid. We'll give a few more examples of sines and cosines in action. The second point, at least as I've stated it, may not be so familiar — "geometry" of a space of functions? — but here's what it means in practice:

- Least squares approximation
- Orthogonality of the complex exponentials (and of the trig functions)

I say "geometry" because what we'll do and what we'll say is analogous to Euclidean geometry as it is expressed (especially for computational purposes) via vectors and dot products. Analogous, not identical. There are differences between a space of functions and a space of (geometric) vectors, but it's almost more a difference of degree than a difference of kind, and your intuition for vectors in $\mathbf{R}^2$ or $\mathbf{R}^3$ can take you quite far. Also, the idea of least squares approximation is closely related to the orthogonality of the complex exponentials.

We'll say less about the third point, though it will figure in our discussion of linear systems.[4] Furthermore, it's the second and third points that are still in force when one wants to work with expansions in functions other than sine and cosine.

1.3.2 The building blocks: a few more examples

The classic example of temporal periodicity is the harmonic oscillator, whether it's a mass on a spring (no friction) or current in an LC circuit (no resistance). The harmonic oscillator is treated in exhaustive detail in just about every physics class. This is so because it is the *only* problem that can be treated in exhaustive detail.

The state of the system is described by a *single* sinusoid, say of the form

$$A\sin(2\pi\nu t + \phi)\,.$$

The parameters in this expression are the *amplitude* A, the *frequency* ν and the *phase* ϕ. The period of this function is $1/\nu$, since

$$A\sin(2\pi\nu\Big(t+\frac{1}{\nu}\Big)+\phi) = A\sin(2\pi\nu t + 2\pi\nu\frac{1}{\nu}+\phi) = A\sin(2\pi\nu t + 2\pi + \phi) = A\sin(2\pi\nu t+\phi)\,.$$

The classic example of spatial periodicity, the example that started the whole subject, is the distribution of heat in a circular ring. A ring is heated up, somehow, and the heat then distributes itself, somehow, through the material. In the long run we expect all points on the ring to be of the same temperature, but they won't be in the short run. At each fixed time, how does the temperature vary around the ring?

In this problem the periodicity comes from the coordinate description of the ring. Think of the ring as a circle. Then a point on the ring is determined by an angle θ and quantities which depend on position are functions of θ. Since θ and $\theta + 2\pi$ are the same point on the circle, any continuous function describing a physical quantity on the circle, e.g., temperature, is a periodic function of θ with period 2π.

The distribution of temperature is not given by a simple sinusoid. It was Fourier's hot idea to consider a *sum* of sinusoids as a model for the temperature distribution:

$$\sum_{n=1}^{N} A_n \sin(n\theta + \phi_n)\,.$$

The dependence on time is in the coefficients A_n. We'll study this problem more completely later, but there are a few points to mention now.

Regardless of the physical context, the individual terms in a trigonometric sum such as the one above are called *harmonics*, terminology that comes from the mathematical representation of musical pitch — more

[4] It is the role of complex exponentials as eigenfunctions that explains why you would *expect* to take only integer multiples of the fundamental period in forming sums of periodic functions.

on this in a moment. The terms contribute to the sum in varying amplitudes and phases, and these can have any values. The frequencies of the terms, on the other hand, are integer multiples of the fundamental frequency $1/2\pi$. Because the frequencies are integer multiples of the fundamental frequency, the sum is also periodic, and the period is 2π. The term $A_n \sin(n\theta + \phi_n)$ has period $2\pi/n$, but the whole sum can't have a shorter cycle than the longest cycle that occurs, and that's 2π. We talked about just this point when we first discussed periodicity.[5]

1.3.3 Musical pitch and tuning

Musical pitch and the production of musical notes is a periodic phenomenon of the same general type as we've been considering. Notes can be produced by vibrating strings or other objects that can vibrate regularly (like lips, reeds, or the bars of a xylophone). The engineering problem is how to tune musical instruments. The subject of tuning has a fascinating history, from the "natural tuning" of the Greeks, based on ratios of integers, to the theory of the "equal tempered scale", which is the system of tuning used today. That system is based on $2^{1/12}$.

There are 12 notes in the equal tempered scale, going from any given note to the same note an octave up, and two adjacent notes have frequencies with ratio $2^{1/12}$. If an A of frequency 440 Hz (concert A) is described by

$$\mathrm{A} = \cos(2\pi \cdot 440\, t)\,,$$

then 6 notes up from A in a well tempered scale is a D♯ given by

$$\mathrm{D\sharp} = \cos(2\pi \cdot 440\sqrt{2}\, t)\,.$$

(The notes in the scale are $\cos(2\pi \cdot 440 \cdot 2^{n/12}t)$ from $n = 0$ to $n = 12$.) Playing the A and the D♯ together gives essentially the signal we had earlier, $\cos t + \cos 2^{1/2}t$. I'll withhold judgment whether or not it sounds any good.

Of course, when you tune a piano you don't tighten the strings irrationally. The art is to make the right approximations. To read more about this, see, for example

`http://www.precisionstrobe.com/`

To read more about tuning in general try

`http://www.wikipedia.org/wiki/Musical_tuning`

Here's a quote from the first reference describing the need for well-tempered tuning:

> Two developments occurred in music technology which necessitated changes from the just toned temperament. With the development of the fretted instruments, a problem occurs when setting the frets for just tuning, that octaves played across two strings around the neck would produce impure octaves. Likewise, an organ set to a just tuning scale would reveal chords with unpleasant properties. A compromise to this situation was the development of the mean toned scale. In this system several of the intervals were adjusted to increase the number of usable keys. With the evolution of composition technique in the 18th century increasing the use of harmonic modulation a change was advocated to the equal tempered scale. Among these

[5] There is another reason that only integer multiples of the fundamental frequency come in. It has to do with the harmonics being eigenfunctions of a differential operator, and the boundary conditions that go with the problem.

advocates was J. S. Bach who published two entire works entitled *The Well-tempered Clavier*. Each of these works contain 24 fugues written in each of twelve major and twelve minor keys and demonstrated that using an equal tempered scale, music could be written in, and shifted to any key.

1.4 It All Adds Up

From simple, single sinusoids we can build up much more complicated periodic functions by taking sums. To highlight the essential ideas it's convenient to standardize a little and consider functions with period 1. This simplifies some of the writing and it will be easy to modify the formulas if the period is not 1. The basic function of period 1 is $\sin 2\pi t$, and so the Fourier-type sum we considered briefly in the previous lecture looks like

$$\sum_{n=1}^{N} A_n \sin(2\pi n t + \phi_n)\,.$$

This form of a general trigonometric sum has the advantage of displaying explicitly the amplitude and phase of each harmonic, but it turns out to be somewhat awkward to calculate with. It's more common to write a general trigonometric sum as

$$\sum_{n=1}^{N} (a_n \cos(2\pi n t) + b_n \sin(2\pi n t))\,,$$

and, if we include a constant term ($n = 0$), as

$$\frac{a_0}{2} + \sum_{n=1}^{N} (a_n \cos(2\pi n t) + b_n \sin(2\pi n t))\,.$$

The reason for writing the constant term with the fraction $1/2$ is because, as you will check in the homework, it simplifies still another expression for such a sum.

In electrical engineering the constant term is often referred to as the *DC component* as in "direct current". The other terms, being periodic, "alternate", as in AC. Aside from the DC component, the harmonics have periods $1, 1/2, 1/3, \ldots, 1/N$, respectively, or frequencies $1, 2, 3, \ldots, N$. Because the frequencies of the individual harmonics are integer multiples of the lowest frequency, the period of the sum is 1.

Algebraic work on such trigonometric sums is made incomparably easier if we use complex exponentials to represent the sine and cosine.[6] I remind you that

$$\cos t = \frac{e^{it} + e^{-it}}{2}\,, \quad \sin t = \frac{e^{it} - e^{-it}}{2i}\,.$$

Hence

$$\cos(2\pi n t) = \frac{e^{2\pi i n t} + e^{-2\pi i n t}}{2}\,, \quad \sin(2\pi n t) = \frac{e^{2\pi i n t} - e^{-2\pi i n t}}{2i}\,.$$

Using this, the sum

$$\frac{a_0}{2} + \sum_{n=1}^{N} (a_n \cos(2\pi n t) + b_n \sin(2\pi n t))$$

[6] See the appendix on complex numbers where there is a discussion of complex exponentials, how they can be used without fear to represent real signals, and an answer to the question of what is meant by a "negative frequency".

can be written as

$$\sum_{n=-N}^{N} c_n e^{2\pi i n t}\,.$$

Sorting out how the a's, b's, and c's are related will be left as a problem. In particular, you'll get $c_0 = a_0/2$, which is the reason we wrote the constant term as $a_0/2$ in the earlier expression.[7]

In this final form of the sum, the coefficients c_n are *complex* numbers, and they satisfy

$$c_{-n} = \overline{c_n}\,.$$

Notice that when $n = 0$ we have

$$c_0 = \overline{c_0}\,,$$

which implies that c_0 is a real number; this jibes with $c_0 = a_0/2$. For any value of n the magnitudes of c_n and c_{-n} are equal:

$$|c_n| = |c_{-n}|\,.$$

The (conjugate) symmetry property, $c_{-n} = \overline{c_n}$, of the coefficients is important. To be explicit: if the signal is real then the coefficients *have* to satisfy it, since $f(t) = \overline{f(t)}$ translates to

$$\sum_{n=-N}^{N} c_n e^{2\pi i n t} = \overline{\sum_{n=-N}^{N} c_n e^{2\pi i n t}} = \sum_{n=-N}^{N} \overline{c_n}\,\overline{e^{2\pi i n t}} = \sum_{n=-N}^{N} \overline{c_n}\, e^{-2\pi i n t}\,,$$

and if we equate like terms we get $c_{-n} = \overline{c_n}$. Conversely, suppose the relation is satisfied. For each n we can group $c_n e^{2\pi i n t}$ with $c_{-n} e^{-2\pi i n t}$, and then

$$c_n e^{2\pi i n t} + c_{-n} e^{-2\pi i n t} = c_n e^{2\pi i n t} + \bar{c_n}\overline{e^{2\pi i n t}} = 2\,\mathrm{Re}\left(c_n e^{2\pi i n t}\right)\,.$$

Therefore the sum is real:

$$\sum_{n=-N}^{N} c_n e^{2\pi i n t} = \sum_{n=0}^{N} 2\,\mathrm{Re}\left(c_n e^{2\pi i n t}\right) = 2\,\mathrm{Re}\left\{\sum_{n=0}^{N} c_n e^{2\pi i n t}\right\}\,.$$

1.5 Lost at *c*

Suppose we have a complicated looking periodic signal; you can think of one varying in time but, again and always, the reasoning to follow applies to any sort of one-dimensional periodic phenomenon. We can scale time to assume that the pattern repeats every 1 second. Call the signal $f(t)$. Can we express $f(t)$ as a sum?

$$f(t) = \sum_{n=-N}^{N} c_n e^{2\pi i n t}$$

In other words, the unknowns in this expression are the coefficients c_n, and the question is can we solve for these coefficients?

[7] When I said that part of your general math know-how should include whipping around sums, this expression in terms of complex exponentials was one of the examples I was thinking of.

Here's a direct approach. Let's take the coefficient c_k for some fixed k. We can isolate it by multiplying both sides by $e^{-2\pi ikt}$:

$$\begin{aligned} e^{-2\pi ikt} f(t) &= e^{-2\pi ikt} \sum_{n=-N}^{N} c_n e^{2\pi int} \\ &= \cdots + e^{-2\pi ikt} c_k e^{2\pi ikt} + \cdots = \cdots + c_k + \cdots \end{aligned}$$

Thus

$$c_k = e^{-2\pi ikt} f(t) - \sum_{n=-N, n\neq k}^{N} c_n e^{-2\pi ikt} e^{2\pi int} = e^{-2\pi ikt} f(t) - \sum_{n=-N, n\neq k}^{N} c_n e^{2\pi i(n-k)t} \,.$$

We've pulled out the coefficient c_k, but the expression on the right involves all the other unknown coefficients. Another idea is needed, and that idea is integrating both sides from 0 to 1. (We take the interval from 0 to 1 as "base" period for the function. Any interval of length 1 would work — that's periodicity.)

Just as in calculus, we can evaluate the integral of a complex exponential by

$$\begin{aligned} \int_0^1 e^{2\pi i(n-k)t}\, dt &= \frac{1}{2\pi i(n-k)} e^{2\pi i(n-k)t} \Big]_{t=0}^{t=1} \\ &= \frac{1}{2\pi i(n-k)} (e^{2\pi i(n-k)} - e^0) = \frac{1}{2\pi i(n-k)} (1-1) = 0 \,. \end{aligned}$$

Note that $n \neq k$ is needed here.

Since the integral of the sum is the sum of the integrals, and the coefficients c_n come out of each integral, all of the terms in the sum integrate to zero and we have a formula for the k-th coefficient:

$$c_k = \int_0^1 e^{-2\pi ikt} f(t)\, dt \,.$$

Let's summarize and be careful to note what we've done here, and what we haven't done. We've shown that *if* we can write a periodic function $f(t)$ of period 1 as a sum

$$f(t) = \sum_{n=-N}^{N} c_n e^{2\pi int} \,,$$

then the coefficients c_n must be given by

$$c_n = \int_0^1 e^{-2\pi int} f(t)\, dt \,.$$

We have *not* shown that every periodic function *can* be expressed this way.

By the way, in none of the preceding calculations did we have to assume that $f(t)$ is a real signal. If, however, we do assume that $f(t)$ is real, then let's see how the formula for the coefficients jibes with $\overline{c_n} = c_{-n}$. We have

$$\begin{aligned} \overline{c_n} &= \overline{\left(\int_0^1 e^{-2\pi int} f(t)\, dt \right)} = \int_0^1 \overline{e^{-2\pi int}}\, \overline{f(t)}\, dt \\ &= \int_0^1 e^{2\pi int} f(t)\, dt \quad \text{(because } f(t) \text{ is real, as are } t \text{ and } dt\text{)} \\ &= c_{-n} \quad \text{(by definition of } c_n\text{)} \end{aligned}$$

The c_n are called the *Fourier coefficients* of $f(t)$, because it was Fourier who introduced these ideas into mathematics and science (but working with the sine and cosine form of the expression). The sum

$$\sum_{n=-N}^{N} c_n e^{2\pi i n t}$$

is called a (finite) *Fourier series*.

If you want to be mathematically hip and impress your friends at cocktail parties, use the notation

$$\hat{f}(n) = \int_0^1 e^{-2\pi i n t} f(t)\, dt$$

for the Fourier coefficients. Always conscious of social status, I will use this notation.

Note in particular that the 0-th Fourier coefficient is the *average* value of the function:

$$\hat{f}(0) = \int_0^1 f(t)\, dt\,.$$

Also note that because of periodicity of $f(t)$, any interval of length 1 will do to calculate $\hat{f}(n)$. Let's check this. To integrate over an interval of length 1 is to integrate from a to $a+1$, where a is any number. Let's compute how this integral varies as a function of a.

$$\begin{aligned}
\frac{d}{da}\Big(\int_a^{a+1} e^{-2\pi i n t} f(t)\, dt\Big) &= e^{-2\pi i n(a+1)} f(a+1) - e^{-2\pi i n a} f(a) \\
&= e^{-2\pi i n a} e^{-2\pi i n} f(a+1) - e^{-2\pi i n a} f(a) \\
&= e^{-2\pi i n a} f(a) - e^{-2\pi i n a} f(a) \quad (\text{using } e^{-2\pi i n} = 1 \text{ and } f(a+1) = f(a)) \\
&= 0\,.
\end{aligned}$$

In other words, the integral

$$\int_a^{a+1} e^{-2\pi i n t} f(t)\, dt$$

is independent of a. So in particular,

$$\int_a^{a+1} e^{-2\pi i n t} f(t)\, dt = \int_0^1 e^{-2\pi i n t} f(t)\, dt = \hat{f}(n)\,.$$

A common instance of this is

$$\hat{f}(n) = \int_{-1/2}^{1/2} e^{-2\pi i n t} f(t)\, dt\,.$$

There are times when such a change is useful.

Finally note that for a given function some coefficients may well be zero. More completely: There may be only a finite number of nonzero coefficients; or maybe all but a finite number of coefficients are nonzero; or maybe none of the coefficients are zero; or there may be an infinite number of nonzero coefficients but also an infinite number of coefficients that are zero — I think that's everything. What's interesting, and important for some applications, is that under some general assumptions one can say something about the *size* of the coefficients. We'll come back to this.

1.6 Period, Frequencies, and Spectrum

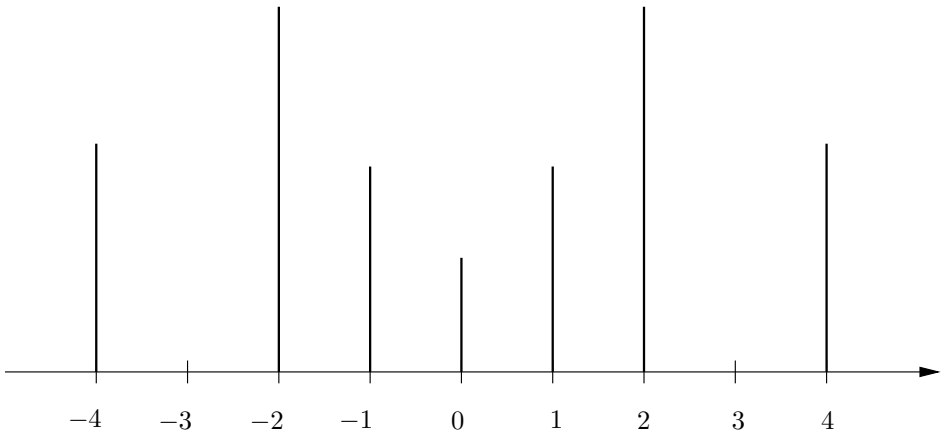
We'll look at some examples and applications in a moment. First I want to make a few more general observations. In the preceding discussion I have more often used the more geometric term *period* instead of the more physical term *frequency*. It's natural to talk about the period for a Fourier series representation of $f(t)$,

$$f(t) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n t}\,.$$

The period is 1. The function repeats according to $f(t+1) = f(t)$ and so do all the individual terms, though the terms for $n \neq 1$ have the strictly shorter period $1/n$.[8] As mentioned earlier, it doesn't seem natural to talk about "the frequency" (should it be 1 Hz?). That misses the point. Rather, being able to write $f(t)$ as a Fourier series means that it is synthesized from many harmonics, many frequencies, positive and negative, perhaps an infinite number. The set of frequencies present in a given periodic signal is the *spectrum* of the signal. Note that it's the frequencies, like ± 2, ± 7, ± 325, that make up the spectrum, *not* the values of the coefficients $\hat{f}(\pm 2)$, $\hat{f}(\pm 7)$, $\hat{f}(\pm 325)$.

Because of the symmetry relation $\hat{f}(-n) = \overline{\hat{f}(n)}$, the coefficients $\hat{f}(n)$ and $\hat{f}(-n) = 0$ are either both zero or both nonzero. Are numbers n where $\hat{f}(n) = 0$ considered to be part of the spectrum? I'd say yes, with the following gloss: if the coefficients are *all* zero from some point on, say $\hat{f}(n) = 0$ for $|n| > N$, then it's common to say that the signal has no spectrum from that point, or that the spectrum of the signal is limited to the points between $-N$ and N. One also says in this case that the *bandwidth* is N (or maybe $2N$ depending to whom you're speaking) and that the signal is *bandlimited*.

Let me also point out a mistake that people sometimes make when thinking too casually about the Fourier coefficients. To represent the spectrum graphically people sometimes draw a bar graph where the heights of the bars are the coefficients. Something like:



Why is this a mistake? Because, remember, the coefficients $\hat{f}(0), \hat{f}(\pm 1), \hat{f}(\pm 2), \ldots$ are *complex numbers* — you can't draw them as a height in a bar graph. (Except for $\hat{f}(0)$ which is real because it's the average value of $f(t)$.) What you're *supposed* to draw to get a picture like the one above is a bar graph of $|\hat{f}(0)|^2, |\hat{f}(\pm 1)|^2, |\hat{f}(\pm 2)|^2, \ldots$, i.e., the squares of the magnitudes of the coefficients. The square magnitudes of the coefficient $|\hat{f}(n)|^2$ can be identified as the energy of the (positive and negative) harmonics $e^{\pm 2\pi i n t}$. (More on this later.) These sorts of plots are what you see produced by a "spectrum analyzer". One could

[8] By convention, here we sort of ignore the constant term c_0 when talking about periods or frequencies. It's obviously periodic of period 1, or any other period for that matter.

also draw just the magnitudes $|\hat{f}(0)|, |\hat{f}(\pm 1)|, |\hat{f}(\pm 2)|, \ldots$, but it's probably more customary to consider the squares of the magnitudes.

The sequence of squared magnitudes $|\hat{f}(n)|^2$ is called the *power spectrum* or the *energy spectrum* (different names in different fields). A plot of the power spectrum gives you a sense of how the coefficients stack up, die off, whatever, and it's a way of comparing two signals. It doesn't give you any idea of the *phases* of the coefficients. I point all this out only because forgetting what quantities are complex and plotting a graph anyway is an easy mistake to make (I've seen it, and not only in student work but in an advanced text on quantum mechanics).

The case when all the coefficients *are* real is when the signal is real and *even*. For then

$$\begin{aligned}
\overline{\hat{f}(n)} = \hat{f}(-n) &= \int_0^1 e^{-2\pi i(-n)t} f(t)\, dt = \int_0^1 e^{2\pi i n t} f(t)\, dt \\
&= -\int_0^{-1} e^{-2\pi i n s} f(-s)\, ds \quad \text{(substituting } t = -s \text{ and changing limits accordingly)} \\
&= \int_{-1}^0 e^{-2\pi i n s} f(s)\, ds \quad \text{(flipping the limits and using that } f(t) \text{ is even)} \\
&= \hat{f}(n) \quad \text{(because you can integrate over any period, in this case from } -1 \text{ to } 0)
\end{aligned}$$

Uniting the two ends of the calculation we get

$$\overline{\hat{f}(n)} = \hat{f}(n),$$

hence $\hat{f}(n)$ is real. Hidden in the middle of this calculation is the interesting fact that if f is even so is $\hat{f}$, i.e.,

$$f(-t) = f(t) \quad \Rightarrow \quad \hat{f}(-n) = \hat{f}(n).$$

It's good to be attuned to these sorts of symmetry results; we'll see their like again for the Fourier transform. What happens if $f(t)$ is odd, for example?

1.6.1 What if the period isn't 1?

Changing to a base period other than 1 does not present too stiff a challenge, and it brings up a very important phenomenon. If we're working with functions $f(t)$ with period T, then

$$g(t) = f(Tt)$$

has period 1. Suppose we have

$$g(t) = \sum_{n=-N}^{N} c_n e^{2\pi i n t},$$

or even, without yet addressing issues of convergence, an infinite series

$$g(t) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n t}.$$

Write $s = Tt$, so that $g(t) = f(s)$. Then

$$f(s) = g(t) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n t} = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n s/T}$$

The harmonics are now $e^{2\pi ins/T}$.

What about the coefficients? If

$$\hat{g}(n) = \int_0^1 e^{-2\pi int} g(t)\, dt$$

then, making the same change of variable $s = Tt$, the integral becomes

$$\frac{1}{T}\int_0^T e^{-2\pi ins/T} f(s)\, ds\,.$$

To wrap up, calling the variable t again, the Fourier series for a function $f(t)$ of period T is

$$\sum_{n=-\infty}^{\infty} c_n e^{2\pi int/T}$$

where the coefficients are given by

$$c_n = \frac{1}{T}\int_0^T e^{-2\pi int/T} f(t)\, dt\,.$$

As in the case of period 1, we can integrate over any interval of length T to find c_n. For example,

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} e^{-2\pi int/T} f(t)\, dt\,.$$

(I didn't use the notation $\hat{f}(n)$ here because I'm reserving that for the case $T = 1$ to avoid any extra confusion — I'll allow that this might be too fussy.)

Remark As we'll see later, there are reasons to consider the harmonics to be

$$\frac{1}{\sqrt{T}} e^{2\pi int/T}$$

and the Fourier coefficients for nonzero n then to be

$$c_n = \frac{1}{\sqrt{T}}\int_0^T e^{-2\pi int/T} f(t)\, dt\,.$$

This makes no difference in the final formula for the series because we have two factors of $1/\sqrt{T}$ coming in, one from the differently normalized Fourier coefficient and one from the differently normalized complex exponential.

Time domain / frequency domain reciprocity Here's the phenomenon that this calculation illustrates. As we've just seen, if $f(t)$ has period T and has a Fourier series expansion then

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi int/T}.$$

We observe from this an important reciprocal relationship between properties of the signal in the time domain (if we think of the variable t as representing time) and properties of the signal as displayed in the frequency domain, i.e., properties of the spectrum. In the time domain the signal repeats after T seconds, while the points in the spectrum are 0, $\pm 1/T$, $\pm 2/T$, ..., which are spaced $1/T$ apart. (Of course for period $T = 1$ the spacing in the spectrum is also 1.) Want an aphorism for this?

> The larger the period in time the smaller the spacing of the spectrum. The smaller the period in time, the larger the spacing of the spectrum.

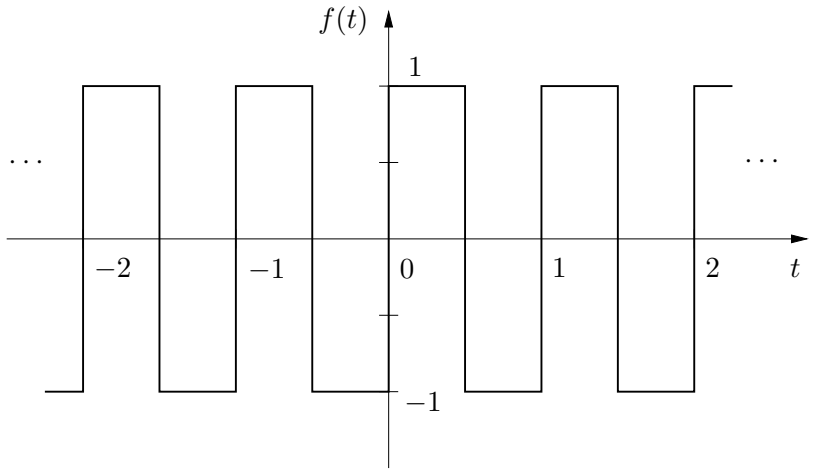
Thinking, loosely, of long periods as slow oscillations and short periods as fast oscillations, convince yourself that the aphorism makes intuitive sense. If you allow yourself to imagine letting $T \to \infty$ you can allow yourself to imagine the discrete set of frequencies becoming a continuum of frequencies.

We'll see many instances of this aphorism. We'll also have other such aphorisms — they're meant to help you organize your understanding and intuition for the subject and for the applications.

1.7 Two Examples and a Warning

All this is fine, but does it really work? That is, *given* a periodic function can we expect to write it as a sum of exponentials in the way we have described? Let's look at an example.

Consider a *square wave* of period 1, such as illustrated below.



Let's calculate the Fourier coefficients. The function is

$$f(t) = \begin{cases} +1 & 0 \le t < \frac{1}{2} \\ -1 & \frac{1}{2} \le t < 1 \end{cases}$$

and then extended to be periodic of period 1. The zeroth coefficient is the average value of the function on $0 \le t \le 1$. Obviously this is zero. For the other coefficients we have

$$\begin{aligned} \hat{f}(n) &= \int_0^1 e^{-2\pi int} f(t)\, dt \\ &= \int_0^{1/2} e^{-2\pi int}\, dt - \int_{1/2}^1 e^{-2\pi int}\, dt \\ &= \Big[-\frac{1}{2\pi in} e^{-2\pi int}\Big]_0^{1/2} - \Big[-\frac{1}{2\pi in} e^{-2\pi int}\Big]_{1/2}^1 = \frac{1}{\pi in}\big(1 - e^{-\pi in}\big) \end{aligned}$$

We should thus consider the *infinite* Fourier series

$$\sum_{n \neq 0} \frac{1}{\pi in}\big(1 - e^{-\pi in}\big) e^{2\pi int}$$

We can write this in a simpler form by first noting that

$$1 - e^{-\pi i n} = \begin{cases} 0 & n \text{ even} \\ 2 & n \text{ odd} \end{cases}$$

so the series becomes

$$\sum_{n \text{ odd}} \frac{2}{\pi i n} e^{2\pi i n t} .$$

Now combine the positive and negative terms and use

$$e^{2\pi i n t} - e^{-2\pi i n t} = 2i \sin 2\pi n t .$$

Substituting this into the series and writing $n = 2k + 1$, our final answer is

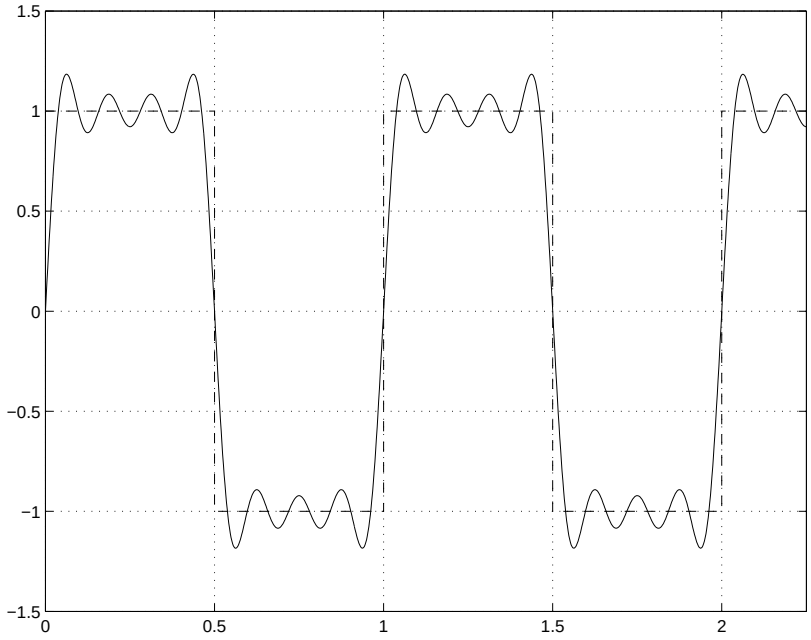
$$\frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1}{2k+1} \sin 2\pi (2k+1) t .$$

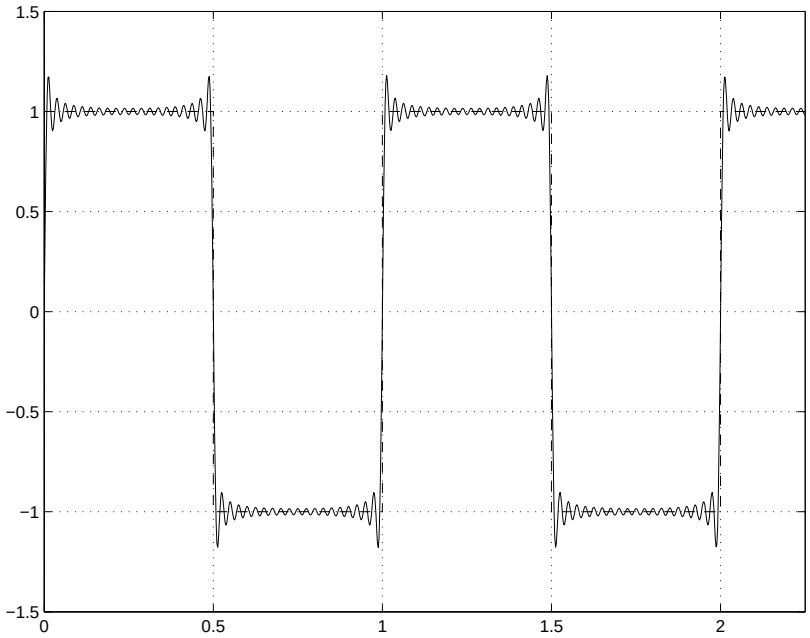
(Note that the function $f(t)$ is *odd* and this jibes with the Fourier series having only sine terms.)

What kind of series is this? In what sense does it converge, if at all, and to what does it converge, i.e, can we represent $f(t)$ as a Fourier series through

$$f(t) = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1}{2k+1} \sin 2\pi (2k+1) t \, ?$$

The graphs below are sums of terms up to frequencies 9 and 39, respectively.





We see a strange phenomenon. We certainly see the general shape of the square wave, but there is trouble at the corners. Clearly, in retrospect, we shouldn't expect to represent a function like the square wave by a *finite* sum of complex exponentials. Why? Because a finite sum of continuous functions is continuous and the square wave has jump discontinuities. Thus, for maybe the first time in your life, one of those theorems from calculus that seemed so pointless at the time makes an appearance: The sum of two (or a finite number) of continuous functions is continuous. Whatever else we may be able to conclude about a Fourier series representation for a square wave, it must contain arbitrarily high frequencies. We'll say what else needs to be said next time.

I picked the example of a square wave because it's easy to carry out the integrations needed to find the Fourier coefficients. However, it's not only a discontinuity that forces high frequencies. Take a triangle wave, say defined by

$$f(t) = \begin{cases} \frac{1}{2} + t & -\frac{1}{2} \le t \le 0 \\ \frac{1}{2} - t & 0 \le t \le +\frac{1}{2} \end{cases}$$

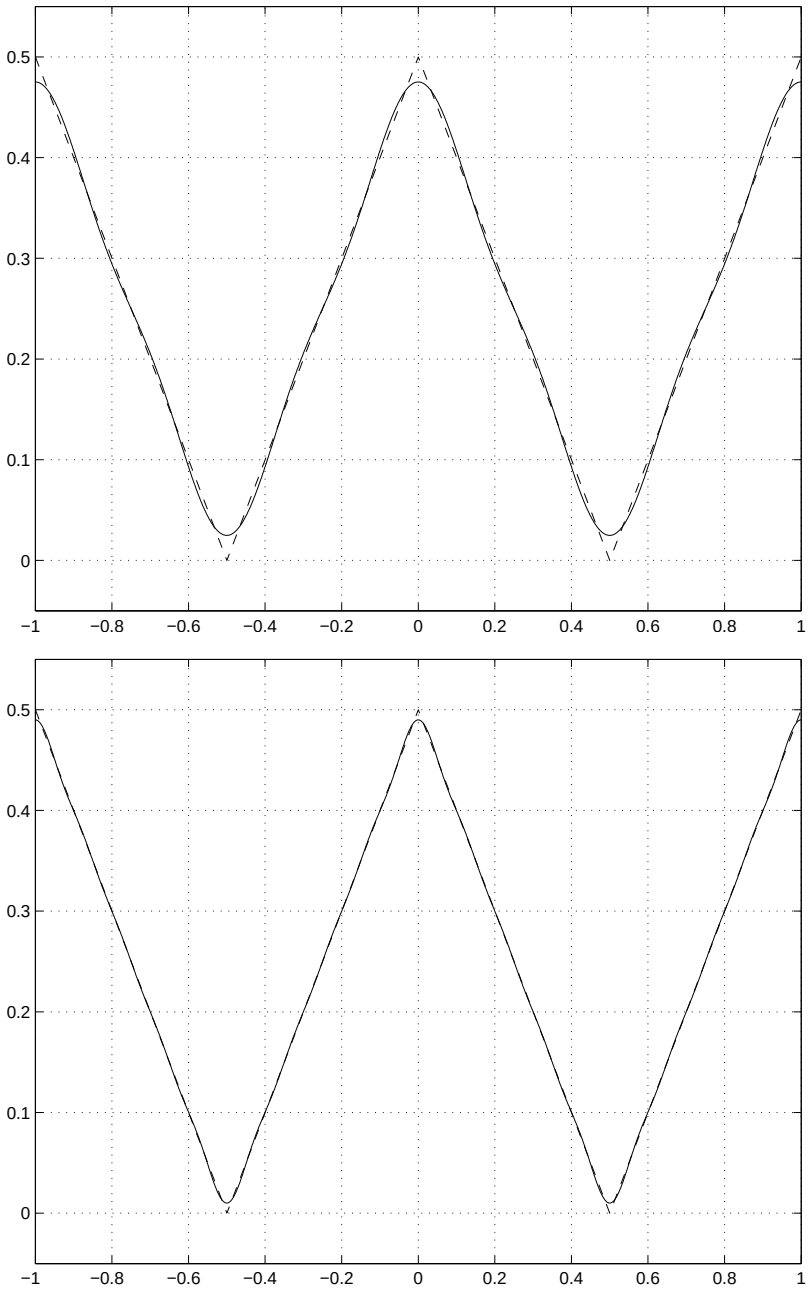
and then extended to be periodic of period 1. This is continuous. There are no jumps, though there are corners. (Draw your own graph!) A little more work than for the square wave shows that we want the infinite Fourier series

$$\frac{1}{4} + \sum_{k=0}^{\infty} \frac{2}{\pi^2(2k+1)^2} \cos(2\pi(2k+1)t)$$

I won't reproduce the calculations in public; the calculation of the coefficients needs integration by parts.

Here, too, there are only odd harmonics and there are infinitely many. This time the series involves only cosines, a reflection of the fact that the triangle wave is an *even* function. Note also that the triangle wave the coefficients decrease like $1/n^2$ while for a square wave they decrease like $1/n$. I alluded to this sort of thing, above (the size of the coefficients); it has exactly to do with the fact that the square wave is discontinuous while the triangle wave is continuous but its *derivative* is discontinuous. So here is yet another occurrence of one of those calculus theorems: The sines and cosines are differentiable to all orders, so any finite sum of them is also differentiable. We therefore should not expect a finite Fourier series to represent the triangle wave, which has corners.

How good a job do the finite sums do in approximating the triangle wave? I'll let you use your favorite software to plot some approximations. You will observe something different from what happened with the square wave. We'll come back to this, too.



One thing common to these two examples might be stated as another aphorism:

It takes high frequencies to make sharp corners.

This particular aphorism is important, for example, in questions of filtering, a topic we'll consider in detail later:

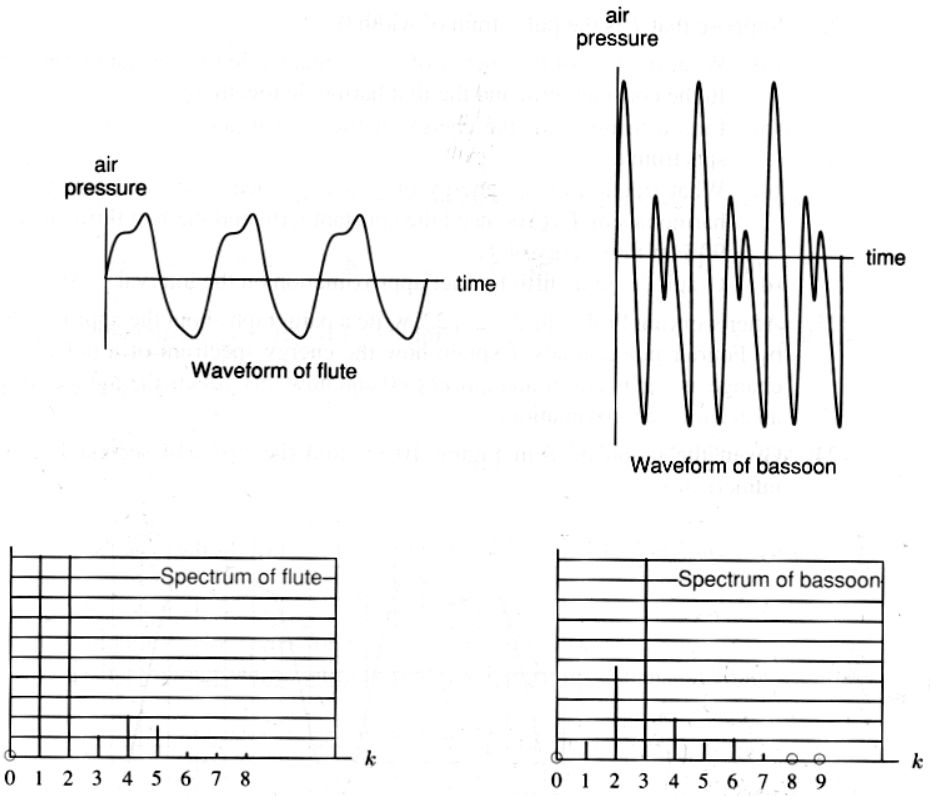
- Filtering means cutting off.

- Cutting off means sharp corners.
- Sharp corners means high frequencies.

This comes up in computer music, for example. If you're not careful to avoid discontinuities in filtering the signal (the music) you'll hear clicks — symptoms of high frequencies — when the signal is played back. A sharp cutoff will inevitably yield an unsatisfactory result, so you have to design your filters to minimize this problem.

Why do instruments sound different? More precisely, why do two instruments sound different even when they are playing the same note? It's because the note they produce is not a single sinusoid of a single frequency, not the A at 440 Hz, for example, but a sum (literally) of many sinusoids, each contributing a different amount. The complex wave that reaches your ear is the combination of many ingredients. Two instruments sound different because of the harmonics they produce and because of the strength of the harmonics.

Shown below are approximately the waveforms (what you'd see on an oscilloscope) for a bassoon and a flute both playing the same note and the power spectrum of the respective waves — what you'd see on a spectrum analyzer, if you've ever worked with one. The height of the bars corresponds to the energy of the individual harmonics, as explained above. Only the positive harmonics are displayed here. The pictures are highly simplified; in reality a spectrum analyzer would display hundreds of frequencies.



The spectral representation — the frequency domain — gives a much clearer explanation of *why* the instruments sound different than does the time domain signal. You can see how the ingredients differ and by how much. The spectral representation also offers many opportunities for varieties of signal processing that would not be so easy to do or even to imagine in the time domain. It's easy to imagine pushing some bars down, pulling others up, or eliminating blocks, operations whose actions in the time domain are far from clear.

As an aside, I once asked Julius Smith, an expert in computer music here at Stanford, why orchestras tune to an oboe playing an A. I thought it might be because the oboe produces a very pure note, mostly a perfect 440 with very few other harmonics, and this would be desirable. In fact, it seems just the opposite is the case. The spectrum of the oboe is very rich, plenty of harmonics. This is good, apparently, because whatever instrument you happen to play, there's a little bit of you in the oboe and vice versa. That helps you tune.

For a detailed discussion of the spectra of musical instruments see

`http://epubs.siam.org/sam-bin/getfile/SIREV/articles/38228.pdf`

1.8 The Math, the Majesty, the End

In previous sections, we worked with the building blocks of periodic functions — sines and cosines and complex exponentials — and considered general sums of such "harmonics". We also showed that *if* a periodic function $f(t)$ — period 1, as a convenient normalization — can be written as a sum

$$f(t) = \sum_{n=-N}^{N} c_n e^{2\pi i n t},$$

then the coefficients are given by the integral

$$c_n = \int_0^1 e^{-2\pi i n t} f(t)\, dt\,.$$

This was a pretty straightforward derivation, isolating c_n and then integrating. When $f(t)$ is real, as in many applications, one has the symmetry relation $c_{-n} = \overline{c_n}$. In a story we'll spin out over the rest of the quarter, we think of this integral as some kind of transform of f, and use the notation

$$\hat{f}(n) = \int_0^1 e^{-2\pi i n t} f(t)\, dt$$

to indicate this relationship.[9]

At this stage, we haven't done much. We have only demonstrated that if it is possible to write a periodic function as a sum of simple harmonics, then it must be done in the way we've just said. We also have some examples that indicate the possible difficulties in this sort of representation; an infinite series may be required and then convergence is certainly an issue. But we're about to do a lot. We're about to answer the question of how far the idea can be pushed: when *can* a periodic signal be written as a sum of simple harmonics?

1.8.1 Square integrable functions

There's much more to the structure of the Fourier coefficients and to the idea of writing a periodic function as a sum of complex exponentials than might appear from our simple derivation. There are:

[9] Notice that although $f(t)$ is defined for a continuous variable t, the transformed function $\hat{f}$ is defined on the integers. There are reasons for this that are much deeper than just solving for the unknown coefficients as we did last time.

- Algebraic and geometric aspects
  - The algebraic and geometric aspects are straightforward extensions of the algebra and geometry of vectors in Euclidean space. The key ideas are the inner product (dot product), orthogonality, and norm. We can pretty much cover the whole thing. I remind you that your job here is to transfer your intuition from geometric vectors to a more general setting where the vectors are signals; at least accept that the words transfer in some kind of meaningful way even if the pictures do not.
- Analytic aspects
  - The analytic aspects are not straightforward and require new ideas on limits and on the nature of integration. The aspect of "analysis" as a field of mathematics distinct from other fields is its systematic use of limiting processes. To define a new kind of limit, or to find new consequences of taking limits (or trying to), is to define a new area of analysis. We really can't cover the whole thing, and it's not appropriate to attempt to. But I'll say a little bit here, and similar issues will come up when we define the Fourier transform.

1.8.2 The punchline revealed

Let me introduce the notation and basic terminology and state what the important results are now, so you can see the point. Then I'll explain where these ideas come from and how they fit together.

Once again, to be definite we're working with periodic functions of period 1. We can consider such a function already to be defined for all real numbers, and satisfying the identity $f(t+1) = f(t)$ for all t, or we can consider $f(t)$ to be defined initially only on the interval from 0 to 1, say, and then extended to be periodic and defined on all of $\mathbf{R}$ by repeating the graph (recall the periodizing operation in the first problem set). In either case, once we know what we need to know about the function on $[0, 1]$ we know everything. All of the action in the following discussion takes place on the interval $[0, 1]$.

When $f(t)$ is a signal defined on $[0, 1]$ the *energy* of the signal is defined to be the integral

$$\int_0^1 |f(t)|^2 \, dt \, .$$

This definition of energy comes up in other physical contexts also; we don't have to be talking about functions of time. (In some areas the integral of the square is identified with power.) Thus

$$\int_0^1 |f(t)|^2 \, dt < \infty$$

means that the signal has *finite energy*, a reasonable condition to expect or to impose.

I'm writing the definition in terms of the integral of the absolute value squared $|f(t)|^2$ rather than just $f(t)^2$ because we'll want to consider the definition to apply to complex valued functions. For real-valued functions it doesn't matter whether we integrate $|f(t)|^2$ or $f(t)^2$.

One further point before we go on. Although our purpose is to use the finite energy condition to work with periodic functions, and though you think of periodic functions as defined for all time, you can see why we have to restrict attention to one period (any period). An integral of a periodic function from $-\infty$ to ∞, for example

$$\int_{-\infty}^{\infty} \sin^2 t \, dt$$

does not exist (or is infinite).

For mathematical reasons, primarily, it's best to take the square root of the integral, and to define

$$\|f\| = \Big(\int_0^1 |f(t)|^2 \, dt \Big)^{1/2}$$

With this definition one has, for example, that

$$\|\alpha f\| = \|\alpha\| \, \|f\| \, ,$$

whereas if we didn't take the square root the constant would come out to the second power — see below. One can also show, though the proof is not so obvious (see Section 1.10), that the triangle inequality holds:

$$\|f + g\| \leq \|f\| + \|g\| \, .$$

Write that out in terms of integrals if you think it's obvious:

$$\Big(\int_0^1 |f(t) + g(t)|^2 \, dt \Big)^{1/2} \leq \Big(\int_0^1 |f(t)|^2 \, dt \Big)^{1/2} + \Big(\int_0^1 |g(t)|^2 \, dt \Big)^{1/2} .$$

We can measure the distance between two functions via

$$\|f - g\| = \Big(\int_0^1 |f(t) - g(t)|^2 \, dt \Big)^{1/2} .$$

Then $\|f - g\| = 0$ if and only if $f = g$.

> **Now get this:** The length of a vector is the square root of the sum of the squares of its components. This norm defined by an integral is the continuous analog of that, and so is the definition of distance.[10] We'll make the analogy even closer when we introduce the corresponding dot product.

We let $L^2([0,1])$ be the set of functions $f(t)$ on $[0,1]$ for which

$$\int_0^1 |f(t)|^2 \, dt < \infty \, .$$

The "L" stands for Lebesgue, the French mathematician who introduced a new definition of the integral that underlies the analytic aspects of the results we're about to talk about. His work was around the turn of the 20-th century. The length we've just introduced, $\|f\|$, is called the *square norm* or the $L^2([0,1])$-*norm* of the function. When we want to distinguish this from other norms that might (and will) come up, we write $\|f\|_2$.

It's true, you'll be relieved to hear, that if $f(t)$ is in $L^2([0,1])$ then the integral defining its Fourier coefficients exists. See Section 1.10 for this. The complex integral

$$\int_0^1 e^{-2\pi i n t} f(t) \, dt$$

can be written in terms of two real integrals by writing $e^{-2\pi i n t} = \cos 2\pi n t - i \sin 2\pi n t$ so everything can be defined and computed in terms of real quantities. There are more things to be said on complex-valued versus real-valued functions in all of this, but it's best to put that off just now.

Here now is the life's work of several generations of mathematicians, all dead, all still revered:

[10] If we've really defined a "length" then scaling $f(t)$ to $\alpha f(t)$ should scale the length of $f(t)$. If we didn't take the square root in defining $\|f\|$ the length wouldn't scale to the first power.

Let $f(t)$ be in $L^2([0,1])$ and let

$$\hat{f}(n) = \int_0^1 e^{-2\pi int} f(t)\, dt, \quad n = 0, \pm 1, \pm 2, \ldots$$

be its Fourier coefficients. Then

1. For any N the finite sum
$$\sum_{n=-N}^{N} \hat{f}(n) e^{2\pi int}$$
is the best approximation to $f(t)$ in $L^2([0,1])$ by a trigonometric polynomial[11] of degree N. (You can think of this as the least squares approximation. I'll explain the phrase "of degree N" in Section 1.12, where we'll prove the statement.)
2. The complex exponentials $e^{2\pi int}$, $(n = 0, \pm 1, \pm 2, \ldots)$ form a basis for $L^2([0,1])$, and the partial sums in statement 1 converge to $f(t)$ in L^2-distance as $N \to \infty$. This means that
$$\lim_{N\to\infty} \left\| \sum_{n=-N}^{N} \hat{f}(n) e^{2\pi int} - f(t) \right\| = 0\,.$$
We write
$$f(t) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi int}\,,$$
where the equals sign is interpreted in terms of the limit.
Once we introduce the inner product on $L^2([0,1])$ a more complete statement will be that the $e^{2\pi int}$ form an *orthonormal* basis. In fact, it's only the orthonormality that we'll establish.
3. The energy of $f(t)$ can be calculated from its Fourier coefficients:
$$\int_0^1 |f(t)|^2\, dt = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2\,.$$
This is known, depending on to whom you are speaking, as Rayleigh's identity or as Parseval's theorem.

To round off the picture, let me add a fourth point that's a sort of converse to items two and three. We won't use this, but it ties things up nicely.

4. If $\{c_n : n = 0, \pm 1, \pm 2, \ldots\}$ is any sequence of complex numbers for which
$$\sum_{n=-\infty}^{\infty} |c_n|^2 < \infty\,,$$
then the function
$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi int}$$
is in $L^2([0,1])$ (meaning the limit of the partial sums converges to a function in $L^2([0,1])$) and $c_n = \hat{f}(n)$.

This last result is often referred to as the Riesz-Fischer theorem.

[11] A trigonometric polynomial is a finite sum of complex exponentials with the same fundamental frequency.

And the point of this is, again . . . One way to think of the formula for the Fourier coefficients is as passing from the "time domain" to the "frequency domain": From a knowledge of $f(t)$ (the time domain) we produce a portrait of the signal in the frequency domain, namely the (complex) coefficients $\hat{f}(n)$ associated with the (complex) harmonics $e^{2\pi int}$. The function $\hat{f}(n)$ is defined on the integers, $n = 0, \pm 1, \pm 2, \ldots$, and the equation

$$f(t) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi int},$$

recovers the time domain representation from the frequency domain representation. At least it does in the L^2 sense of equality. The extent to which equality holds in the usual, pointwise sense (plug in a value of t and the two sides agree) is a question we will address later.

The magnitude $|\hat{f}(n)|^2$ is the energy contributed by the n-th harmonic. We really have equal contributions from the "positive" and "negative" harmonics $e^{2\pi int}$ and $e^{-2\pi int}$ since $|\hat{f}(-n)| = |\hat{f}(n)|$ (note the absolute values here). As you will show in the first problem set, in passing between the complex exponential form

$$\sum_{n=-\infty}^{\infty} c_n e^{2\pi int}, \quad c_n = \hat{f}(n)$$

and the sine-cosine form

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos 2\pi nt + \sum_{n=1}^{\infty} b_n \sin 2\pi nt$$

of the Fourier series, we have $|c_n| = \frac{1}{2}\sqrt{a_n^2 + b_n^2}$, so $\hat{f}(n)$ and $\hat{f}(-n)$ together contribute a total energy of $\sqrt{a_n^2 + b_n^2}$.

Rayleigh's identity says that we can compute the energy of the signal by adding up the energies of the individual harmonics. That's quite a satisfactory state of affairs — and an extremely useful result. You'll see an example of its use in the first problem set.

Here are a few more general comments on these results.

- The first point, on best approximation in $L^2([0,1])$ by a finite sum, is a purely algebraic result. This is of practical value since, in any real application you're always making finite approximations, and this result gives guidance on how well you're doing. We'll have a more precise statement (in Appendix 3) after we set up the necessary ingredients on inner products and orthogonality.

  Realize that this gives an alternative characterization of the Fourier coefficients. Originally we said: if we can express $f(t)$ as a sum of complex exponentials, then the unknown coefficients in the expression must be given by the integral formula we found. Instead, we could have asked: What is the "least squares" approximation to the function? And again we would be led to the same integral formula for the coefficients.

- Rayleigh's identity is also an algebraic result. Once we have the proper setup it will follow effortlessly.

- The remaining statements, points 2 and 4, involve some serious analysis and we won't go into the proofs. The crisp statements that we have given are true *provided* one adopts a more general theory of integration, Lebesgue's theory. In particular, one must allow for much wilder functions to be integrated than those that are allowed for the Riemann integral, which is the integral you saw in calculus courses. This is not to say that the Riemann integral is "incorrect", rather it is incomplete — it does not allow for integrating functions that one needs to integrate *in order to get an adequate theory of Fourier series*, among other things.

> These are mathematical issues only. They have no practical value. To paraphrase John Tukey, a mathematician who helped to invent the FFT: "I wouldn't want to fly in a plane whose design depended on whether a function was Riemann or Lebesgue integrable."

So do you have to worry about this? Not really, but do take note of the examples we looked at in the previous lecture. Suppose a periodic signal has even a single discontinuity or a corner, like a square wave, a sawtooth wave or a triangle wave for example. Or think of taking a smooth signal and cutting it off (using a window), thus inducing a discontinuity or a corner. The Fourier series for such a signal *must* have infinitely many terms, and thus arbitrarily high frequencies in the spectrum. This is so, recall, because if

$$f(t) = \sum_{n=-N}^{N} \hat{f}(n) e^{2\pi i n t}$$

for some finite N then $f(t)$ would be the finite sum of smooth functions, hence smooth itself. It's the possibility (the reality) of representing discontinuous (or wilder) functions by an infinite sum of smooth functions that's really quite a strong result. This was anticipated, and even stated by Fourier, but people didn't believe him. The results we've stated above are Fourier's vindication, but probably not in a form he would have recognized.

1.9 Orthogonality

The aspect of Euclidean geometry that sets it apart from geometries which share most of its other features is *perpendicularity* and its consequences. To set up a notion of perpendicularity in settings other than the familiar Euclidean plane or three dimensional space is to try to copy the Euclidean properties that go with it.

Perpendicularity becomes operationally useful, especially for applications, when it's linked to measurement, i.e., to length. This link is the Pythagorean theorem. [12] Perpendicularity becomes austere when mathematicians start referring to it as *orthogonality*, but that's what I'm used to and it's another term you can throw around to impress your friends.

Vectors To fix ideas, I want to remind you briefly of vectors and geometry in Euclidean space. We write vectors in $\mathbf{R}^n$ as n-tuples of real numbers:

$$\mathbf{v} = (v_1, v_2, \ldots, v_n)$$

The v_i are called the components of $\mathbf{v}$. The length, or *norm* of $\mathbf{v}$ is

$$\|\mathbf{v}\| = (v_1^2 + v_2^2 + \cdots + v_n^2)^{1/2}\,.$$

[12] How do you lay out a big rectangular field of specified dimensions? You use the Pythagorean theorem. I had an encounter with this a few summers ago when I volunteered to help lay out soccer fields. I was only asked to assist, because evidently I could not be trusted with the details. Put two stakes in the ground to determine one side of the field. That's one leg of what is to become a right triangle — half the field. I hooked a tape measure on one stake and walked off in a direction generally perpendicular to the first leg, stopping when I had gone the regulation distance for that side of the field, or when I needed rest. The chief of the crew hooked another tape measure on the other stake and walked approximately along the diagonal of the field — the hypotenuse. We adjusted our positions — but not the length we had walked off — to meet up, so that the Pythagorean theorem was satisfied; he had a chart showing what this distance should be. Hence at our meeting point the leg I determined must be perpendicular to the first leg we laid out. This was my first practical use of the Pythagorean theorem, and so began my transition from a pure mathematician to an engineer.

The distance between two vectors $\mathbf{v}$ and $\mathbf{w}$ is $\|\mathbf{v}-\mathbf{w}\|$.

How does the Pythagorean theorem look in terms of vectors? Let's just work in $\mathbf{R}^2$. Let $\mathbf{u} = (u_1, u_2)$, $\mathbf{v} = (v_1, v_2)$, and $\mathbf{w} = \mathbf{u}+\mathbf{v} = (u_1+v_1, u_2+v_2)$. If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ form a right triangle with $\mathbf{w}$ the hypotenuse, then

$$\begin{aligned} \|\mathbf{w}\|^2 = \|\mathbf{u}+\mathbf{v}\|^2 &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 \\ (u_1+v_1)^2 + (u_2+v_2)^2 &= (u_1^2+u_2^2) + (v_1^2+v_2^2) \\ (u_1^2 + 2u_1v_1 + v_1^2) + (u_2^2 + 2u_2v_2 + v_2^2) &= u_1^2 + u_2^2 + v_1^2 + v_2^2 \end{aligned}$$

The squared terms cancel and we conclude that

$$u_1v_1 + u_2v_2 = 0$$

is a necessary and sufficient condition for $\mathbf{u}$ and $\mathbf{v}$ to be perpendicular.

And so we introduce the (algebraic) definition of the *inner product* or *dot product* of two vectors. We give this in $\mathbf{R}^n$:

If $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ and $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ then the inner product is

$$\mathbf{v}\cdot\mathbf{w} = v_1w_1 + v_2w_2 + \cdots + v_nw_n$$

Other notations for the inner product are $(\mathbf{v}, \mathbf{w})$ (just parentheses; we'll be using this notation) and $\langle \mathbf{v}, \mathbf{w}\rangle$ (angle brackets for those who think parentheses are not fancy enough; the use of angle brackets is especially common in physics where it's also used to denote more general *pairings* of vectors that produce real or complex numbers.)

Notice that

$$(\mathbf{v}, \mathbf{v}) = v_1^2 + v_2^2 + \cdots + v_n^2 = \|\mathbf{v}\|^2 .$$

Thus

$$\|\mathbf{v}\| = (\mathbf{v}, \mathbf{v})^{1/2} .$$

There is also a geometric approach to the inner product, which leads to the formula

$$(\mathbf{v}, \mathbf{w}) = ||\mathbf{v}||\,||\mathbf{w}||\cos\theta$$

where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$. This is sometimes taken as an alternate definition of the inner product, though we'll stick with the algebraic definition. For a few comments on this see Section 1.10.

We see that $(\mathbf{v}, \mathbf{w}) = 0$ if and only if $\mathbf{v}$ and $\mathbf{w}$ are orthogonal. This was the point, after all, and it is a truly helpful result, especially because it's so easy to verify when the vectors are given in coordinates. The inner product does more than identify orthogonal vectors, however. When it's nonzero it tells you how much of one vector is in the direction of another. That is, the vector

$$\frac{(\mathbf{v}, \mathbf{w})}{||\mathbf{w}||}\frac{\mathbf{w}}{||\mathbf{w}||} \quad \text{also written as} \quad \frac{(\mathbf{v}, \mathbf{w})}{(\mathbf{w}, \mathbf{w})}\mathbf{w} \ ,$$

is the projection of $\mathbf{v}$ onto the unit vector $\mathbf{w}/||\mathbf{w}||$, or, if you prefer, $(\mathbf{v},\mathbf{w})/||\mathbf{w}||$ is the (scalar) component of $\mathbf{v}$ in the direction of $\mathbf{w}$. I think of the inner product as measuring how much one vector "knows" another; two orthogonal vectors don't know each other.

Finally, I want to list the main algebraic properties of the inner product. I won't give the proofs; they are straightforward verifications. We'll see these properties again — modified slightly to allow for complex numbers — a little later.

1. $(\mathbf{v},\mathbf{v}) \geq 0$ and $(\mathbf{v},\mathbf{v}) = 0$ if and only if $\mathbf{v} = 0$ (positive definiteness)
2. $(\mathbf{v},\mathbf{w}) = (\mathbf{w},\mathbf{v})$ (symmetry)
3. $(\alpha\mathbf{v},\mathbf{w}) = \alpha(\mathbf{v},\mathbf{w})$ for any scalar α (homogeneity)
4. $(\mathbf{v}+\mathbf{w},\mathbf{u}) = (\mathbf{v},\mathbf{u}) + (\mathbf{w},\mathbf{u})$ (additivity)

In fact, these are exactly the properties that ordinary multiplication has.

Orthonormal basis The natural basis for $\mathbf{R}^n$ are the vectors of length 1 in the n "coordinate directions":

$$\mathbf{e}_1 = (1,0,\dots,0)\,,\ \mathbf{e}_2 = (0,1,\dots,0)\,,\ \dots,\ \mathbf{e}_n = (0,0,\dots,1).$$

These vectors are called the "natural" basis because a vector $\mathbf{v} = (v_1, v_2, \dots, v_n)$ is expressed "naturally" in terms of its components as

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \cdots + v_n\mathbf{e}_n\,.$$

One says that the natural basis $\mathbf{e}_1$, $\mathbf{e}_2$, …, $\mathbf{e}_n$ are an *orthonormal basis* for $\mathbf{R}^n$, meaning

$$(\mathbf{e}_i,\mathbf{e}_j) = \delta_{ij}\,,$$

where δ_{ij} is the *Kronecker delta* defined by

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Notice that

$$(\mathbf{v},\mathbf{e}_k) = v_k\,,$$

and hence that

$$\mathbf{v} = \sum_{k=1}^{n} (\mathbf{v},\mathbf{e}_k)\mathbf{e}_k\,.$$

In words:

> When $\mathbf{v}$ is decomposed as a sum of vectors in the directions of the orthonormal basis vectors, the components are given by the inner product of $\mathbf{v}$ with the basis vectors.
>
> Since the $\mathbf{e}_k$ have length 1, the inner products $(\mathbf{v},\mathbf{e}_k)$ are the projections of $\mathbf{v}$ onto the basis vectors.[13]

[13] Put that the other way I like so much, the inner product $(\mathbf{v},\mathbf{e}_k)$ is how much $\mathbf{v}$ and $\mathbf{e}_k$ know each other.

Functions All of what we've just done can be carried over to $L^2([0,1])$, including the same motivation for orthogonality and defining the inner product. When are two functions "perpendicular"? Answer: when the Pythagorean theorem is satisfied. Thus if we are to have

$$\|f+g\|^2 = \|f\|^2 + \|g\|^2$$

then

$$\begin{aligned}
\int_0^1 (f(t)+g(t))^2\,dt &= \int_0^1 f(t)^2\,dt + \int_0^1 g(t)^2\,dt \\
\int_0^1 (f(t)^2 + 2f(t)g(t) + g(t)^2)\,dt &= \int_0^1 f(t)^2\,dt + \int_0^1 g(t)^2\,dt \\
\int_0^1 f(t)^2\,dt + 2\int_0^1 f(t)g(t)\,dt + \int_0^1 g(t)^2\,dt &= \int_0^1 f(t)^2\,dt + \int_0^1 g(t)^2\,dt
\end{aligned}$$

If you buy the premise, you have to buy the conclusion — we conclude that the condition to adopt to define when two functions are perpendicular (or as we'll now say, orthogonal) is

$$\int_0^1 f(t)g(t)\,dt = 0\,.$$

So we *define* the inner product of two functions in $L^2([0,1])$ to be.

$$(f,g) = \int_0^1 f(t)g(t)\,dt\,.$$

(See Section 1.10 for a discussion of why $f(t)g(t)$ is integrable if $f(t)$ and $g(t)$ are each square integrable.)

This inner product has all of the algebraic properties of the dot product of vectors. We list them, again.

1. $(f,f) \geq 0$ and $(f,f) = 0$ if and only if $f = 0$.
2. $(f,g) = (g,f)$
3. $(f+g,h) = (f,h) + (g,h)$
4. $(\alpha f, g) = \alpha(f,g)$

In particular, we have

$$(f,f) = \int_0^1 f(t)^2\,dt = \|f\|^2\,.$$

Now, let me relieve you of a burden that you may feel you must carry. There is no reason on earth why you should have any pictorial intuition for the inner product of two functions, and for when two functions are orthogonal. How can you picture the condition $(f,g) = 0$? In terms of the graphs of f and g? I don't think so. And if (f,g) is not zero, how are you to picture how much f and g know each other? Don't be silly.

We're working by analogy here. It's a very strong analogy, but that's not to say that the two settings — functions and geometric vectors — are identical. They aren't. As I have said before, what you should do is draw pictures in $\mathbf{R}^2$ and $\mathbf{R}^3$, see, somehow, what algebraic or geometric idea may be called for, and *using the same words* make the attempt to carry that over to $L^2([0,1])$. It's surprising how often and how well this works.

There's a catch There's always a catch. In the preceding discussion we've been working with the *real* vector space $\mathbf{R}^n$, as motivation, and with real-valued functions in $L^2([0,1])$. But, of course, the definition of the Fourier coefficients involves complex functions in the form of the complex exponential, and the Fourier series is a sum of complex terms. We could avoid this catch by writing everything in terms of sine and cosine, a procedure you may have followed in an earlier course. However, we don't want to sacrifice the algebraic dexterity we can show by working with the complex form of the Fourier sums, and a more effective and encompassing choice is to consider *complex-valued* square integrable functions and the *complex inner product*.

Here are the definitions. For the definition of $L^2([0,1])$ we assume again that

$$\int_0^1 |f(t)|^2\,dt < \infty\,.$$

The definition looks the same as before, but $|f(t)|^2$ is now the magnitude of the (possibly) complex number $f(t)$.

The inner product of complex-valued functions $f(t)$ and $g(t)$ in $L^2([0,1])$ is defined to be

$$(f,g) = \int_0^1 f(t)\overline{g(t)}\,dt\,.$$

The complex conjugate in the second slot causes a few changes in the algebraic properties. To wit:

1. $(f,g) = \overline{(g,f)}$ (*Hermitian* symmetry)
2. $(f,f) \geq 0$ and $(f,f) = 0$ if and only if $f = 0$ (positive definiteness — same as before)
3. $(\alpha f, g) = \alpha(f,g)$, $(f,\alpha g) = \overline{\alpha}(f,g)$ (homogeneity — same as before in the first slot, conjugate scalar comes out if it's in the second slot)
4. $(f+g,h) = (f,h) + (g,h)$, $(f,g+h) = (f,g) + (f,h)$ (additivity — same as before, no difference between additivity in first or second slot)

I'll say more about the reason for the definition in Appendix 2. As before,

$$(f,f) = \int_0^1 f(t)\overline{f(t)}\,dt = \int_0^1 |f(t)|^2\,dt = \|f\|^2\,.$$

From now on, when we talk about $L^2([0,1])$ and the inner product on $L^2([0,1])$ we will always assume the complex inner product. If the functions happen to be real-valued then this reduces to the earlier definition.

The complex exponentials are an orthonormal basis Number two in our list of the greatest hits of the theory of Fourier series says that the complex exponentials form a basis for $L^2([0,1])$. This is *not* a trivial statement. In many ways it's the whole ball game, for in establishing this fact one sees why $L^2([0,1])$ is the natural space to work with, and why convergence in $L^2([0,1])$ is the right thing to ask for in asking for the convergence of the partial sums of Fourier series.[14] But it's too much for us to do.

Instead, we'll be content with the news that, just like the natural basis of $\mathbf{R}^n$, the complex exponentials are *orthonormal*. Here's the calculation; in fact, it's the same calculation we did when we first solved for the Fourier coefficients. Write

$$e_n(t) = e^{2\pi i n t}\,.$$

[14] An important point in this development is understanding what happens to the usual kind of pointwise convergence *vis à vis* $L^2([0,1])$ convergence when the functions are smooth enough.

The inner product of two of them, $e_n(t)$ and $e_m(t)$, when $n \neq m$ is

$$(e_n, e_m) = \int_0^1 e^{2\pi int}\overline{e^{2\pi imt}}\,dt = \int_0^1 e^{2\pi int}e^{-2\pi imt}\,dt = \int_0^1 e^{2\pi i(n-m)t}\,dt$$
$$= \frac{1}{2\pi i(n-m)}e^{2\pi i(n-m)t}\Big]_0^1 = \frac{1}{2\pi i(n-m)}\big(e^{2\pi i(n-m)} - e^0\big) = \frac{1}{2\pi i(n-m)}(1-1) = 0\,.$$

They are orthogonal. And when $n = m$

$$(e_n, e_n) = \int_0^1 e^{2\pi int}\overline{e^{2\pi int}}\,dt = \int_0^1 e^{2\pi int}e^{-2\pi int}\,dt = \int_0^1 e^{2\pi i(n-n)t}\,dt = \int_0^1 1\,dt = 1\,.$$

Therefore the functions $e_n(t)$ are *orthonormal*:

$$(e_n, e_m) = \delta_{nm} = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$$

What is the component of a function $f(t)$ "in the direction" $e_n(t)$? By analogy to the Euclidean case, it is given by the inner product

$$(f, e_n) = \int_0^1 f(t)\overline{e_n(t)}\,dt = \int_0^1 f(t)e^{-2\pi int}\,dt\,,$$

precisely the n-th Fourier coefficient $\hat{f}(n)$. (Note that e_n really does have to be in the second slot here.)

Thus writing the Fourier series

$$f = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{2\pi int}\,,$$

as we did earlier, is exactly like the decomposition in terms of an orthonormal basis and associated inner product:

$$f = \sum_{n=-\infty}^{\infty} (f, e_n)e_n\,.$$

What we haven't done is to show that this really works — that the complex exponentials are a *basis* as well as being orthonormal. We would be required to show that

$$\lim_{N\to\infty} \Big\| f - \sum_{n=-N}^{N} (f, e_n)e_n \Big\| = 0\,.$$

We're not going to do that. It's hard.

What if the period isn't 1? Remember how we modified the Fourier series when the period is T rather than 1. We were led to the expansion

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi int/T}\,.$$

where

$$c_n = \frac{1}{T}\int_0^T e^{-2\pi int/T} f(t)\,dt\,.$$

The whole setup we've just been through can be easily modified to cover this case. We work in the space $L^2([0,T])$ of square integrable functions on the interval $[0,T]$. The (complex) inner product is

$$(f,g) = \int_0^T f(t)\overline{g(t)}\,dt\,.$$

What happens with the T-periodic complex exponentials $e^{2\pi int/T}$? If $n \neq m$ then, much as before,

$$\begin{aligned}(e^{2\pi int/T}\,,\, e^{2\pi imt/T}) &= \int_0^T e^{2\pi int/T}\,\overline{e^{2\pi imt/T}}\,dt = \int_0^T e^{2\pi int/T}\,e^{-2\pi imt/T}\,dt \\ &= \int_0^T e^{2\pi i(n-m)t/T}\,dt = \frac{1}{2\pi i(n-m)/T}\,e^{2\pi i(n-m)t/T}\Big]_0^T \\ &= \frac{1}{2\pi i(n-m)/T}\,(e^{2\pi i(n-m)} - e^0) = \frac{1}{2\pi i(n-m)/T}\,(1-1) = 0\end{aligned}$$

And when $n = m$:

$$\begin{aligned}(e^{2\pi int/T}\,,\, e^{2\pi int/T}) &= \int_0^T e^{2\pi int/T}\,\overline{e^{2\pi int/T}}\,dt \\ &= \int_0^T e^{2\pi int/T}\,e^{-2\pi int/T}\,dt = \int_0^T 1\,dt = T\,.\end{aligned}$$

Aha — it's not 1, it's T. The complex exponentials with period T are orthogonal but not orthonormal. To get the latter property we scale the complex exponentials to

$$e_n(t) = \frac{1}{\sqrt{T}}e^{2\pi int/T}\,,$$

for then

$$(e_n, e_m) = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$$

This is where the factor $1/\sqrt{T}$ comes from, the factor mentioned earlier in this chapter. The inner product of f with e_n is

$$(f, e_n) = \frac{1}{\sqrt{T}}\int_0^T f(t)e^{-2\pi int/T}\,dt\,.$$

Then

$$\sum_{n=-\infty}^{\infty}(f,e_n)e_n = \sum_{n=-\infty}^{\infty}\left(\frac{1}{\sqrt{T}}\int_0^T f(s)e^{-2\pi ins/T}\,ds\right)\frac{1}{\sqrt{T}}e^{2\pi int/T} = \sum_{n=-\infty}^{\infty} c_n e^{2\pi int/T},$$

where

$$c_n = \frac{1}{T}\int_0^T e^{-2\pi int/T}f(t)\,dt\,,$$

as above. We're back to our earlier formula.

Rayleigh's identity As a last application of these ideas, let's derive Rayleigh's identity, which states that

$$\int_0^1 |f(t)|^2\,dt = \sum_{n=-\infty}^{\infty}|\hat{f}(n)|^2\,.$$

This is a cinch! Expand $f(t)$ as a Fourier series:

$$f(t) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n t} = \sum_{n=-\infty}^{\infty} (f, e_n) e_n \,.$$

Then

$$\begin{aligned}
\int_0^1 |f(t)|^2 \, dt = \|f\|^2 &= (f, f) \\
&= \Big(\sum_{n=-\infty}^{\infty} (f, e_n) e_n, \sum_{m=-\infty}^{\infty} (f, e_m) e_m \Big) \\
&= \sum_{n,m} (f, e_n) \overline{(f, e_m)} (e_n, e_m) = \sum_{n,m=-\infty}^{\infty} (f, e_n) \overline{(f, e_m)} \delta_{nm} \\
&= \sum_{n=-\infty}^{\infty} (f, e_n) \overline{(f, e_n)} = \sum_{n=-\infty}^{\infty} |(f, e_n)|^2 = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2
\end{aligned}$$

The above derivation used

1. The algebraic properties of the complex inner product;
2. The fact that the $e_n(t) = e^{2\pi i n t}$ are orthonormal with respect to this inner product;
3. Know-how in whipping around sums

Do not go to sleep until you can follow every line in this derivation.

Writing Rayleigh's identity as

$$\|f\|^2 = \sum_{n=-\infty}^{\infty} |(f, e_n)|^2$$

again highlights the parallels between the geometry of L^2 and the geometry of vectors: How do you find the squared length of a vector? By adding the squares of its components with respect to an orthonormal basis. That's exactly what Rayleigh's identity is saying.

1.10 Appendix: The Cauchy-Schwarz Inequality and its Consequences

The Cauchy-Schwarz inequality is a relationship between the inner product of two vectors and their norms. It states

$$|(\mathbf{v}, \mathbf{w})| \leq \|\mathbf{v}\| \, \|\mathbf{w}\| \,.$$

This is trivial to see from the geometric formula for the inner product:

$$|(\mathbf{v}, \mathbf{w})| = \|\mathbf{v}\| \, \|\mathbf{w}\| \, |\cos\theta| \leq \|\mathbf{v}\| \, \|\mathbf{w}\| \,,$$

because $|\cos\theta| \leq 1$. In fact, the rationale for the geometric formula of the inner product will *follow* from the Cauchy-Schwarz inequality.

It's certainly not obvious how to derive the inequality from the algebraic definition. Written out in components, the inequality says that

$$\Big| \sum_{k=1}^{n} v_k w_k \Big| \;\leq\; \Big(\sum_{k=1}^{n} v_k^2 \Big)^{1/2} \Big(\sum_{k=1}^{n} w_k^2 \Big)^{1/2} .$$

Sit down and try that one out sometime.

In fact, the proof of the Cauchy-Schwarz inequality in general uses only the four algebraic properties of the inner product listed earlier. Consequently the same argument applies to any sort of "product" satisfying these properties. It's such an elegant argument (due to John von Neumann, I believe) that I'd like to show it to you. We'll give this for the real inner product here, with comments on the complex case to follow in the next appendix.

Any inequality can ultimately be written in a way that says that some quantity is positive. Some things that we know are positive: the square of a real number; the area of something; and the length of something are examples.[15] For this proof we use that the norm of a vector is positive, but we throw in a parameter.[16] Let t be any real number. Then $\|\mathbf{v} - t\mathbf{w}\|^2 \geq 0$. Write this in terms of the inner product and expand using the algebraic properties; because of homogeneity, symmetry, and additivity, this is just like multiplication — that's important to realize:

$$\begin{aligned} 0 &\leq \|\mathbf{v} - t\mathbf{w}\|^2 \\ &= (\mathbf{v} - t\mathbf{w}, \mathbf{v} - t\mathbf{w}) \\ &= (\mathbf{v}, \mathbf{v}) - 2t(\mathbf{v}, \mathbf{w}) + t^2(\mathbf{w}, \mathbf{w}) \\ &= \|\mathbf{v}\|^2 - 2t(\mathbf{v}, \mathbf{w}) + t^2\|\mathbf{w}\|^2 \end{aligned}$$

This is a quadratic equation in t, of the form $at^2 + bt + c$, where $a = \|\mathbf{w}\|^2$, $b = -2(\mathbf{v}, \mathbf{w})$, and $c = \|\mathbf{v}\|^2$. The first inequality, and the chain of equalities that follow, says that this quadratic is *always nonnegative*. Now a quadratic that's always nonnegative has to have a *non-positive* discriminant: The discriminant, $b^2 - 4ac$ determines the nature of the roots of the quadratic — if the discriminant is positive then there are two real roots, but if there are two real roots, then the quadratic must be negative somewhere.

Therefore $b^2 - 4ac \leq 0$, which translates to

$$4(\mathbf{v}, \mathbf{w})^2 - 4\|\mathbf{w}\|^2 \|\mathbf{v}\|^2 \leq 0 \quad \text{or} \quad (\mathbf{v}, \mathbf{w})^2 \leq \|\mathbf{w}\|^2 \|\mathbf{v}\|^2 .$$

Take the square root of both sides to obtain

$$|(\mathbf{v}, \mathbf{w})| \leq \|\mathbf{v}\| \, \|\mathbf{w}\| ,$$

as desired. (Amazing, isn't it — a nontrivial application of the *quadratic formula*!)[17] This proof also shows when equality holds in the Cauchy-Schwarz inequality. When is that?

To get back to geometry, we now know that

$$-1 \leq \frac{(\mathbf{v}, \mathbf{w})}{\|\mathbf{v}\| \, \|\mathbf{w}\|} \leq 1 .$$

Therefore there is a unique angle θ with $0 \leq \theta \leq \pi$ such that

$$\cos\theta = \frac{(\mathbf{v}, \mathbf{w})}{\|\mathbf{v}\| \, \|\mathbf{w}\|} ,$$

[15] This little riff on the nature of inequalities qualifies as a minor secret of the universe. More subtle inequalities sometimes rely on convexity, as in the center of gravity of a system of masses is contained within the convex hull of the masses.

[16] "Throwing in a parameter" goes under the heading of dirty tricks of the universe.

[17] As a slight alternative to this argument, if the quadratic $f(t) = at^2 + bt + c$ is everywhere nonnegative then, in particular, its minimum value is nonnegative. This minimum occurs at $t = -b/2a$ and leads to the same inequality, $4ac - b^2 \geq 0$.

i.e.,

$$(\mathbf{v}, \mathbf{w}) = \|\mathbf{v}\| \, \|\mathbf{w}\| \cos\theta \, .$$

Identifying θ as the angle between $\mathbf{v}$ and $\mathbf{w}$, we have now reproduced the geometric formula for the inner product. What a relief.

The triangle inequality,

$$\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$$

follows directly from the Cauchy-Schwarz inequality. Here's the argument.

$$\begin{aligned}
\|\mathbf{v} + \mathbf{w}\|^2 &= (\mathbf{v} + \mathbf{w}, \mathbf{v} + \mathbf{w}) \\
&= (\mathbf{v}, \mathbf{v}) + 2(\mathbf{v}, \mathbf{w}) + (\mathbf{w}, \mathbf{w}) \\
&\leq (\mathbf{v}, \mathbf{v}) + 2|(\mathbf{v}, \mathbf{w})| + (\mathbf{w}, \mathbf{w}) \\
&\leq (\mathbf{v}, \mathbf{v}) + 2\|\mathbf{v}\| \, \|\mathbf{w}\| + (\mathbf{w}, \mathbf{w}) \quad \text{(by Cauchy-Schwarz)} \\
&= \|\mathbf{v}\|^2 + 2\|\mathbf{v}\| \, \|\mathbf{w}\| + \|\mathbf{w}\|^2 = (\|\mathbf{v}\| + \|\mathbf{w}\|)^2.
\end{aligned}$$

Now take the square root of both sides to get $\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$. In coordinates this says that

$$\left(\sum_{k=1}^{n} (v_k + w_k)^2 \right)^{1/2} \leq \left(\sum_{k=1}^{n} v_k^2 \right)^{1/2} + \left(\sum_{k=1}^{n} w_k^2 \right)^{1/2} .$$

For the inner product on $L^2([0,1])$, the Cauchy-Schwarz inequality takes the impressive form

$$\left| \int_0^1 f(t) g(t) \, dt \right| \leq \left(\int_0^1 f(t)^2 \, dt \right)^{1/2} \left(\int_0^1 g(t)^2 \, dt \right)^{1/2} .$$

You can think of this as a limiting case of the Cauchy-Schwarz inequality for vectors — sums of products become integrals of products on taking limits, an ongoing theme — but it's better to think in terms of general inner products and their properties. For example, we now also know that

$$\|f + g\| \leq \|f\| + \|g\| \, ,$$

i.e.,

$$\left(\int_0^1 (f(t) + g(t))^2 \, dt \right)^{1/2} \leq \left(\int_0^1 f(t)^2 \, dt \right)^{1/2} + \left(\int_0^1 g(t)^2 \, dt \right)^{1/2} .$$

Once again, one could, I suppose, derive this from the corresponding inequality for sums, but why keep going through that extra work?

Incidentally, I have skipped over something here. If $f(t)$ and $g(t)$ are square integrable, then in order to get the Cauchy-Schwarz inequality working, one has to know that the inner product (f, g) makes sense, i.e.,

$$\int_0^1 f(t) g(t) \, dt < \infty \, .$$

(This isn't an issue for vectors in $\mathbf{R}^n$, of course. Here's an instance when something more needs to be said for the case of functions.) To deduce this you can first observe that[18]

$$f(t) g(t) \leq f(t)^2 + g(t)^2 \, .$$

[18] And where does that little observation come from? From the same positivity trick used to prove Cauchy-Schwarz:

$$0 \leq (f(t) - g(t))^2 = f(t)^2 - 2f(t)g(t) + g(t)^2 \quad \Rightarrow \quad 2f(t)g(t) \leq f(t)^2 + g(t)^2 \, .$$

This is the inequality between the arithmetic and geometric mean.

With this

$$\int_0^1 f(t)g(t)\,dt \le \int_0^1 f(t)^2\,dt + \int_0^1 g(t)^2\,dt < \infty\,,$$

since we started by assuming that $f(t)$ and $g(t)$ are square integrable.

Another consequence of this last argument is the fortunate fact that the Fourier coefficients of a function in $L^2([0,1])$ exist. That is, we're wondering about the existence of

$$\int_0^1 e^{-2\pi int} f(t)\,dt\,,$$

allowing for integrating complex functions. Now

$$\left|\int_0^1 e^{-2\pi int} f(t)\,dt\right| \le \int_0^1 \left|e^{-2\pi int} f(t)\right| dt = \int_0^1 |f(t)|\,dt\,,$$

so we're wondering whether

$$\int_0^1 |f(t)|\,dt < \infty\,,$$

i.e., is $f(t)$ *absolutely integrable* given that it is *square* integrable. But $f(t) = f(t)\cdot 1$, and both $f(t)$ and the constant function 1 are square integrable on $[0,1]$, so the result follows from Cauchy-Schwartz. We wonder no more.

> **Warning:** This casual argument *would not work* if the interval $[0,1]$ were replaced by the entire real line. The constant function 1 has an infinite integral on $\mathbf{R}$. You may think we can get around this little inconvenience, but it is *exactly* the sort of trouble that comes up in trying to apply Fourier *series* ideas (where functions are defined on finite intervals) to Fourier *transform* ideas (where functions are defined on all of $\mathbf{R}$).

1.11 Appendix: More on the Complex Inner Product

Here's an argument why the conjugate comes in in defining a complex inner product. Let's go right to the case of integrals. What if we apply the Pythagorean Theorem to deduce the condition for perpendicularity in the complex case, just as we did in the real case? We have

$$\int_0^1 |f(t)+g(t)|^2 = \int_0^1 |f(t)|^2\,dt + \int_0^1 |g(t)|^2\,dt$$

$$\int_0^1 (|f(t)|^2 + 2\,\mathrm{Re}\{f(t)\overline{g(t)}\} + |g(t)|^2)\,dt = \int_0^1 |f(t)|^2\,dt + \int_0^1 |g(t)|^2\,dt$$

$$\int_0^1 |f(t)|^2\,dt + 2\,\mathrm{Re}\left(\int_0^1 f(t)\overline{g(t)}\,dt\right) + \int_0^1 |g(t)|^2\,dt = \int_0^1 |f(t)|^2\,dt + \int_0^1 |g(t)|^2\,dt$$

So it looks like the condition should be

$$\mathrm{Re}\left(\int_0^1 f(t)\overline{g(t)}\,dt\right) = 0\,.$$

Why doesn't this determine the definition of the inner product of two complex functions? That is, why don't we define

$$(f,g) = \mathrm{Re}\left(\int_0^1 f(t)\overline{g(t)}\,dt\right)?$$

This definition has a nicer symmetry property, for example, than the definition we used earlier. Here we have

$$(f,g) = \mathrm{Re}\left(\int_0^1 f(t)\overline{g(t)}\,dt\right) = \mathrm{Re}\left(\int_0^1 \overline{f(t)}g(t)\,dt\right) = (g,f)\,,$$

so none of that Hermitian symmetry that we always have to remember.

The problem is that this definition doesn't give any kind of homogeneity when multiplying by a *complex* scalar. If α is a complex number then

$$(\alpha f,g) = \mathrm{Re}\left(\int_0^1 \alpha f(t)\overline{g(t)}\,dt\right) = \mathrm{Re}\left(\alpha\int_0^1 f(t)\overline{g(t)}\,dt\right).$$

But we can't pull the α out of taking the real part unless it's real to begin with. If α is not real then

$$(\alpha f,g) \neq \alpha(f,g)\,.$$

Not having equality here is too much to sacrifice. (Nor do we have anything good for $(f,\alpha g)$, despite the natural symmetry $(f,g) = (g,f)$.) We adopt the definition

$$(f,g) = \int_0^1 f(t)\overline{g(t)}\,dt\,.$$

A helpful identity A frequently employed identity for the complex inner product is:

$$\|f+g\|^2 = \|f\|^2 + 2\,\mathrm{Re}(f,g) + \|g\|^2\,.$$

We more or less used this, above, and I wanted to single it out. The verification is:

$$\begin{aligned}\|f+g\|^2 &= (f+g,f+g) = (f,f+g) + (g,f+g)\\ &= (f,f) + (f,g) + (g,f) + (g,g)\\ &= (f,f) + (f,g) + \overline{(f,g)} + (g,g) = \|f\|^2 + 2\,\mathrm{Re}(f,g) + \|g\|^2\,.\end{aligned}$$

Similarly,

$$\|f-g\|^2 = \|f\|^2 - 2\,\mathrm{Re}(f,g) + \|g\|^2\,.$$

Here's how to get the Cauchy-Schwarz inequality for complex inner products from this. The inequality states

$$|(f,g)| \leq \|f\|\,\|g\|\,.$$

On the left hand side we have the magnitude of the (possibly) complex number (f,g). As a slight twist on what we did in the real case, let $\alpha = te^{i\theta}$ be a complex number (t real) and consider

$$\begin{aligned}0 \leq \|f-\alpha g\|^2 &= \|f\|^2 - 2\,\mathrm{Re}(f,\alpha g) + \|\alpha g\|^2\\ &= \|f\|^2 - 2\,\mathrm{Re}\left(\overline{\alpha}(f,g)\right) + \|\alpha g\|^2\\ &= \|f\|^2 - 2t\,\mathrm{Re}\left(e^{-i\theta}(f,g)\right) + t^2\|g\|^2\,.\end{aligned}$$

Now we can choose θ here, and we do so to make

$$\mathrm{Re}\left(e^{-i\theta}(f,g)\right) = |(f,g)|\,.$$

Multiplying (f,g) by $e^{-i\theta}$ rotates the complex number (f,g) clockwise by θ, so choose θ to rotate (f,g) to be real and positive. From here the argument is the same as it was in the real case.

It's worth writing out the Cauchy-Schwarz inequality in terms of integrals:

$$\left|\int_0^1 f(t)\overline{g(t)}\,dt\right| \leq \left(\int_0^1 |f(t)|^2\,dt\right)^{1/2}\left(\int_0^1 |g(t)|^2\,dt\right)^{1/2}.$$

1.12 Appendix: Best L^2 Approximation by Finite Fourier Series

Here's a precise statement, and a proof, that a finite Fourier series of degree N gives the best (trigonometric) approximation of that order in $L^2([0,1])$ to a function.

Theorem If $f(t)$ is in $L^2([0,1])$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$ are any complex numbers, then

$$\left\| f - \sum_{n=-N}^{N} (f, e_n) e_n \right\| \leq \left\| f - \sum_{n=-N}^{N} \alpha_n e_n \right\| .$$

Furthermore, equality holds only when $\alpha_n = (f, e_n)$ for every n.

It's the last statement, on the case of equality, that leads to the Fourier coefficients in a different way than solving for them directly as we did originally. Another way of stating the result is that the *orthogonal projection* of f onto the subspace of $L^2([0,1])$ spanned by the e_n, $n = -N, \ldots, N$ is

$$\sum_{n=-N}^{N} \hat{f}(n) e^{2\pi i n t} .$$

Here comes the proof. Hold on. Write

$$\begin{aligned} \left\| f - \sum_{n=-N}^{N} \alpha_n e_n \right\|^2 &= \left\| f - \sum_{n=-N}^{N} (f, e_n) e_n + \sum_{n=-N}^{N} (f, e_n) e_n - \sum_{n=-N}^{N} \alpha_n e_n \right\|^2 \\ &= \left\| \left(f - \sum_{n=-N}^{N} (f, e_n) e_n \right) + \sum_{n=-N}^{N} ((f, e_n) - \alpha_n) e_n \right\|^2 \end{aligned}$$

We squared all the norms because we want to use the properties of inner products to expand the last line. Using the identity we derived earlier, the last line equals

$$\begin{aligned} &\left\| \left(f - \sum_{n=-N}^{N} (f, e_n) e_n \right) + \sum_{n=-N}^{N} ((f, e_n) - \alpha_n) e_n \right\|^2 = \\ &\qquad \left\| f - \sum_{n=-N}^{N} (f, e_n) e_n \right\|^2 + \\ &\qquad\qquad 2 \operatorname{Re} \left(f - \sum_{n=-N}^{N} (f, e_n) e_n, \sum_{m=-N}^{N} ((f, e_m) - \alpha_m) e_m \right) + \left\| \sum_{n=-N}^{N} ((f, e_n) - \alpha_n) e_n \right\|^2 . \end{aligned}$$

This looks complicated, but the middle term is just a sum of multiples of terms of the form

$$\left(f - \sum_{n=-N}^{N} (f, e_n) e_n, e_m \right) = (f, e_m) - \sum_{n=-N}^{N} (f, e_n)(e_n, e_m) = (f, e_m) - (f, e_m) = 0 ,$$

so the whole thing drops out! The final term is

$$\left\| \sum_{n=-N}^{N} \big((f, e_n) - \alpha_n \big) e_n \right\|^2 = \sum_{n=-N}^{N} \big| (f, e_n) - \alpha_n \big|^2 .$$

We are left with

$$\left\| f - \sum_{n=-N}^{N} \alpha_n e_n \right\|^2 = \left\| f - \sum_{n=-N}^{N} (f, e_n) e_n \right\|^2 + \sum_{n=-N}^{N} |(f, e_n) - \alpha_n|^2 .$$

This completely proves the theorem, for the right hand side is the sum of two positive terms and hence

$$\left\| f - \sum_{n=-N}^{N} \alpha_n e_n \right\|^2 \geq \left\| f - \sum_{n=-N}^{N} (f, e_n) e_n \right\|^2$$

with equality holding if and only if

$$\sum_{n=-N}^{N} |(f, e_n) - \alpha_n|^2 = 0 .$$

The latter holds if and only if $\alpha_n = (f, e_n)$ for all n.

The preceding argument may have seemed labor intensive, but it was all *algebra* based on the properties of the inner product. Imagine trying to write all of it out in terms of integrals.

1.13 Fourier Series in Action

We've had a barrage of general information and structure, and it's time to pass to the particular and put some of these ideas to work. In these notes I want to present a few model cases of how Fourier series can be applied. The range of applications is vast, so my principle of selection has been to choose examples that are both interesting in themselves and have connections with different areas.

The first applications are to heat flow; these are classical, celebrated problems and should be in your storehouse of general knowledge. Another reason for including them is the *form* that one of the solutions takes as a convolution integral — you'll see why this is interesting. We'll also look briefly at how the differential equation governing heat flow comes up in other areas. The key word is *diffusion*.

The second application is not classical at all; in fact, it does not fit into the L^2-theory as we laid it out last time. It has to do, on the one hand, with sound synthesis, and on the other, as we'll see later, with sampling theory. Later in the course, when we do higher dimensional Fourier analysis, we'll have an application of higher dimensional Fourier series to random walks on a lattice. It's cool, and, with a little probability thrown in the analysis of the problem is not beyond what we know to this point, but enough is enough.

1.13.1 Hot enough for ya?

The study of how temperature varies over a region was the first use by Fourier in the 1820's of the method of expanding a function into a series of trigonometric functions. The physical phenomenon is described, at least approximately, by a partial differential equation, and Fourier series can be used to write down solutions.

We'll give a brief, standard derivation of the differential equation in one spatial dimension, so the configuration to think of is a one-dimensional rod. The argument involves a number of common but difficult, practically undefined terms, first among them the term "heat", followed closely by the term "temperature".

As it is usually stated, heat is a transfer of "energy" (another undefined term, thank you) due to temperature difference; the transfer process is called "heat". What gets transferred is energy. Because of this,

heat is usually identified as a form of energy and has units of energy. We talk of heat as a 'transfer of energy', and hence of 'heat flow', because, like so many other physical quantities heat is only interesting if it's associated with a change. Temperature, more properly called "thermodynamic temperature" (formerly "absolute temperature"), is a derived quantity. The temperature of a substance is proportional to the kinetic energy of the atoms in the substance.[19] A substance at temperature 0 (absolute zero) cannot transfer energy — it's not "hot". The principle at work, essentially stated by Newton, is:

> A temperature difference between two substances in contact with each other causes a *transfer* of energy from the substance of higher temperature to the substance of lower temperature, and that's heat, or heat flow. No temperature difference, no heat.

Back to the rod. The temperature is a function of both the spatial variable x giving the position along the rod and of the time t. We let $u(x,t)$ denote the temperature, and the problem is to find it. The description of heat, just above, with a little amplification, is enough to propose a partial differential equation that $u(x,t)$ should satisfy.[20] To derive it, we introduce $q(x,t)$, the amount of heat that "flows" per second at x and t (so $q(x,t)$ is the rate at which energy is transfered at x and t). Newton's law of cooling says that this is proportional to the gradient of the temperature:

$$q(x,t) = -ku_x(x,t)\,, \quad k > 0\,.$$

The reason for the minus sign is that if $u_x(x,t) > 0$, i.e., if the temperature is increasing at x, then the rate at which heat flows at x is negative — from hotter to colder, hence back from x. The constant k can be identified with the reciprocal of "thermal resistance" of the substance. For a given temperature gradient, the higher the resistance the smaller the heat flow per second, and similarly the smaller the resistance the greater the heat flow per second.

As the heat flows from hotter to colder, the temperature rises in the colder part of the substance. The rate at which the temperature rises at x, given by $u_t(x,t)$, is proportional to the rate at which heat "accumulates" per unit length. Now $q(x,t)$ is already a rate — the heat flow per second — so the rate at which heat accumulates per unit length is the rate in minus the rate out per length, which is (if the heat is flowing from left to right)

$$\frac{q(x,t) - q(x+\Delta x,t)}{\Delta x}\,.$$

Thus in the limit

$$u_t(x,t) = -k'q_x(x,t)\,, \quad k' > 0\,.$$

The constant k' can be identified with the reciprocal of the "thermal capacity" per unit length. Thermal resistance and thermal capacity are not the standard terms, but they can be related to standard terms, e.g., specific heat. They are used here because of the similarity of heat flow to electrical phenomena — see the discussion of the mathematical analysis of telegraph cables, below.

Next, differentiate the first equation with respect to x to get

$$q_x(x,t) = -ku_{xx}(x,t)\,,$$

and substitute this into the second equation to obtain an equation involving $u(x,t)$ alone:

$$u_t(x,t) = kk'u_{xx}(x,t)\,.$$

This is the *heat equation*.

To summarize, in whatever particular context it's applied, the setup for a problem based on the heat equation involves:

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[19] With this (partial) definition the unit of temperature is the Kelvin.

[20] This follows Bracewell's presentation.

- A region in space.
- An initial distribution of temperature on that region.

It's natural to think of fixing one of the variables and letting the other change. Then the solution $u(x,t)$ tells you

- For each fixed time t how the temperature is distributed on the region.
- At each fixed point x how the temperature is changing over time.

We want to look at two examples of using Fourier series to solve such a problem: heat flow on a circle and, more dramatically, the temperature of the earth. These are nice examples because they show different aspects of how the methods can be applied and, as mentioned above, they exhibit forms of solutions, especially for the circle problem, of a type we'll see frequently.

Why a circle, why the earth — and why Fourier methods? Because in each case the function $u(x,t)$ will be *periodic* in one of the variables. In one case we work with periodicity in space and in the other periodicity in time.

Heating a circle Suppose a circle is heated up, not necessarily uniformly. This provides an initial distribution of temperature. Heat then flows around the circle and the temperature changes over time. At any fixed time the temperature must be a periodic function of the position on the circle, for if we specify points on the circle by an angle θ then the temperature, as a function of θ, is the same at θ and at $\theta + 2\pi$, since these are the same points.

We can imagine a circle as an interval with the endpoints identified, say the interval $0 \le x \le 1$, and we let $u(x,t)$ be the temperature as a function of position and time. Our analysis will be simplified if we choose units so the heat equation takes the form

$$u_t = \tfrac{1}{2} u_{xx}\,,$$

that is, so the constant depending on physical attributes of the wire is $1/2$. The function $u(x,t)$ is periodic in the spatial variable x with period 1, i.e., $u(x+1,t) = u(x,t)$, and we can try expanding it as a Fourier series with coefficients that depend on time:

$$u(x,t) = \sum_{n=-\infty}^{\infty} c_n(t) e^{2\pi i n x} \quad \text{where} \quad c_n(t) = \int_0^1 e^{-2\pi i n x} u(x,t)\, dx\,.$$

This representation of $c_n(t)$ as an integral together with the heat equation for $u(x,t)$ will allow us to find $c_n(t)$ explicitly. Differentiate $c_n(t)$ with respect to t by differentiating under the integral sign:

$$c_n'(t) = \int_0^1 u_t(x,t) e^{-2\pi i n x}\, dx;$$

Now using $u_t = \frac{1}{2} u_{xx}$ we can write this as

$$c_n'(t) = \int_0^1 \tfrac{1}{2} u_{xx}(x,t) e^{-2\pi i n x}\, dx$$

and integrate by parts (twice) to get the derivatives off of u (the function we don't know) and put them onto $e^{-2\pi i n x}$ (which we can certainly differentiate). Using the facts that $e^{-2\pi i n} = 1$ and $u(0,t) = u(1,t)$

(both of which come in when we plug in the limits of integration when integrating by parts) we get

$$\begin{aligned} c_n'(t) &= \int_0^1 \tfrac{1}{2} u(x,t) \frac{d^2}{dx^2} e^{-2\pi i n x}\, dx \\ &= \int_0^1 \tfrac{1}{2} u(x,t)(-4\pi^2 n^2) e^{-2\pi i n x}\, dx \\ &= -2\pi^2 n^2 \int_0^1 u(x,t) e^{-2\pi i n x}\, dx = -2\pi^2 n^2\, c_n(t). \end{aligned}$$

We have found that $c_n(t)$ satisfies a simple *ordinary* differential equation

$$c_n'(t) = -2\pi^2 n^2\, c_n(t)\,,$$

whose solution is

$$c_n(t) = c_n(0) e^{-2\pi^2 n^2 t}\,.$$

The solution involves the initial value $c_n(0)$ and, in fact, this initial value should be, and will be, incorporated into the formulation of the problem in terms of the initial distribution of heat.

At time $t = 0$ we assume that the temperature $u(x, 0)$ is specified by some (periodic!) function $f(x)$:

$$u(x,0) = f(x)\,, \quad f(x+1) = f(x) \quad \text{for all } x.$$

Then using the integral representation for $c_n(t)$,

$$\begin{aligned} c_n(0) &= \int_0^1 u(x,0) e^{-2\pi i n x}\, dx \\ &= \int_0^1 f(x) e^{-2\pi i n x}\, dx = \hat{f}(n)\,, \end{aligned}$$

the n-th Fourier coefficient of f! Thus we can write

$$c_n(t) = \hat{f}(n) e^{-2\pi^2 n^2 t}\,,$$

and the general solution of the heat equation is

$$u(x,t) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{-2\pi^2 n^2 t} e^{2\pi i n x}\,.$$

This is a neat way of writing the solution and we could leave it at that, but for reasons we're about to see it's useful to bring back the integral definition of $\hat{f}(n)$ and write the expression differently.

Write the formula for $\hat{f}(n)$ as

$$\hat{f}(n) = \int_0^1 f(y) e^{-2\pi i n y}\, dy\,.$$

(Don't use x as the variable of integration since it's already in use in the formula for $u(x, t)$.) Then

$$\begin{aligned} u(x,t) &= \sum_{n=-\infty}^{\infty} e^{-2\pi^2 n^2 t} e^{2\pi i n x} \int_0^1 f(y) e^{-2\pi i n y}\, dy \\ &= \int_0^1 \sum_{n=-\infty}^{\infty} e^{-2\pi^2 n^2 t} e^{2\pi i n (x-y)} f(y)\, dy\,, \end{aligned}$$

or, with

$$g(x-y,t) = \sum_{n=-\infty}^{\infty} e^{-2\pi^2 n^2 t} e^{2\pi i n(x-y)},$$

we have

$$u(x,t) = \int_0^1 g(x-y,t) f(y)\, dy.$$

The function

$$g(x,t) = \sum_{n=-\infty}^{\infty} e^{-2\pi^2 n^2 t} e^{2\pi i n x}$$

is called *Green's function*, or the *fundamental solution* for the heat equation for a circle. Note that g is a periodic function of period 1 in the spatial variable. The expression for the solution $u(x,t)$ is a *convolution integral*, a term you have probably heard from earlier classes, but new here. In words, $u(x,t)$ is given by the convolution of the initial temperature $f(x)$ with Green's function $g(x,t)$. This is a very important fact.

In general, whether or not there is extra time dependence as in the present case, the integral

$$\int_0^1 g(x-y) f(y)\, dy$$

is called the *convolution* of f and g. Observe that the integral makes sense only if g is periodic. That is, for a given x between 0 and 1 and for y varying from 0 to 1 (as the variable of integration) $x-y$ will assume values outside the interval $[0,1]$. If g were not periodic it wouldn't make sense to consider $g(x-y)$, but the periodicity is just what allows us to do that.

To think more in EE terms, if you know the terminology coming from linear systems, the Green's function $g(x,t)$ is the "impulse response" associated with the linear system "heat flow on a circle", meaning

- Inputs go in: the initial heat distribution $f(x)$.
- Outputs come out: the temperature $u(x,t)$.
- Outputs are given by the convolution of g with the input: $u(x,t) = \displaystyle\int_0^1 g(x-y,t) f(y)\, dy$.

Convolutions occur absolutely everywhere in Fourier analysis and we'll be spending a lot of time with them this quarter. In fact, an important result states that convolutions *must* occur in relating outputs to inputs for linear time invariant systems. We'll see this later.

In our example, as a formula for the solution, the convolution may be interpreted as saying that for each time t the temperature $u(x,t)$ at a point x is a kind of smoothed average of the initial temperature distribution $f(x)$. In other settings a convolution integral may have different interpretations.

Heating the earth, storing your wine The wind blows, the rain falls, and the temperature at any particular place on earth changes over the course of a year. Let's agree that the way the temperature varies is pretty much the same year after year, so that the temperature at any particular place on earth is roughly a periodic function of time, where the period is 1 year. What about the temperature x-meters *under* that particular place? How does the temperature depend on x and t?[21]

[21] This example is taken from *Fourier Series and Integrals* by H. Dym & H. McKean, who credit Sommerfeld.

Fix a place on earth and let $u(x,t)$ denote the temperature x meters underground at time t. We assume again that u satisfies the heat equation, $u_t = \frac{1}{2}u_{xx}$. This time we try a solution of the form

$$u(x,t) = \sum_{n=-\infty}^{\infty} c_n(x)e^{2\pi int}\,,$$

reflecting the periodicity in time.

Again we have an integral representation of $c_n(x)$ as a Fourier coefficient,

$$c_n(x) = \int_0^1 u(x,t)e^{-2\pi int}\,dt\,,$$

and again we want to plug into the heat equation and find a differential equation that the coefficients satisfy. The heat equation involves a second (partial) derivative with respect to the spatial variable x, so we differentiate c_n twice and differentiate u under the integral sign twice with respect to x:

$$c_n''(x) = \int_0^1 u_{xx}(x,t)e^{-2\pi int}\,dt\,.$$

Using the heat equation and integrating by parts (once) gives

$$\begin{aligned} c_n''(x) &= \int_0^1 2u_t(x,t)e^{-2\pi int}\,dt \\ &= \int_0^1 4\pi inu(x,t)e^{-2\pi int}\,dt = 4\pi inc_n(x)\,. \end{aligned}$$

We can solve this second-order differential equation in x easily on noting that

$$(4\pi in)^{1/2} = \pm(2\pi|n|)^{1/2}(1\pm i)\,,$$

where we take $1+i$ when $n>0$ and $1-i$ when $n<0$. I'll leave it to you to decide that the root to take is $-(2\pi|n|)^{1/2}(1\pm i)$, thus

$$c_n(x) = A_n e^{-(2\pi|n|)^{1/2}(1\pm i)x}\,.$$

What is the initial value $A_n = c_n(0)$? Again we assume that at $x=0$ there is a periodic function of t that models the temperature (at the fixed spot on earth) over the course of the year. Call this $f(t)$. Then $u(0,t) = f(t)$, and

$$c_n(0) = \int_0^1 u(0,t)e^{-2\pi int}\,dt = \hat{f}(n)\,.$$

Our solution is then

$$u(x,t) = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{-(2\pi|n|)^{1/2}(1\pm i)x}e^{2\pi int}\,.$$

That's not a beautiful expression, but it becomes more interesting if we rearrange the exponentials to isolate the periodic parts (the ones that have an i in them) from the nonperiodic part that remains. The latter is $e^{-(2\pi|n|)^{1/2}x}$. The terms then look like

$$\hat{f}(n)\,e^{-(2\pi|n|)^{1/2}x}\,e^{2\pi int\mp(2\pi|n|)^{1/2}ix}\,.$$

What's interesting here? The dependence on the depth, x. Each term is *damped* by the exponential

$$e^{-(2\pi|n|)^{1/2}x}$$

and *phase shifted* by the amount $(2\pi|n|)^{1/2}x$.

Take a simple case. Suppose that the temperature at the surface $x = 0$ is given just by $\sin 2\pi t$ and that the mean annual temperature is 0, i.e.,

$$\int_0^1 f(t)\,dt = \hat{f}(0) = 0\,.$$

All Fourier coefficients other than the first (and minus first) are zero, and the solution reduces to

$$u(x,t) = e^{-(2\pi)^{1/2}x}\sin(2\pi t - (2\pi)^{1/2}x)\,.$$

Take the depth x so that $(2\pi)^{1/2}x = \pi$. Then the temperature is damped by $e^{-\pi} = 0.04$, quite a bit, and it is half a period (six months) out of phase with the temperature at the surface. The temperature x-meters below stays pretty constant because of the damping, and because of the phase shift it's cool in the summer and warm in the winter. There's a name for a place like that. It's called a cellar.

The first shot in the second industrial revolution Many types of diffusion processes are similar enough in principle to the flow of heat that they are modeled by the heat equation, or a variant of the heat equation, and Fourier analysis is often used to find solutions. One celebrated example of this was the paper by William Thomson (later Lord Kelvin): "On the theory of the electric telegraph" published in 1855 in the Proceedings of the Royal Society.

The high tech industry of the mid to late 19th century was submarine telegraphy. Sharp pulses were sent at one end, representing the dots and dashes of Morse code, and in transit, if the cable was very long and if pulses were sent in too rapid a succession, the pulses were observed to smear out and overlap to the degree that at the receiving end it was impossible to resolve them. The commercial success of telegraph transmissions between continents depended on undersea cables reliably handling a large volume of traffic. How should cables be designed? The stakes were high and a quantitative analysis was needed.

A qualitative explanation of signal distortion was offered by Michael Faraday, who was shown the phenomenon by Latimer Clark. Clark, an official of the Electric and International Telegraph Company, had observed the blurring of signals on the Dutch-Anglo line. Faraday surmised that a cable immersed in water became in effect an enormous capacitor, consisting as it does of two conductors — the wire and the water — separated by insulating material (gutta-percha in those days). When a signal was sent, part of the energy went into charging the capacitor, which took time, and when the signal was finished the capacitor discharged and that also took time. The delay associated with both charging and discharging distorted the signal and caused signals sent too rapidly to overlap.

Thomson took up the problem in two letters to G. Stokes (of Stokes' theorem fame), which became the published paper. We won't follow Thomson's analysis at this point, because, with the passage of time, it is more easily understood via Fourier transforms rather than Fourier series. However, here are some highlights. Think of the whole cable as a (flexible) cylinder with a wire of radius a along the axis and surrounded by a layer of insulation of radius b (thus of thickness $b-a$). To model the electrical properties of the cable, Thomson introduced the "electrostatic capacity per unit length" depending on a and b and ϵ, the permittivity of the insulator. His formula was

$$C = \frac{2\pi\epsilon}{\ln(b/a)}\,.$$

(You may have done just this calculation in an EE or physics class.) He also introduced the "resistance per unit length", denoting it by K. Imagining the cable as a series of infinitesimal pieces, and using Kirchhoff's circuit law and Ohm's law on each piece, he argued that the voltage $v(x,t)$ at a distance x from the end

of the cable and at a time t must satisfy the partial differential equation

$$v_t = \frac{1}{KC} v_{xx} \, .$$

Thomson states: "This equation agrees with the well-known equation of the linear motion of heat in a solid conductor, and various forms of solution which Fourier has given are perfectly adapted for answering practical questions regarding the use of the telegraph wire."

After the fact, the basis of the analogy is that charge diffusing through a cable may be described in the same way as heat through a rod, with a gradient in electric potential replacing gradient of temperature, etc. (Keep in mind, however, that the electron was not discovered till 1897.) Here we see K and C playing the role of thermal resistance and thermal capacity in the derivation of the heat equation.

The result of Thomson's analysis that had the greatest practical consequence was his demonstration that "...the time at which the maximum electrodynamic effect of connecting the battery for an instant ..." [sending a sharp pulse, that is] occurs for

$$t_{\max} = \tfrac{1}{6} KCx^2 \, .$$

The number $t_{\max}$ is what's needed to understand the delay in receiving the signal. It's the fact that the distance from the end of the cable, x, comes in *squared* that's so important. This means, for example, that the delay in a signal sent along a 1000 mile cable will be 100 times as large as the delay along a 100 mile cable, and not 10 times as large, as was thought. This was Thomson's "Law of squares."

Thomson's work has been called "The first shot in the second industrial revolution."[22] This was when electrical engineering became decidedly mathematical. His conclusions did not go unchallenged, however. Consider this quote of Edward Whitehouse, chief electrician for the Atlantic Telegraph Company, speaking in 1856

> I believe nature knows no such application of this law [the law of squares] and I can only regard it as a fiction of the schools; a forced and violent application of a principle in Physics, good and true under other circumstances, but misapplied here.

Thomson's analysis did not prevail and the first transatlantic cable was built without regard to his specifications. Thomson said they had to design the cable to make KC small. They thought they could just crank up the power. The continents were joined August 5, 1858, after four previous failed attempts. The first successful sent message was August 16. The cable failed three weeks later. Too high a voltage. They fried it.

Rather later, in 1876, Oliver Heaviside greatly extended Thomson's work by including the effects of induction. He derived a more general differential equation for the voltage $v(x,t)$ in the form

$$v_{xx} = KCv_t + SCv_{tt} \, ,$$

where S denotes the inductance per unit length and, as before, K and C denote the resistance and capacitance per unit length. The significance of this equation, though not realized till later still, is that it allows for solutions that represent propagating *waves*. Indeed, from a PDE point of view the equation looks like a mix of the heat equation and the wave equation. (We'll study the wave equation later.) It is Heaviside's equation that is now usually referred to as the "telegraph equation".

[22] See *Getting the Message: A History of Communications* by L. Solymar.

The last shot in the second World War Speaking of high stakes diffusion processes, in the early stages of the theoretical analysis of atomic explosives it was necessary to study the diffusion of neutrons produced by fission as they worked their way through a mass of uranium. The question: How much mass is needed so that enough uranium nuclei will fission in a short enough time to produce an explosion.[23]

An analysis of this problem was carried out by Robert Serber and some students at Berkeley in the summer of 1942, preceding the opening of the facilities at Los Alamos (where the bulk of the work was done and the bomb was built). They found that the so-called "critical mass" needed for an explosive chain reaction was about 60 kg of U^{235}, arranged in a sphere of radius about 9 cm (together with a tamper surrounding the Uranium). A less careful model of how the diffusion works gives a critical mass of 200 kg. As the story goes, in the development of the German atomic bomb project (which predated the American efforts), Werner Heisenberg worked with a less accurate model and obtained too high a number for the critical mass. This set their program back.

For a fascinating and accessible account of this and more, see Robert Serber's *The Los Alamos Primer*. These are the notes of the first lectures given by Serber at Los Alamos on the state of knowledge on atomic bombs, annotated by him for this edition. For a dramatized account of Heisenberg's role in the German atomic bomb project — including the misunderstanding of diffusion — try Michael Frayn's play *Copenhagen*.

1.13.2 A nonclassical example: What's the buzz?

We model a musical tone as a periodic wave. A pure tone is a single sinusoid, while more complicated tones are sums of sinusoids. The frequencies of the higher harmonics are integer multiples of the fundamental harmonic and the harmonics will typically have different energies. As a model of the most "complete" and "uniform" tone we might take a sum of *all* harmonics, each sounded with the same energy, say 1. If we further assume that the period is 1 (i.e., that the fundamental harmonic has frequency 1) then we're looking at the signal

$$f(t) = \sum_{n=-\infty}^{\infty} e^{2\pi i n t}\,.$$

What does this sound like? Not very pleasant, depending on your tastes. It's a buzz; all tones are present and the sum of all of them together is "atonal". I'd like to hear this sometime, so if any of you can program it I'd appreciate it. Of course if you program it then: (1) you'll have to use a finite sum; (2) you'll have to use a discrete version. In other words, you'll have to come up with the "discrete-time buzz", where what we've written down here is sometimes called the "continuous-time buzz". We'll talk about the discrete time buzz later, but you're welcome to figure it out now.

The expression for $f(t)$ is not a classical Fourier series in any sense. It does not represent a signal with finite energy and the series does not converge in L^2 or in any other easily defined sense. Nevertheless, the buzz is an important signal for several reasons. What does it look like in the time domain?

In the first problem set you are asked to find a closed form expression for the partial sum

$$D_N(t) = \sum_{n=-N}^{N} e^{2\pi i n t}\,.$$

Rather than giving it away, let's revert to the real form. Isolating the $n = 0$ term and combining positive

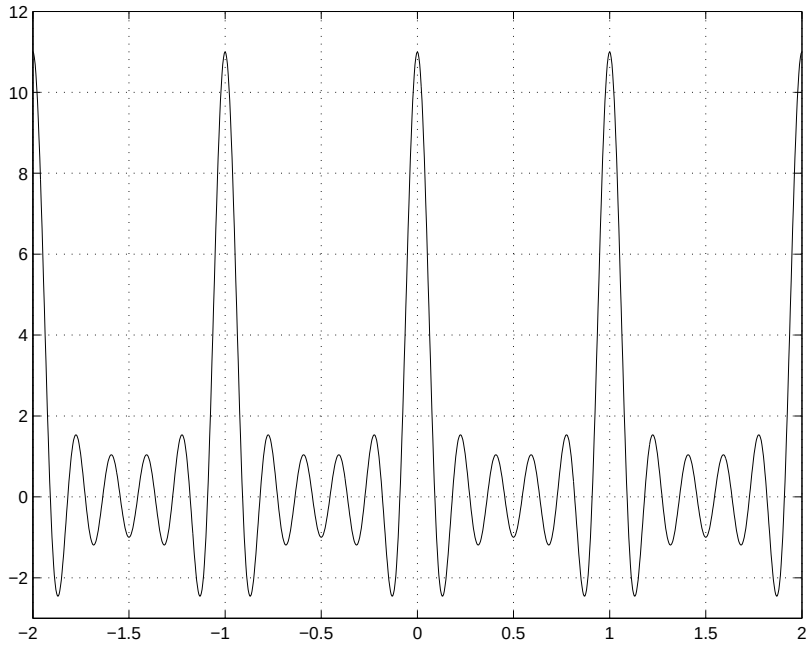
[23] The explosive power of an atomic bomb comes from the electrostatic repulsion between the protons in the nucleus when enough energy is added for it to fission. It doesn't have anything to do with $E = mc^2$.

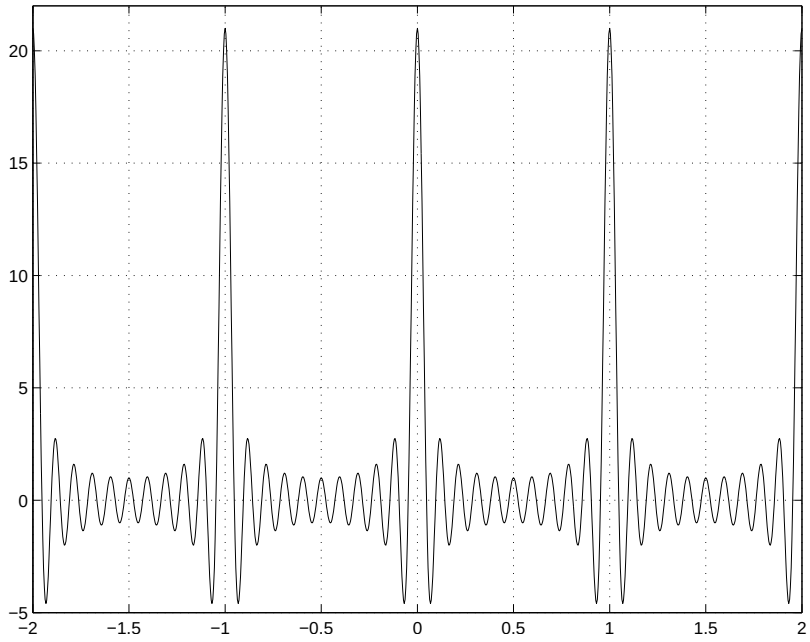
and negative terms we get

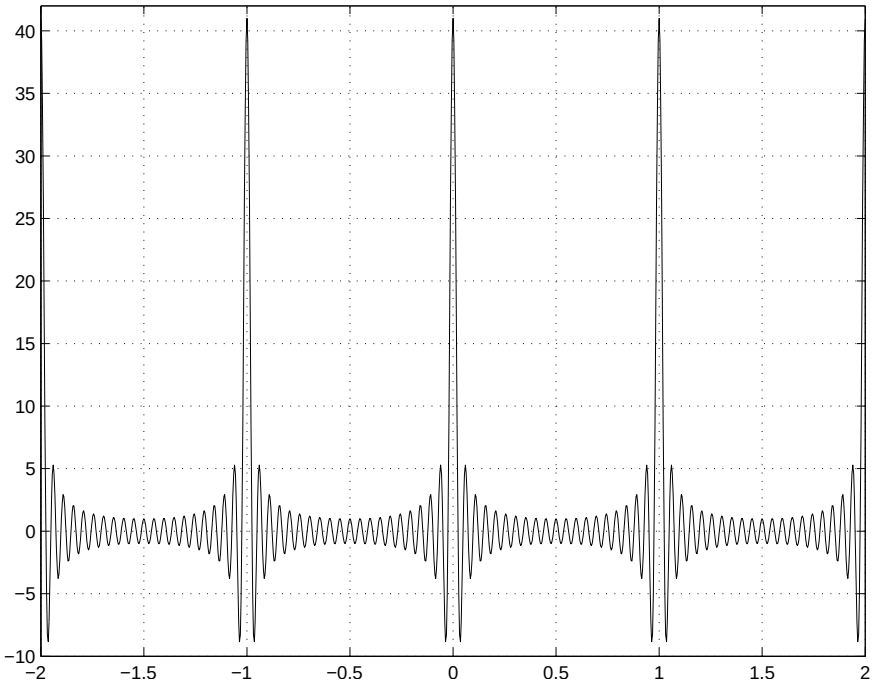
$$\sum_{n=-N}^{N} e^{2\pi int} = 1 + \sum_{n=1}^{N} (e^{2\pi int} + e^{-2\pi int}) = 1 + 2\sum_{n=1}^{N} \cos 2\pi nt\,.$$

One thing to note is that the value at the origin is $1+2N$; by periodicity this is the value at all the integers, and with a little calculus you can check that $1 + 2N$ is the maximum. It's getting bigger and bigger with N. (What's the minimum, by the way?)

Here are some plots (not terribly good ones) for $N = 5$, 10, and 20:







We see that the signal becomes more and more concentrated at the integers, with higher and higher peaks. In fact, as we'll show later, the sequence of signals $D_N(t)$ tends to a sum of δ's at the integers as $N \to \infty$:

$$D_N(t) \to \sum_{n=-\infty}^{\infty} \delta(t-n)\,.$$

In what sense the convergence takes place will also have to wait till later. This all goes to show you that L^2 is not the last word in the development and application of Fourier series (even if I made it seem that way).

The sum of regularly spaced δ's is sometimes called an *impulse train*, and we'll have other descriptive names for it. It is a fundamental object in sampling, the first step in turning an analog signal into a digital signal. The finite sum, $D_N(t)$, is called the *Dirichlet kernel* by mathematicians and it too has a number of applications, one of which we'll see in the notes on convergence of Fourier series.

In digital signal processing, particularly computer music, it's the discrete form of the impulse train — the discrete time buzz — that's used. Rather than create a sound by adding (sampled) sinusoids one works in the frequency domain and synthesizes the sound from its spectrum. Start with the discrete impulse train, which has all frequencies in equal measure. This is easy to generate. Then shape the spectrum by increasing or decreasing the energies of the various harmonics, perhaps decreasing some to zero. The sound is synthesized from this shaped spectrum, and other operations are also possible. See, for example, *A Digital Signal Processing Primer* by Ken Steiglitz.

One final look back at heat. Green's function for the heat equation had the form

$$g(x,t) = \sum_{n=-\infty}^{\infty} e^{-2\pi^2 n^2 t} e^{2\pi i n x}\,.$$

Look what happens as $t \to 0$. This tends to

$$\sum_{n=-\infty}^{\infty} e^{2\pi i n x},$$

the continuous buzz. Just thought you'd find that provocative.

1.14 Notes on Convergence of Fourier Series

My first comment on convergence is — don't go there. Recall that we get tidy mathematical results on convergence of Fourier series if we consider L^2-convergence, or "convergence in mean square". Unpacking the definitions, that's convergence of the integral of the square of the difference between a function and its finite Fourier series approximation:

$$\lim_{N\to\infty} \int_0^1 \left| f(t) - \sum_{n=-N}^{N} \hat{f}(n) e^{2\pi i n t} \right|^2 dt = 0\,.$$

While this is quite satisfactory in many ways, you might want to know, for computing *values* of a function, that if you plug a value of t into some finite approximation

$$\sum_{n=-N}^{N} \hat{f}(n) e^{2\pi i n t}$$

you'll be close to the value of the function $f(t)$. And maybe you'd like to know how big you have to take N to get a certain desired accuracy.

All reasonable wishes, but starting to ask about convergence of Fourier series, beyond the L^2-convergence, is starting down a road leading to endless complications, details, and, in the end, probably madness. Actually — and calmly — for the kinds of functions that come up in applications the answers are helpful and not really so difficult to establish. It's when one inquires into convergence of Fourier series for the most general functions that the trouble really starts. With that firm warning understood, there are a few basic things you ought to know about, if only to know that this can be dangerous stuff.

In the first part of these notes my intention is to summarize the main facts together with some examples and simple arguments. I'll give careful statements, but we won't enjoy the complete proofs that support them, though in the appendices I'll fill in more of the picture. There we'll sketch the argument for the result at the heart of the L^2-theory of Fourier series, that the complex exponentials form a basis for $L^2([0,1])$. For more and more and much more see Dym and McKean's *Fourier Series and Integrals*.

1.14.1 How big are the Fourier coefficients?

Suppose that $f(t)$ is square integrable, and let

$$f(t) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n t}$$

be its Fourier series. Rayleigh's identity says

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 = \int_0^1 |f(t)|^2\, dt < \infty\,.$$

In particular the series

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2$$

converges, and it follows that

$$|\hat{f}(n)|^2 \to 0 \quad \text{as} \quad n \to \pm\infty\,.$$

This is a general result on convergent series from good old calculus days — if the series converges the general term must tend to zero.[24] Knowing that the coefficients tend to zero, can we say how fast?

Here's a simple minded approach that gives some sense of the answer, and shows how the answer depends on discontinuities in the function or its derivatives. All of this discussion is based on integration by parts with definite integrals.[25] Suppose, as always, that $f(t)$ is periodic of period 1. By the periodicity condition we have $f(0) = f(1)$. Let's assume for this discussion that the function doesn't jump at the endpoints 0 and 1 (like the saw tooth function, below) and that any "problem points" are inside the interval. (This really isn't a restriction. I just want to deal with a single discontinuity for the argument to follow.) That is, we're imagining that there may be trouble at a point t_0 with $0 < t_0 < 1$; maybe $f(t)$ jumps there, or maybe $f(t)$ is continuous at t_0 but there's a corner, so $f'(t)$ jumps at t_0, and so on.

The n-th Fourier coefficient is given by

$$\hat{f}(n) = \int_0^1 e^{-2\pi int} f(t)\, dt\,.$$

To analyze the situation near t_0 write this as the sum of two integrals:

$$\hat{f}(n) = \int_0^{t_0} e^{-2\pi int} f(t)\, dt + \int_{t_0}^1 e^{-2\pi int} f(t)\, dt\,.$$

Apply integration by parts to each of these integrals. In doing so, we're going to suppose that at least away from t_0 the function has as many derivatives as we want. Then, on a first pass,

$$\int_0^{t_0} e^{-2\pi int} f(t)\, dt = \left[\frac{e^{-2\pi int} f(t)}{-2\pi in}\right]_0^{t_0} - \int_0^{t_0} \frac{e^{-2\pi int} f'(t)}{-2\pi in}\, dt$$

$$\int_{t_0}^1 e^{-2\pi int} f(t)\, dt = \left[\frac{e^{-2\pi int} f(t)}{-2\pi in}\right]_{t_0}^1 - \int_{t_0}^1 \frac{e^{-2\pi int} f'(t)}{-2\pi in}\, dt$$

Add these together. Using $f(0) = f(1)$, this results in

$$\hat{f}(n) = \left[\frac{e^{-2\pi int} f(t)}{-2\pi in}\right]_{t_0^-}^{t_0^+} - \int_0^1 \frac{e^{-2\pi int} f'(t)}{-2\pi in}\, dt\,,$$

where the notation t_0^- and t_0^+ means to indicate we're looking at the values of $f(t)$ as we take left hand and right hand limits at t_0. If $f(t)$ is continuous at t_0 then the terms in brackets cancel and we're left with just the integral as an expression for $\hat{f}(n)$. But if $f(t)$ is not continuous at t_0 — if it jumps, for example — then we don't get cancellation, and we expect that the Fourier coefficient will be of order $1/n$ in magnitude.[26]

---

[24] In particular, $\sum_{n=-\infty}^{\infty} e^{2\pi int}$, the buzz example, cannot converge for any value of t since $|e^{2\pi int}| = 1$.

[25] On the off chance that you're rusty on this, here's what the formula looks like, as it's usually written:

$$\int_a^b u\, dv = [uv]_a^b - \int_a^b v\, du\,.$$

To apply integration by parts in a given problem is to decide which part of the integrand is u and which part is dv.

[26] If we had more jump discontinuities we'd split the integral up going over several subintervals and we'd have several terms of order $1/n$. The combined result would still be of order $1/n$. This would also be true if the function jumped at the endpoints 0 and 1.

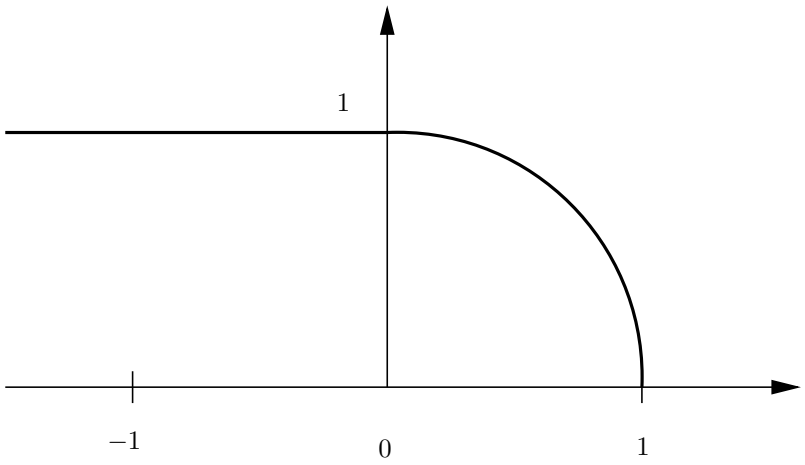
Now suppose that $f(t)$ *is* continuous at t_0, and integrate by parts a second time. In the same manner as above, this gives

$$\hat{f}(n) = \left[\frac{e^{-2\pi int} f'(t)}{(-2\pi in)^2}\right]_{t_0^-}^{t_0^+} - \int_0^1 \frac{e^{-2\pi int} f'(t)}{(-2\pi in)^2}\, dt\,,$$

If $f'(t)$ (the *derivative*) is continuous at t_0 then the bracketed part disappears. If $f'(t)$ is not continuous at t_0, for example if there is a corner at t_0, then the terms do not cancel and we expect the Fourier coefficient to be of size $1/n^2$.

We can continue in this way. The rough rule of thumb may be stated as:

- If $f(t)$ is not continuous then the Fourier coefficients should have some terms like $1/n$.
- If $f(t)$ is differentiable except for corners ($f(t)$ is continuous but $f'(t)$ is not) then the Fourier coefficients should have some terms like $1/n^2$.
- If $f''(t)$ exists but is not continuous then the Fourier coefficients should have some terms like $1/n^3$.
    - A discontinuity in $f''(t)$ is harder to visualize; typically it's a discontinuity in the curvature. For example, imagine a curve consisting of an arc of a circle and a line segment tangent to the circle at their endpoints. Something like



The curve and its first derivative are continuous at the point of tangency, but the second derivative has a jump. If you rode along this path at constant speed you'd feel a jerk — a discontinuity in the acceleration — when you passed through the point of tangency.

Obviously this result extends to discontinuities in higher order derivatives. It also jibes with some examples we had earlier. The square wave

$$f(t) = \begin{cases} +1 & 0 \le t < \frac{1}{2} \\ -1 & \frac{1}{2} \le t < 1 \end{cases}$$

has jump discontinuities, and its Fourier series is

$$\sum_{n \text{ odd}} \frac{2}{\pi i n} e^{2\pi int} = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1}{2k+1} \sin 2\pi(2k+1)t\,.$$

The triangle wave

$$g(t) = \begin{cases} \frac{1}{2} + t & -\frac{1}{2} \le t \le 0 \\ \frac{1}{2} - t & 0 \le t \le +\frac{1}{2} \end{cases}$$

is continuous but the derivative is discontinuous. (In fact the derivative is the square wave.) Its Fourier series is

$$\tfrac{1}{4} + \sum_{k=0}^{\infty} \frac{2}{\pi^2(2k+1)^2} \cos(2\pi(2k+1)t)\,.$$

1.14.2 Rates of convergence and smoothness

The size of the Fourier coefficients tells you something about the rate of convergence of the Fourier series. There is a precise result on the rate of convergence, which we'll state but not prove:

> **Theorem** Suppose that $f(t)$ is p-times continuously differentiable, where p is at least 1. Then the partial sums
>
> $$S_N(t) = \sum_{n=-N}^{N} \hat{f}(n) e^{2\pi i n t}$$
>
> converge to $f(t)$ pointwise and uniformly on $[0,1]$ as $N \to \infty$. Furthermore
>
> $$\max |f(t) - S_N(t)| \leq \text{constant} \frac{1}{N^{p-\frac{1}{2}}}$$
>
> for $0 \leq t \leq 1$.

We won't prove it, but I do want to explain a few things. First, at a meta level, this result has to do with how local properties of the function are reflected in global properties of its Fourier series.[27] In the present setting, "local properties" of a function refers to how smooth it is, i.e., how many times it's continuously differentiable. About the only kind of "global question" one can ask about series is how fast they converge, and that's what is estimated here. The essential point is that the error in the approximation (and indirectly the rate at which the coefficients decrease) is governed by the smoothness (the degree of differentiability) of the signal. The smoother the function — a "local" statement — the better the approximation, and this is not just in the mean, L^2 sense, but uniformly over the interval — a "global" statement.

Let me explain the two terms "pointwise" and "uniformly"; the first is what you think you'd like, but the second is better. "Pointwise" convergence means that if you plug in a particular value of t the series converges *at that point* to the value of the signal at that point. "Uniformly" means that the rate at which the series converges is the same for all points in $[0,1]$. There are several ways of rephrasing this. Analytically, the way of capturing the property of uniformity is by making a statement, as we did above, on the maximum amount the function $f(t)$ can differ from its sequence of approximations $S_N(t)$ for any t in the interval. The "constant" in the inequality will depend on f (typically the maximum of some derivative of some order over the interval, which regulates how much the function wiggles) but *not* on t — that's uniformity. A geometric picture of uniform convergence may be clearer. A sequence of functions $f_n(t)$ converges uniformly to a function $f(t)$ if the graphs of the $f_n(t)$ get uniformly close to the graph of $f(t)$. I'll leave that second "uniformly" in the sentence to you to specify more carefully (it would force you to restate the analytic condition) but the picture should be clear. If the picture isn't clear, see Appendix 1.16, and think about graphs staying close to graphs if you're puzzling over our later discussion of Gibbs' phenomenon.

---

[27] We will also see "local" — "global" interplay at work in properties of the Fourier transform, which is one reason I wanted us to see this result for Fourier series.

Interestingly, in proving the theorem it's not so hard to show that the partial sums themselves are converging, and how fast. The trickier part is to show that the sums are converging *to* the value $f(t)$ of the function at every t! At any rate, the takeaway headline from this is:

> If the function is smooth, the Fourier series converges in every sense you could want; L^2, pointwise, uniformly.

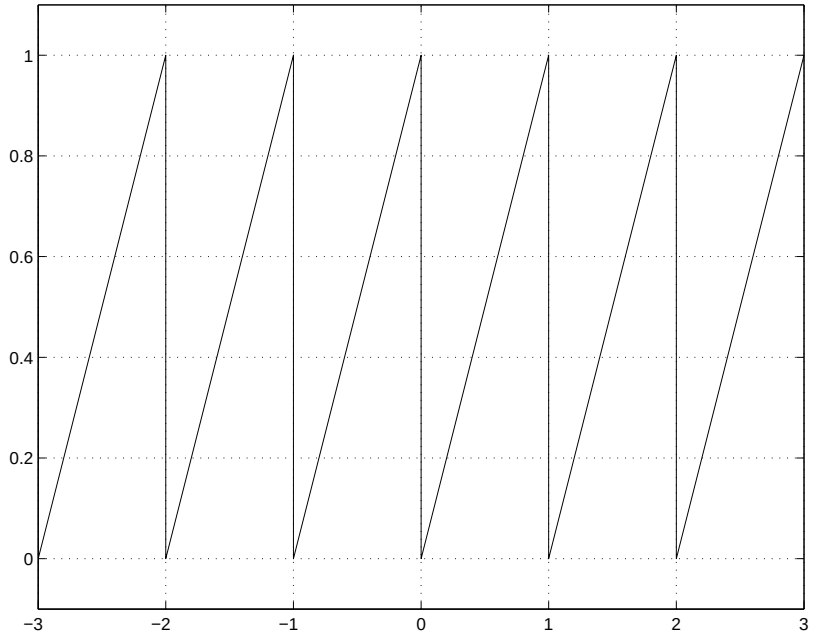
So don't bother me or anyone else about this, anymore.

1.14.3 Convergence if it's not continuous?

Let's consider the sawtooth signal from the homework, say

$$f(t) = \begin{cases} t & 0 \le t < 1 \\ 0 & \text{otherwise} \end{cases}$$

and extended to be periodic of period 1.



The Fourier coefficients are given by

$$\hat{f}(n) = \int_0^1 te^{-2\pi int}\,dt\,.$$

Integrating by parts gives, when $n \neq 0$,

$$\hat{f}(n) = \left[\frac{te^{-2\pi int}}{-2\pi in}\right]_0^1 - \int_0^1 \frac{1}{-2\pi in}e^{-2\pi int}\,dt = \frac{i}{2\pi n} \quad \text{(use } 1/i = -i\text{; the integral is 0.)}$$

Notice a few things.

- The coefficients are of the order $1/n$, just as they're supposed to be.
- The term with $n = 0$ is $1/2$, which we have to get directly, not from the integration by parts step.

(You might also notice the conjugate symmetry in the coefficients, $\hat{f}(-n) = \overline{\hat{f}(n)}$. This is often a good check on your calculations.)

So the Fourier series is

$$f(t) = \tfrac{1}{2} + \sum_{n=-\infty}^{\infty} \frac{i}{2\pi n} e^{2\pi i n t}\,.$$

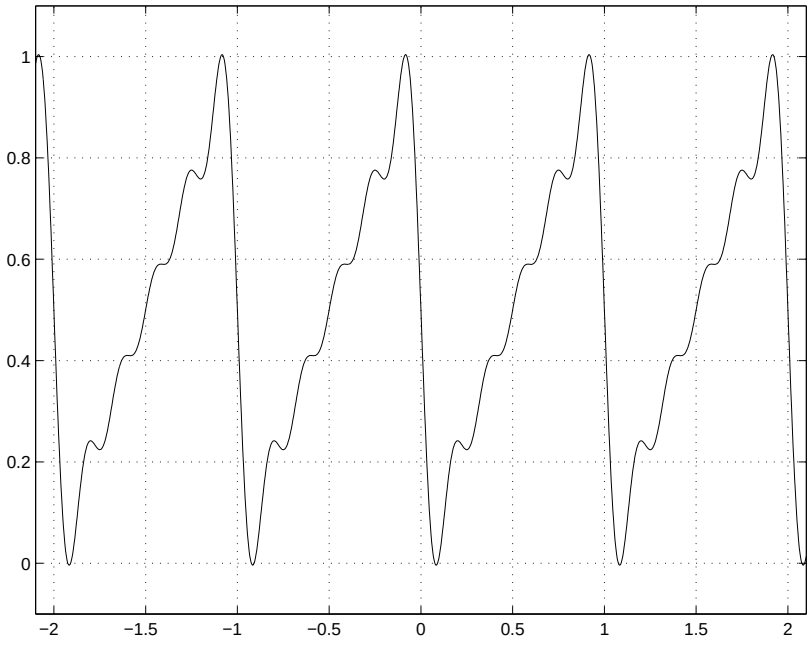
which means that

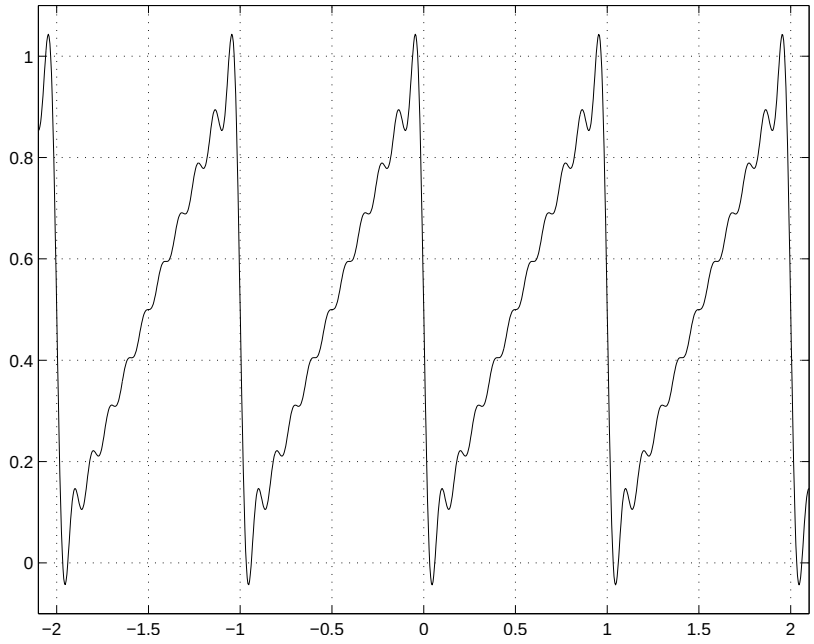
$$\lim_{N\to\infty} \left\| f(t) - \left(\tfrac{1}{2} + \sum_{n=-N}^{N} \frac{i}{2\pi n} e^{2\pi i n t} \right) \right\| = 0$$

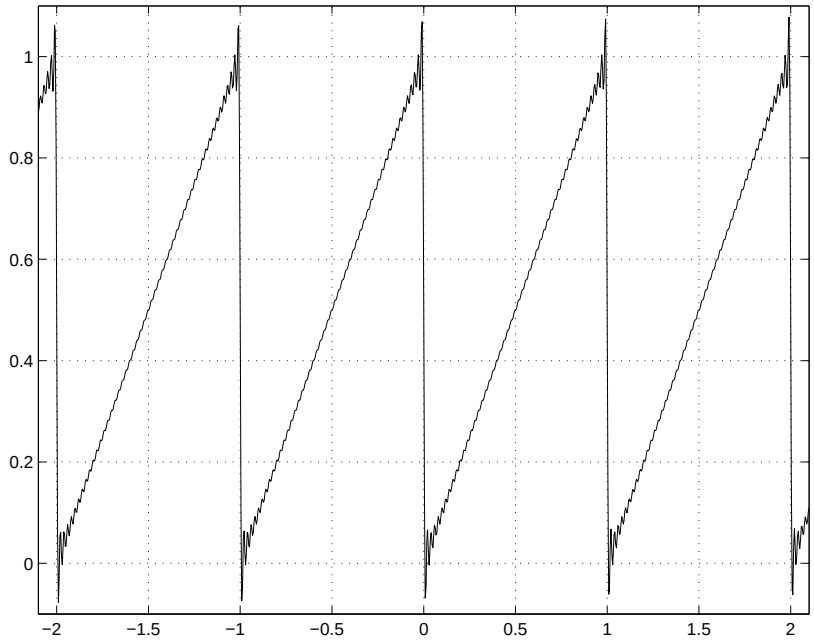
in the L^2 norm. But what do we get when we plug in a value of t and compute the sum, even setting aside the obvious problem of adding up an infinite number of terms?

Here are the plots for $N = 5$, 10, and 50 of partial sums

$$S_N(t) = \tfrac{1}{2} + \sum_{n=-N}^{N} \frac{i}{2\pi n} e^{2\pi i n t}\,.$$







There are two questions:

1. What does the series converge to, if it converges at all, at a point of discontinuity?
2. What's that funny behavior at the corners?

Here's the answer to the first question.

Theorem At a jump discontinuity (such as occurs in the sawtooth) the partial sums

$$S_N(t) = \sum_{n=-N}^{N} \hat{f}(n) e^{2\pi i n t}$$

converge to the average of the upper and lower values at the discontinuities.

For example, for the sawtooth the partial sums converge to 1/2 at the points $t = 0, \pm 1, \pm 2, \ldots$.

Because of this result some people *define* a value of a function at a jump discontinuity to be the average of the upper and lower values. That's reasonable in many contexts — this is one context and we'll see others — but it becomes a religious issue to some so I'll pass without further comment.

We can combine this theorem with the previous theorem to state a useful result that's easy to apply in practice:

> **Theorem on pointwise convergence** Suppose that $f(t)$ is continuous with a continuous derivative except at perhaps a finite number of points (in a period). Then for each $a \in [0, 1]$,
>
> $$S_N(a) \to \tfrac{1}{2}(\lim_{t \to a^-} f(t) + \lim_{t \to a^+} f(t))$$
>
> as $N \to \infty$.

If $f(t)$ is continuous at a then the left and right hand limits are equal and we just have $S_n(a) \to f(a)$. If $f(t)$ has a jump at a then we're in the situation in the theorem just above and $S_N(a)$ converges to the average of the left and right hand limits.

The funny behavior near the corners, where it seems that the approximations are overshooting the signal, is more interesting. We saw this also with the approximations to the square wave. This is the *Gibbs phenomenon*, named after J. W. Gibbs. It really happens, and it's time to come to terms with it. It was observed experimentally by Michelson and Stratton (that's the same Albert Michelson as in the famous "Michelson and Morley" experiment) who designed a mechanical device to draw finite Fourier series. Michelson and Stratton assumed that the extra wiggles they were seeing at jumps was a mechanical problem with the machine. But Gibbs, who used the sawtooth as an example, showed that the phenomenon is real and *does not go away* even in the limit. The oscillations may become more compressed, but they don't go away. (However, they do contribute zero in the limit of the L^2 norm of the difference between the function and its Fourier series.)

A standard way to formulate Gibbs's phenomenon precisely is for a square wave that jumps from -1 to $+1$ at $t = 0$ when t goes from negative to positive. Away from the single jump discontinuity, $S_N(t)$ tends uniformly to the values, $+1$ or -1 as the case may be, as $N \to \infty$. Hence the precise statement of Gibbs's phenomenon will be that the maximum of $S_N(t)$ remains greater than 1 as $N \to \infty$. And that's what is proved:

$$\lim_{N \to \infty} \max S_N(t) = 1.089490\ldots$$

So the overshoot is almost 9% — quite noticeable! See Section 1.18 of these notes for an outline of the derivation.

Now, there's something here that may bother you. We have the theorem on pointwise convergence that says at a jump discontinuity the partial sums converge to the average of the values at the jump. We also have Gibbs" phenomenon and the picture of an overshooting oscillation that doesn't go away. How can these two pictures coexist? If you're confused it's because you're thinking that convergence of $S_N(t)$, at, say, $t = 0$ in the sawtooth example, is the same as convergence of the graphs of the $S_N(t)$ to the graph of the sawtooth function. But they are *not* the same thing. It's the distinction between pointwise and uniform convergence — see Section 1.15.

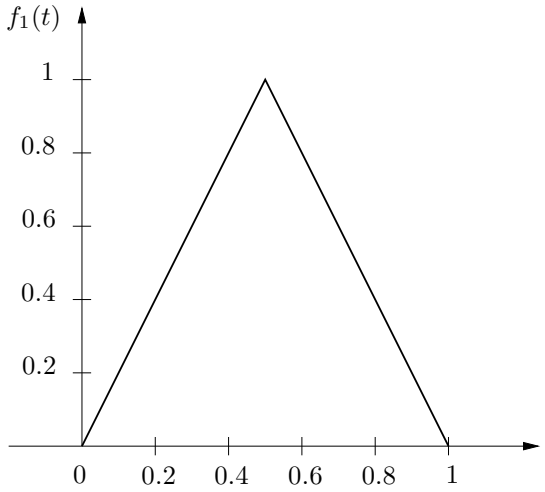
Finally, you should be aware that discontinuities are not at all uncommon. You might introduce jumps via windows or filters, for example. I mentioned earlier that this can be a problem in computer music,

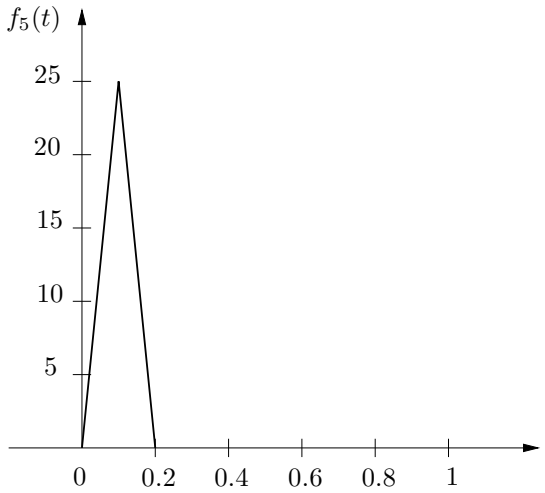
and *images* as two-dimensional signals, often have edges.[28] Remember, as we said in an earlier lecture, a discontinuity or a corner means that you *must* have infinitely high frequencies in the spectrum, so cutting off the approximation at a certain point is sure to introduce ripples in the computation of values of the function by means of a finite Fourier series approximation.

1.15 Appendix: Pointwise Convergence vs. Uniform Convergence

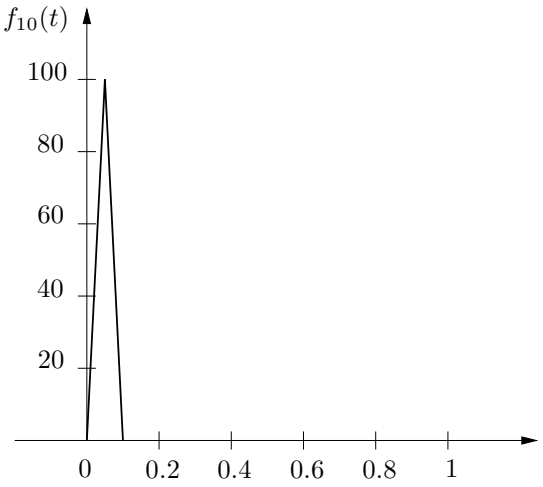
Here's an example, a classic of its type, to show that pointwise convergence is not the same as uniform convergence, or what amounts to the same thing, that we can have a sequence of functions $f_n(t)$ with the property that $f_n(t) \to f(t)$ for every value of t as $n \to \infty$ but the graphs of the $f_n(t)$ *do not* ultimately look like the graph of $f(t)$. Let me describe such a sequence of functions in words, draw a few pictures, and leave it to you to write down a formula.

The $f_n(t)$ will all be defined on $0 \le t \le 1$. For each n the graph of the function $f_n(t)$ is zero from $1/n$ to 1 and for $0 \le t \le 1/n$ it's an isosceles triangle with height n^2. Here are pictures of $f_1(t)$, $f_5(t)$ and $f_{10}(t)$.





[28] In fact, it's an important problem to *detect* edges in images.



The peak slides to the left and gets higher and higher as n increases. It's clear that for each t the sequence $f_n(t)$ tends to 0. This is so because $f_n(0) = 0$ for all n, and for any $t \neq 0$ eventually, that is, for large enough n, the peak is going to slide to the left of t and $f_n(t)$ will be zero from that n on out. Thus $f_n(t)$ converges pointwise to the constant 0. But the graphs of the $f_n(t)$ certainly are not uniformly close to 0!

1.16 Appendix: Studying Partial Sums via the Dirichlet Kernel: The Buzz Is Back

There are some interesting mathematical tools used to study the partial sums of Fourier series and their convergence properties, as in the theorem we stated earlier on the rate of convergence of the partial sums for p times continuously differentiable functions. In fact, we've already seen the main tool — it's the *Dirichlet kernel*

$$D_N(t) = \sum_{n=-N}^{N} e^{2\pi int}$$

that we introduced in Section 1.13.2 in the context of the "buzz signal". Here's how it's used.

We can write a partial sum in what turns out to be a helpful way by bringing back in the definition of the Fourier coefficients as an integral.

$$\begin{aligned}
S_N(t) &= \sum_{n=-N}^{N} \hat{f}(n) e^{2\pi int} \\
&= \sum_{n=-N}^{N} \left(\int_0^1 f(s) e^{-2\pi ins}\, ds \right) e^{2\pi int} \\
&\qquad \text{(calling the variable of integration } s \text{ since we're already using } t) \\
&= \sum_{n=-N}^{N} \left(\int_0^1 e^{2\pi int} f(s) e^{-2\pi ins}\, ds \right) \\
&= \int_0^1 \left(\sum_{n=-N}^{N} e^{2\pi int} e^{-2\pi ins} f(s) \right) ds \\
&= \int_0^1 \left(\sum_{n=-N}^{N} e^{2\pi in(t-s)} \right) f(s)\, ds = \int_0^1 D_N(t-s) f(s)\, ds\,.
\end{aligned}$$

Just as we saw in the solution of the heat equation, we have produced a convolution! The integral

$$\int_0^1 D_N(t-s)f(s)\,ds$$

is the convolution of $f(t)$ with the function $D_N(t)$ and it produces the partial sum $S_N(t)$.[29]

Why is this helpful? By means of the convolution integral, estimates for $S_N(t)$ involve both properties of f (on which we make certain assumptions) *together with* properties of $D_N(t)$, for which we can find an explicit expression. Here's how it goes, just to see if anyone's reading these notes. The idea is to view $D_N(t)$ as a geometric series. We can simplify the algebra by factoring out the term corresponding to $-N$, thereby writing the sum as going from 0 to $2N$:

$$\begin{aligned}\sum_{n=-N}^{N} e^{2\pi inp} &= e^{-2\pi iNp}\sum_{n=0}^{2N} e^{2\pi inp}\\ &= e^{-2\pi iNp}\,\frac{e^{2\pi i(2N+1)p}-1}{e^{2\pi ip}-1}\\ &\qquad\text{(using the formula for the sum of a geometric series } \textstyle\sum r^n \text{ with } r=e^{2\pi ip})\end{aligned}$$

It's most common to write this in terms of the sine function. Recall that

$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}\,.$$

To bring the sine into the expression, above, there's a further little factoring trick that's used often:

$$\begin{aligned}e^{2\pi i(2N+1)p}-1 &= e^{\pi i(2N+1)p}(e^{\pi i(2N+1)p}-e^{-\pi i(2N+1)p})\\ &= 2ie^{\pi i(2N+1)p}\sin(\pi(2N+1)p)\\ e^{2\pi ip}-1 &= e^{\pi ip}(e^{\pi ip}-e^{-\pi ip})\\ &= 2ie^{\pi ip}\sin(\pi p)\end{aligned}$$

Therefore

$$e^{-2\pi iNp}\,\frac{e^{2\pi i(2N+1)p}-1}{e^{2\pi ip}-1} = e^{-2\pi iNp}\,\frac{e^{\pi i(2N+1)p}}{e^{\pi ip}}\,\frac{2i\sin(\pi(2N+1)p)}{2i\sin(\pi p)} = \frac{\sin(\pi(2N+1)p)}{\sin(\pi p)}\,.$$

Nice.

Recall from Section 1.13.2 that as N gets larger $D_N(t)$ becomes more and more sharply peaked at the integers, and $D_N(0)\to\infty$ as $N\to\infty$. Forming the convolution, as in

$$S_N(t) = \int_0^1 D_N(t-s)f(s)\,ds\,,$$

above, shifts the peak at 0 to t, and integrates. The integrand is concentrated around t (as it turns out the peaks at the other integers don't contribute) and in the limit as $N\to\infty$ the integral tends to $f(t)$.[30]

Carrying this out in detail — which we are not going to do — depends on the explicit formula for $D_N(t)$. The more one assumes about the signal $f(t)$ the more the argument can produce. This is how one gets the theorem on order of differentiability and rate of convergence of partial sums of Fourier series.

---

[29] It's no accident that convolution comes in and we'll understand this thoroughly when we develop some properties of the Fourier transform. The moral of the story will then be that while math majors take the appearance of $D_N(t)$ to be a mystical revelation, to any EE it's just meeting an old friend on the corner. You'll see.

[30] Those of you who have worked with δ functions may think you recognize this sort of thing:

$$\int \delta(t-s)f(s)\,ds = f(t)$$

and you'd be right. We'll do plenty of this.

1.17 Appendix: The Complex Exponentials Are a Basis for $L^2([0,1])$

Remember the second point in our hit parade of the L^2-theory of Fourier series:

> The complex exponentials $e^{2\pi int}$, $n = 0, \pm 1, \pm 2, \dots$ form a basis for $L^2([0,1])$, and the partial sums converge to $f(t)$ as $N \to \infty$ in the L^2-distance. This means that
>
> $$\lim_{N\to\infty} \left\| \sum_{n=-N}^{N} \hat{f}(n) e^{2\pi int} - f(t) \right\| = 0 \,.$$

I said earlier that we wouldn't attempt a complete proof of this, and we won't. But with the discussion just preceding we can say more precisely how the proof goes, and what the issues are that we cannot get into. The argument is in three steps.

Let $f(t)$ be a square integrable function and let $\epsilon > 0$.

Step 1 Any function in $L^2([0,1])$ can be approximated in the L^2-norm by a continuously differentiable function.[31] That is, starting with a given f in $L^2([0,1])$ and any $\epsilon > 0$ we can find a function $g(t)$ that is continuously differentiable on $[0,1]$ for which

$$\|f - g\| < \epsilon \,.$$

This is the step we cannot do! It's here, in proving this statement, that one needs the more general theory of integration and the limiting processes that go with it. Let it rest.

Step 2 From the discussion above, we now know (at least we've now been told, with some indication of why) that the Fourier partial sums for a continuously differentiable function ($p = 1$ in the statement of the theorem) converge uniformly to the function. Thus, with $g(t)$ as in Step 1, we can choose N so large that

$$\max \left| g(t) - \sum_{n=-N}^{N} \hat{g}(n) e^{2\pi int} \right| < \epsilon \,.$$

Then for the L^2-norm,

$$\int_0^1 \left| g(t) - \sum_{n=-N}^{N} \hat{g}(n) e^{2\pi int} \right|^2 dt \leq \int_0^1 \left(\max \left| g(t) - \sum_{n=-N}^{N} \hat{g}(n) e^{2\pi int} \right| \right)^2 dt < \int_0^1 \epsilon^2 \, dt = \epsilon^2 \,.$$

Hence

$$\left\| g(t) - \sum_{n=-N}^{N} \hat{g}(n) e^{2\pi int} \right\| < \epsilon \,.$$

Step 3 Remember that the Fourier coefficients provide the best finite approximation in L^2 to the function, that is, as we'll need it,

$$\left\| f(t) - \sum_{n=-N}^{N} \hat{f}(n) e^{2\pi int} \right\| \leq \left\| f(t) - \sum_{n=-N}^{N} \hat{g}(n) e^{2\pi int} \right\| .$$

[31] Actually, it's true that any function in $L^2([0,1])$ can be approximated by an *infinitely* differentiable function.

And at last

$$\begin{aligned}\left\|f(t)-\sum_{n=-N}^{N}\hat{f}(n)e^{2\pi int}\right\| &\le \left\|f(t)-\sum_{n=-N}^{N}\hat{g}(n)e^{2\pi int}\right\| \\ &= \left\|f(t)-g(t)+g(t)-\sum_{n=-N}^{N}\hat{g}(n)e^{2\pi int}\right\| \\ &\le \|f(t)-g(t)\| + \left\|g(t)-\sum_{n=-N}^{N}\hat{g}(n)e^{2\pi int}\right\| < 2\epsilon\,.\end{aligned}$$

This shows that

$$\left\|f(t)-\sum_{n=-N}^{N}\hat{f}(n)e^{2\pi int})\right\|$$

can be made arbitrarily small by taking N large enough, which is what we were required to do.

1.18 Appendix: More on the Gibbs Phenomenon

Here's what's involved in establishing the Gibbs' phenomenon for the square wave

$$f(t)=\begin{cases}-1 & -\frac{1}{2}\le t<0\\ +1 & 0\le t\le+\frac{1}{2}\end{cases}$$

We're supposed to show that

$$\lim_{N\to\infty}\max S_N(t)=1.089490\ldots$$

Since we've already introduced the Dirichlet kernel, let's see how it can be used here. I'll be content with showing the approach and the outcome, and won't give the somewhat tedious detailed estimates. As in Appendix 2, the partial sum $S_N(t)$ can be written as a convolution with D_N. In the case of the square wave, as we've set it up here,

$$\begin{aligned}S_N(t) &= \int_{-1/2}^{1/2} D_N(t-s)f(s)\,ds \\ &= -\int_{-1/2}^{0} D_N(t-s)\,ds + \int_{0}^{1/2} D_N(t-s)\,ds \\ &= -\int_{-1/2}^{0} D_N(s-t)\,ds + \int_{0}^{1/2} D_N(s-t)\,ds \quad \text{(using that } D_N \text{ is even.)}\end{aligned}$$

The idea next is to try to isolate, and estimate, the behavior near the origin by getting an integral from $-t$ to t. We can do this by first making a change of variable $u=s-t$ in both integrals. This results in

$$-\int_{-1/2}^{0} D_N(s-t)\,ds+\int_{0}^{1/2} D_N(s-t)\,ds=-\int_{-\frac{1}{2}-t}^{-t} D_N(u)\,du+\int_{-t}^{\frac{1}{2}-t} D_N(u)\,du\,.$$

To this last expression add and subtract

$$\int_{-t}^{t} D_N(u)\,du$$

and combine integrals to further obtain

$$-\int_{-\frac{1}{2}-t}^{-t} D_N(u)\,du + \int_{-t}^{\frac{1}{2}-t} D_N(u)\,du = -\int_{-\frac{1}{2}-t}^{t} D_N(u)\,du + \int_{-t}^{\frac{1}{2}-t} D_N(u)\,du + \int_{-t}^{t} D_N(u)\,du$$

Finally, make a change of variable $w = -u$ in the first integral and use the evenness of D_N. Then the first two integrals combine and we are left with, again letting s be the variable of integration in both integrals,

$$S_N(t) = \int_{-t}^{t} D_N(s)\,ds - \int_{\frac{1}{2}-t}^{\frac{1}{2}+t} D_N(s)\,ds\,.$$

The reason that this is helpful is that using the explicit formula for D_N one can show (this takes some work — integration by parts) that

$$\left|S_N(t) - \int_{-t}^{t} D_N(s)\,ds\right| = \left|\int_{\frac{1}{2}-t}^{\frac{1}{2}+t} D_N(s)\,ds\right| \le \frac{\text{constant}}{n}\,,$$

and hence

$$\lim_{N\to\infty}\left|S_N(t) - \int_{-t}^{t} D_N(s)\,ds\right| = 0\,.$$

This means that if we can establish a max for $\int_{-t}^{t} D_N(s)\,ds$ we'll also get one for $S_N(t)$. That, too, takes some work, but the fact that one has an explicit formula for D_N makes it possible to deduce for $|t|$ small and N large that $\int_{-t}^{t} D_N(t)\,dt$, and hence $S_N(t)$ is well approximated by

$$\frac{2}{\pi}\int_0^{(2N+1)\pi t} \frac{\sin s}{s}\,ds\,.$$

This integral has a maximum at the first place where $\sin((2N+1)\pi t) = 0$, i.e., at $t = 1/(2N+1)$. At this point the value of the integral (found via numerical approximations) is

$$\frac{2}{\pi}\int_0^{\pi} \frac{\sin s}{s}\,ds = 1.09940\ldots\,,$$

and that's where the 9% overshoot figure comes from.

Had enough?

Chapter 2

Fourier Transform

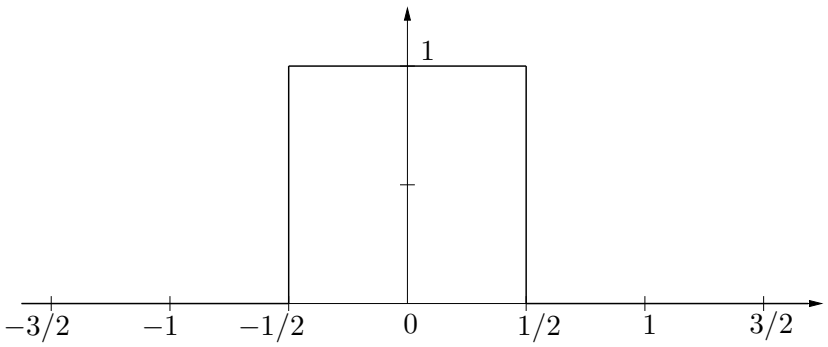
2.1 A First Look at the Fourier Transform

We're about to make the transition from Fourier series to the Fourier transform. "Transition" is the appropriate word, for in the approach we'll take the Fourier transform emerges as we pass from periodic to nonperiodic functions. To make the trip we'll view a nonperiodic function (which can be just about anything) as a limiting case of a periodic function as the period becomes longer and longer. Actually, this process doesn't immediately produce the desired result. It takes a little extra tinkering to coax the Fourier transform out of the Fourier series, but it's an interesting approach.[1]

Let's take a specific, simple, and important example. Consider the "rect" function ("rect" for "rectangle") defined by

$$\Pi(t) = \begin{cases} 1 & |t| < 1/2 \\ 0 & |t| \geq 1/2 \end{cases}$$

Here's the graph, which is not very complicated.



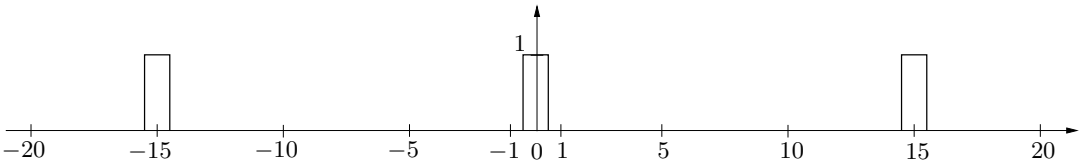
$\Pi(t)$ is even — centered at the origin — and has width 1. Later we'll consider shifted and scaled versions. You can think of $\Pi(t)$ as modeling a switch that is on for one second and off for the rest of the time. Π is also

[1] As an aside, I don't know if this is the best way of motivating the definition of the Fourier transform, but I don't know a better way and most sources you're likely to check will just present the formula as a done deal. It's true that, in the end, it's the formula and what we can do with it that we want to get to, so if you don't find the (brief) discussion to follow to your tastes, I am not offended.

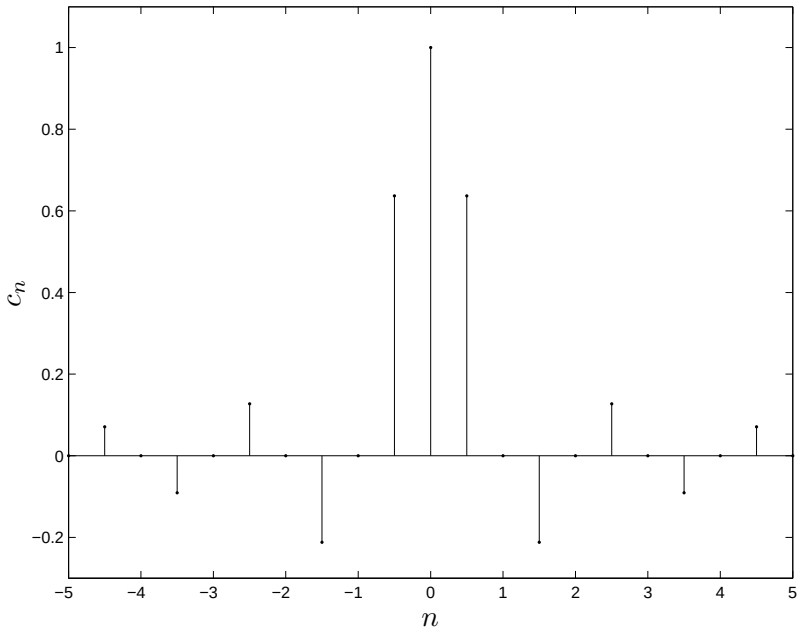
called, variously, the *top hat* function (because of its graph), the *indicator* function, or the *characteristic* function for the interval $(-1/2, 1/2)$.

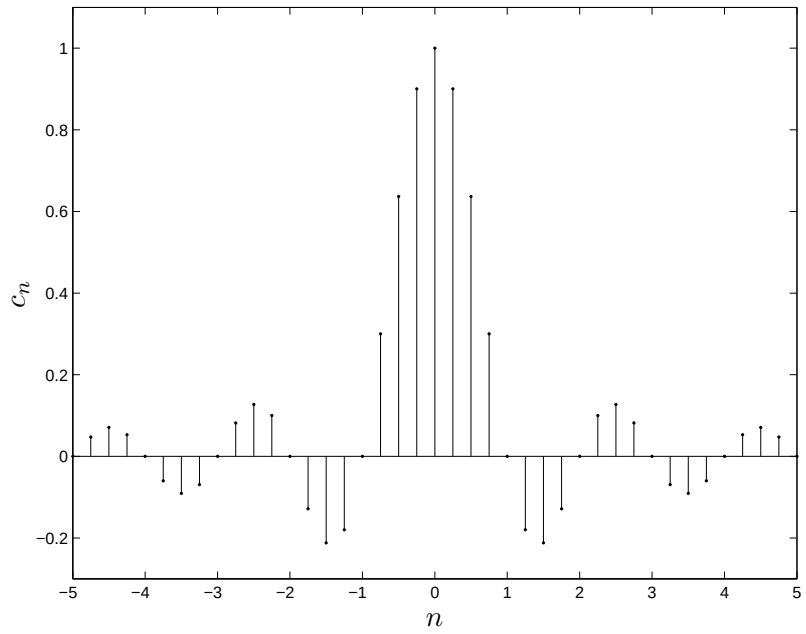
While we have defined $\Pi(\pm 1/2) = 0$, other common conventions are either to have $\Pi(\pm 1/2) = 1$ or $\Pi(\pm 1/2) = 1/2$. And some people don't define Π at $\pm 1/2$ at all, leaving two holes in the domain. I don't want to get dragged into this dispute. It almost never matters, though for some purposes the choice $\Pi(\pm 1/2) = 1/2$ makes the most sense. We'll deal with this on an exceptional basis if and when it comes up.

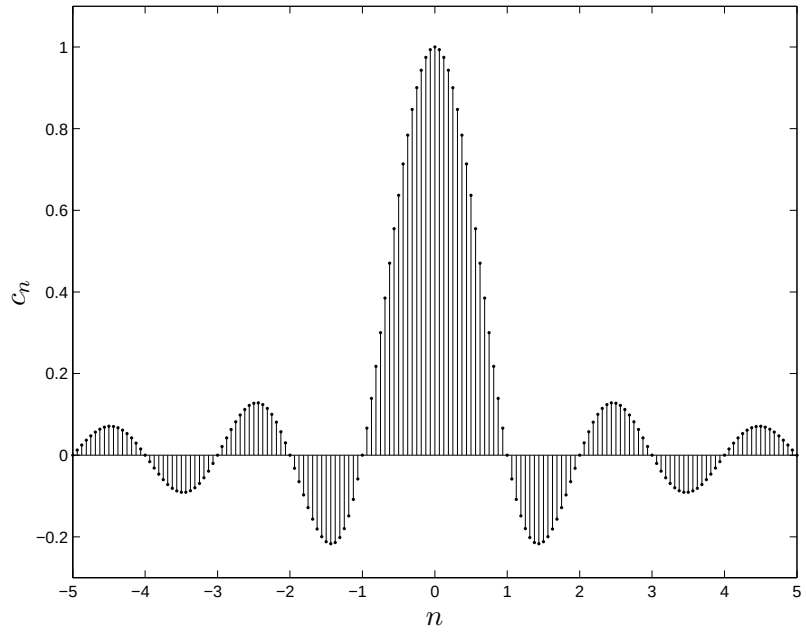
$\Pi(t)$ is not periodic. It doesn't have a Fourier series. In problems you experimented a little with periodizations, and I want to do that with Π but for a specific purpose. As a periodic version of $\Pi(t)$ we repeat the nonzero part of the function at regular intervals, separated by (long) intervals where the function is zero. We can think of such a function arising when we flip a switch on for a second at a time, and do so repeatedly, and we keep it off for a long time in between the times it's on. (One often hears the term *duty cycle* associated with this sort of thing.) Here's a plot of $\Pi(t)$ periodized to have period 15.



Here are some plots of the Fourier coefficients of periodized rectangle functions with periods 2, 4, and 16, respectively. Because the function is real and even, in each case the Fourier coefficients are real, so these are plots of the actual coefficients, not their square magnitudes.







We see that as the period increases the frequencies are getting closer and closer together and it looks as though the coefficients are tracking some definite curve. (But we'll see that there's an important issue here of vertical scaling.) We can analyze what's going on in this particular example, and combine that with some general statements to lead us on.

Recall that for a general function $f(t)$ of period T the Fourier series has the form

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n t/T}$$

so that the frequencies are $0, \pm 1/T, \pm 2/T, \ldots$. Points in the spectrum are spaced $1/T$ apart and, indeed, in the pictures above the spectrum is getting more tightly packed as the period T increases. The n-th Fourier coefficient is given by

$$c_n = \frac{1}{T}\int_0^T e^{-2\pi i n t/T} f(t)\, dt = \frac{1}{T}\int_{-T/2}^{T/2} e^{-2\pi i n t/T} f(t)\, dt\,.$$

We can calculate this Fourier coefficient for $\Pi(t)$:

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} e^{-2\pi int/T}\Pi(t)\,dt = \frac{1}{T}\int_{-1/2}^{1/2} e^{-2\pi int/T}\cdot 1\,dt$$
$$= \frac{1}{T}\Big[\frac{1}{-2\pi in/T}e^{-2\pi int/T}\Big]_{t=-1/2}^{t=1/2} = \frac{1}{2\pi in}\left(e^{\pi in/T} - e^{-\pi in/T}\right) = \frac{1}{\pi n}\sin\left(\frac{\pi n}{T}\right).$$

Now, although the spectrum is *indexed* by n (it's a discrete set of points), the points in the spectrum are n/T ($n = 0, \pm1, \pm2, \ldots$), and it's more helpful to think of the "spectral information" (the value of c_n) as a transform of Π *evaluated* at the points n/T. Write this, provisionally, as

$$(\text{Transform of periodized }\Pi)\left(\frac{n}{T}\right) = \frac{1}{\pi n}\sin\left(\frac{\pi n}{T}\right).$$

We're almost there, but not quite. If you're dying to just take a limit as $T \to \infty$ consider that, for each n, if T is very large then n/T is very small and

$$\frac{1}{\pi n}\sin\left(\frac{\pi n}{T}\right) \text{ is about size } \frac{1}{T} \quad (\text{remember } \sin\theta \approx \theta \text{ if } \theta \text{ is small}).$$

In other words, for each n this so-called transform,

$$\frac{1}{\pi n}\sin\left(\frac{\pi n}{T}\right),$$

tends to 0 like $1/T$. To compensate for this we scale up by T, that is, we consider instead

$$(\text{Scaled transform of periodized }\Pi)\left(\frac{n}{T}\right) = T\,\frac{1}{\pi n}\sin\left(\frac{\pi n}{T}\right) = \frac{\sin(\pi n/T)}{\pi n/T}.$$

In fact, the plots of the *scaled* transforms are what I showed you, above.

Next, if T is large then we can think of replacing the closely packed discrete points n/T by a continuous variable, say s, so that with $s = n/T$ we would then write, approximately,

$$(\text{Scaled transform of periodized }\Pi)(s) = \frac{\sin\pi s}{\pi s}.$$

What does this procedure look like in terms of the integral formula? Simply

$$(\text{Scaled transform of periodized }\Pi)\left(\frac{n}{T}\right) = T\cdot c_n$$
$$= T\cdot\frac{1}{T}\int_{-T/2}^{T/2} e^{-2\pi int/T}f(t)\,dt = \int_{-T/2}^{T/2} e^{-2\pi int/T}f(t)\,dt.$$

If we now think of $T \to \infty$ as having the effect of replacing the discrete variable n/T by the continuous variable s, as well as pushing the limits of integration to $\pm\infty$, then we may write for the (limiting) transform of Π the integral expression

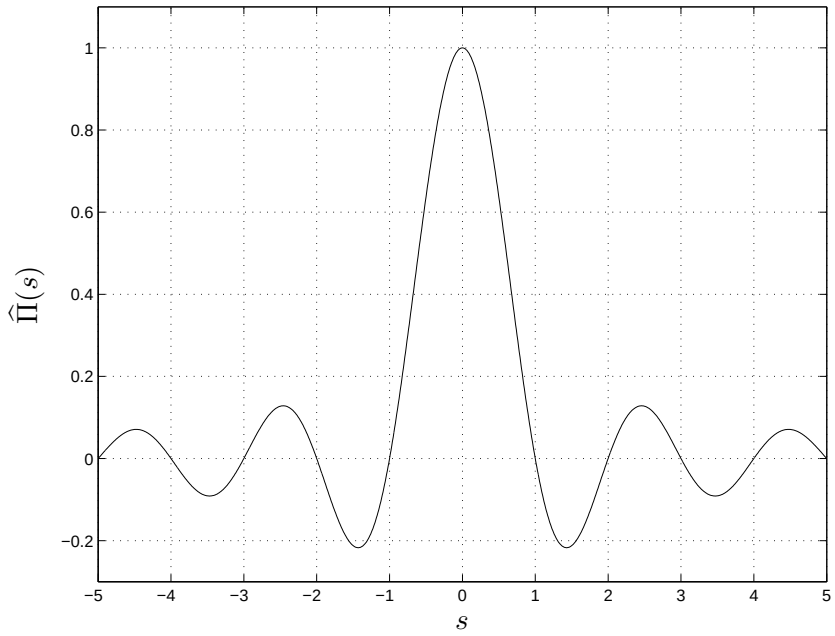
$$\widehat{\Pi}(s) = \int_{-\infty}^{\infty} e^{-2\pi ist}\,\Pi(t)\,dt.$$

Behold, the Fourier transform is born!

Let's calculate the integral. (We know what the answer is, because we saw the discrete form of it earlier.)

$$\widehat{\Pi}(s) = \int_{-\infty}^{\infty} e^{-2\pi ist}\Pi(t)\,dt = \int_{-1/2}^{1/2} e^{-2\pi ist}\cdot 1\,dt = \frac{\sin\pi s}{\pi s}.$$

Here's a graph. You can now certainly see the continuous curve that the plots of the discrete, scaled Fourier coefficients are shadowing.



The function $\sin \pi x / \pi x$ (written now with a generic variable x) comes up so often in this subject that it's given a name, sinc:

$$\operatorname{sinc} x = \frac{\sin \pi x}{\pi x}$$

pronounced "sink". Note that

$$\operatorname{sinc} 0 = 1$$

by virtue of the famous limit

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \, .$$

It's fair to say that many EE's see the sinc function in their dreams.





How general is this? We would be led to the same idea — scale the Fourier coefficients by T — if we had started off periodizing just about any function with the intention of letting $T \to \infty$. Suppose $f(t)$ is zero outside of $|t| \leq 1/2$. (Any interval will do, we just want to suppose a function is zero outside some interval so we can periodize.) We periodize $f(t)$ to have period T and compute the Fourier coefficients:

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} e^{-2\pi int/T} f(t)\,dt = \frac{1}{T}\int_{-1/2}^{1/2} e^{-2\pi int/T} f(t)\,dt\,.$$

How big is this? We can estimate

$$|c_n| = \frac{1}{T}\left|\int_{-1/2}^{1/2} e^{-2\pi int/T} f(t)\,dt\right|$$
$$\leq \frac{1}{T}\int_{-1/2}^{1/2} |e^{-2\pi int/T}|\,|f(t)|\,dt = \frac{1}{T}\int_{-1/2}^{1/2} |f(t)|\,dt = \frac{A}{T}\,,$$

where

$$A = \int_{-1/2}^{1/2} |f(t)|\,dt\,,$$

which is some *fixed number independent* of n and T. Again we see that c_n tends to 0 like $1/T$, and so again we scale back up by T and consider

$$(\text{Scaled transform of periodized } f)\left(\frac{n}{T}\right) = Tc_n = \int_{-T/2}^{T/2} e^{-2\pi int/T} f(t)\,dt\,.$$

In the limit as $T \to \infty$ we replace n/T by s and consider

$$\hat{f}(s) = \int_{-\infty}^{\infty} e^{-2\pi ist} f(t)\,dt\,.$$

We're back to the same integral formula.

Fourier transform defined There you have it. We now *define* the *Fourier transform* of a function $f(t)$ to be

$$\hat{f}(s) = \int_{-\infty}^{\infty} e^{-2\pi ist} f(t)\,dt\,.$$

For now, just take this as a formal definition; we'll discuss later when such an integral exists. We assume that $f(t)$ is defined for all real numbers t. For any $s \in \mathbf{R}$, integrating $f(t)$ against $e^{-2\pi ist}$ with respect to t produces a *complex valued* function of s, that is, the Fourier transform $\hat{f}(s)$ is a complex-valued function of $s \in \mathbf{R}$. If t has dimension time then to make st dimensionless in the exponential $e^{-2\pi ist}$ s must have dimension 1/time.

While the Fourier transform takes flight from the desire to find spectral information on a nonperiodic function, the extra complications and extra richness of what results will soon make it seem like we're in a much different world. The definition just given is a good one *because* of the richness and *despite* the complications. Periodic functions are great, but there's more bang than buzz in the world to analyze.

The spectrum of a periodic function is a discrete set of frequencies, possibly an infinite set (when there's a corner) but always a discrete set. By contrast, the Fourier transform of a nonperiodic signal produces a continuous spectrum, or a continuum of frequencies.

It may be that $\hat{f}(s)$ is identically zero for $|s|$ sufficiently large — an important class of signals called *bandlimited* — or it may be that the nonzero values of $\hat{f}(s)$ extend to $\pm\infty$, or it may be that $\hat{f}(s)$ is zero for just a few values of s.

The Fourier transform analyzes a signal into its frequency components. We haven't yet considered how the corresponding synthesis goes. How can we recover $f(t)$ in the time domain from $\hat{f}(s)$ in the frequency domain?

Recovering $f(t)$ from $\hat{f}(s)$ We can push the ideas on nonperiodic functions as limits of periodic functions a little further and discover how we might obtain $f(t)$ from its transform $\hat{f}(s)$. Again suppose $f(t)$ is zero outside some interval and periodize it to have (large) period T. We expand $f(t)$ in a Fourier series,

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n t/T}\,.$$

The Fourier *coefficients* can be written via the Fourier *transform* of f evaluated at the points $s_n = n/T$.

$$\begin{aligned} c_n &= \frac{1}{T}\int_{-T/2}^{T/2} e^{-2\pi i n t/T} f(t)\,dt = \frac{1}{T}\int_{-\infty}^{\infty} e^{-2\pi i n t/T} f(t)\,dt \\ &\qquad \text{(we can extend the limits to } \pm\infty \text{ since } f(t) \text{ is zero outside of } [-T/2, T/2]) \\ &= \frac{1}{T}\hat{f}\left(\frac{n}{T}\right) = \frac{1}{T}\hat{f}(s_n)\,. \end{aligned}$$

Plug this into the expression for $f(t)$:

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{1}{T}\hat{f}(s_n) e^{2\pi i s_n t}\,.$$

Now, the points $s_n = n/T$ are spaced $1/T$ apart, so we can think of $1/T$ as, say Δs, and the sum above as a Riemann sum approximating an integral

$$\sum_{n=-\infty}^{\infty} \frac{1}{T}\hat{f}(s_n) e^{2\pi i s_n t} = \sum_{n=-\infty}^{\infty} \hat{f}(s_n) e^{2\pi i s_n t}\Delta s \approx \int_{-\infty}^{\infty} \hat{f}(s) e^{2\pi i s t}\,ds\,.$$

The limits on the integral go from $-\infty$ to ∞ because the sum, and the points s_n, go from $-\infty$ to ∞. Thus as the period $T \to \infty$ we would expect to have

$$f(t) = \int_{-\infty}^{\infty} \hat{f}(s) e^{2\pi i s t}\,ds$$

and we have recovered $f(t)$ from $\hat{f}(s)$. We have found the *inverse Fourier transform* and *Fourier inversion.*

The inverse Fourier transform defined, and Fourier inversion, too The integral we've just come up with can stand on its own as a "transform", and so we define the *inverse Fourier transform* of a function $g(s)$ to be

$$\check{g}(t) = \int_{-\infty}^{\infty} e^{2\pi i s t} g(s)\,ds \quad \text{(upside down hat — cute)}\,.$$

Again, we're treating this formally for the moment, withholding a discussion of conditions under which the integral makes sense. In the same spirit, we've also produced the *Fourier inversion theorem.* That is

$$f(t) = \int_{-\infty}^{\infty} e^{2\pi i s t} \hat{f}(s)\,ds\,.$$

Written very compactly,

$$(\hat{f})^{\check{}} = f\,.$$

The inverse Fourier transform looks just like the Fourier transform except for the minus sign. Later we'll say more about the remarkable symmetry between the Fourier transform and its inverse.

By the way, we could have gone through the whole argument, above, starting with $\hat{f}$ as the basic function instead of f. If we did that we'd be led to the complementary result on Fourier inversion,

$$(\check{g})^{\hat{}} = g\,.$$

A quick summary Let's summarize what we've done here, partly as a guide to what we'd like to do next. There's so much involved, all of importance, that it's hard to avoid saying everything at once. Realize that it will take some time before everything is in place.

- The Fourier transform of the signal $f(t)$ is

$$\hat{f}(s) = \int_{-\infty}^{\infty} f(t)e^{-2\pi ist}\,dt\,.$$

  This is a complex-valued function of s.

  One value is easy to compute, and worth pointing out, namely for $s = 0$ we have

$$\hat{f}(0) = \int_{-\infty}^{\infty} f(t)\,dt\,.$$

  In calculus terms this is the area under the graph of $f(t)$. If $f(t)$ is real, as it most often is, then $\hat{f}(0)$ is real even though other values of the Fourier transform may be complex.

- The domain of the Fourier transform is the set of real numbers s. One says that $\hat{f}$ is defined on the *frequency domain*, and that the original signal $f(t)$ is defined on the *time domain* (or the *spatial domain*, depending on the context). For a (nonperiodic) signal defined on the whole real line we generally do not have a discrete set of frequencies, as in the periodic case, but rather a *continuum* of frequencies.[2] (We still do call them "frequencies", however.) The set of all frequencies is the *spectrum* of $f(t)$.

  - Not all frequencies need occur, i.e., $\hat{f}(s)$ might be zero for some values of s. Furthermore, it might be that there aren't any frequencies outside of a certain range, i.e.,

$$\hat{f}(s) = 0 \quad \text{for } |s| \text{ large}\,.$$

    These are called *bandlimited signals* and they are an important special class of signals. They come up in sampling theory.

- The inverse Fourier transform is defined by

$$\check{g}(t) = \int_{-\infty}^{\infty} e^{2\pi ist}g(s)\,ds\,.$$

  Taken together, the Fourier transform and its inverse provide a way of passing between two (equivalent) representations of a signal via the Fourier inversion theorem:

$$(\hat{f})^{\check{}} = f\,, \quad (\check{g})^{\hat{}} = g\,.$$

  We note one consequence of Fourier inversion, that

$$f(0) = \int_{-\infty}^{\infty} \hat{f}(s)\,ds\,.$$

  There is no quick calculus interpretation of this result. The right hand side is an integral of a complex-valued function (generally), and result is real (if $f(0)$ is real).

[2] A periodic function *does* have a Fourier transform, but it's a sum of δ functions. We'll have to do that, too, and it will take some effort.

Now remember that $\hat{f}(s)$ is a transformed, complex-valued function, and while it may be "equivalent" to $f(t)$ it has very different properties. Is it really true that when $\hat{f}(s)$ exists we can just plug it into the formula for the inverse Fourier transform — which is also an improper integral that looks the same as the forward transform except for the minus sign — and really get back $f(t)$? Really? That's worth wondering about.

- The square magnitude $|\hat{f}(s)|^2$ is called the *power spectrum* (especially in connection with its use in communications) or the *spectral power density* (especially in connection with its use in optics) or the *energy spectrum* (especially in every other connection).

  An important relation between the energy of the signal in the time domain and the energy spectrum in the frequency domain is given by Parseval's identity for Fourier transforms:

$$\int_{-\infty}^{\infty} |f(t)|^2\,dt = \int_{-\infty}^{\infty} |\hat{f}(s)|^2\,ds\,.$$

  This is also a future attraction.

A warning on notations: None is perfect, all are in use Depending on the operation to be performed, or on the context, it's often useful to have alternate notations for the Fourier transform. But here's a warning, which is the start of a complaint, which is the prelude to a full blown rant. Diddling with notation seems to be an unavoidable hassle in this subject. Flipping back and forth between a transform and its inverse, naming the variables in the different domains (even writing or not writing the variables), changing plus signs to minus signs, taking complex conjugates, these are all routine day-to-day operations and they can cause endless muddles if you are not careful, and sometimes even if you are careful. You will believe me when we have some examples, and you will hear me complain about it frequently.

Here's one example of a common convention:

> If the function is called f then one often uses the corresponding capital letter, F, to denote the Fourier transform. So one sees a and A, z and Z, and everything in between. Note, however, that one typically uses different names for the variable for the two functions, as in $f(x)$ (or $f(t)$) and $F(s)$. This 'capital letter notation' is very common in engineering but often confuses people when 'duality' is invoked, to be explained below.

And then there's this:

> Since taking the Fourier transform is an operation that is applied to a function to produce a new function, it's also sometimes convenient to indicate this by a kind of "operational" notation. For example, it's common to write $\mathcal{F}f(s)$ for $\hat{f}(s)$, and so, to repeat the full definition
>
> $$\mathcal{F}f(s) = \int_{-\infty}^{\infty} e^{-2\pi ist} f(t)\,dt\,.$$
>
> This is often the most unambiguous notation. Similarly, the operation of taking the inverse Fourier transform is then denoted by $\mathcal{F}^{-1}$, and so
>
> $$\mathcal{F}^{-1}g(t) = \int_{-\infty}^{\infty} e^{2\pi ist} g(s)\,ds\,.$$

We will use the notation $\mathcal{F}f$ more often than not. It, too, is far from ideal, the problem being with keeping variables straight — you'll see.

Finally, a function and its Fourier transform are said to constitute a "Fourier pair", ; this is concept of 'duality' to be explained more precisely later. There have been various notations devised to indicate this sibling relationship. One is

$$f(t) \rightleftharpoons F(s)$$

Bracewell advocated the use of

$$F(s) \supset f(t)$$

and Gray and Goodman also use it. I hate it, personally.

A warning on definitions Our definition of the Fourier transform is a standard one, but it's not the only one. The question is where to put the 2π: in the exponential, as we have done; or perhaps as a factor out front; or perhaps left out completely. There's also a question of which is the Fourier transform and which is the inverse, i.e., which gets the minus sign in the exponential. All of the various conventions are in day-to-day use in the professions, and I only mention this now because when you're talking with a friend over drinks about the Fourier transform, be sure you both know which conventions are being followed. I'd hate to see that kind of misunderstanding get in the way of a beautiful friendship.

Following the helpful summary provided by T. W. Körner in his book *Fourier Analysis*, I will summarize the many irritating variations. To be general, let's write

$$\mathcal{F}f(s) = \frac{1}{A}\int_{-\infty}^{\infty} e^{iBst} f(t)\, dt\,.$$

The choices that are found in practice are

$$\begin{array}{ll} A = \sqrt{2\pi} & B = \pm 1 \\ A = 1 & B = \pm 2\pi \\ A = 1 & B = \pm 1 \end{array}$$

The definition we've chosen has $A = 1$ and $B = -2\pi$.

Happy hunting and good luck.

2.2 Getting to Know Your Fourier Transform

In one way, at least, our study of the Fourier transform will run the same course as your study of calculus. When you learned calculus it was necessary to learn the derivative and integral formulas for specific functions and types of functions (powers, exponentials, trig functions), and also to learn the general principles and rules of differentiation and integration that allow you to work with combinations of functions (product rule, chain rule, inverse functions). It will be the same thing for us now. We'll need to have a storehouse of specific functions and their transforms that we can call on, and we'll need to develop general principles and results on how the Fourier transform operates.

2.2.1 Examples

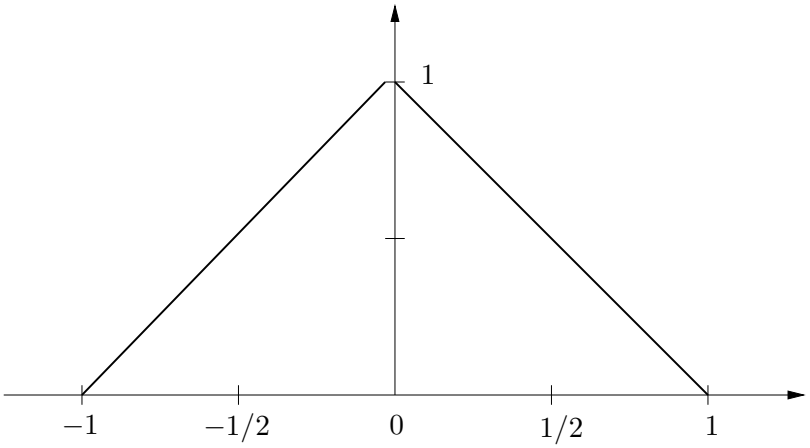
We've already seen the example

$$\widehat{\Pi} = \operatorname{sinc} \quad \text{or} \mathcal{F}\Pi(s) = \operatorname{sinc} s$$

using the $\mathcal{F}$ notation. Let's do a few more examples.

The triangle function Consider next the "triangle function", defined by

$$\Lambda(x) = \begin{cases} 1 - |x| & |x| \le 1 \\ 0 & \text{otherwise} \end{cases}$$

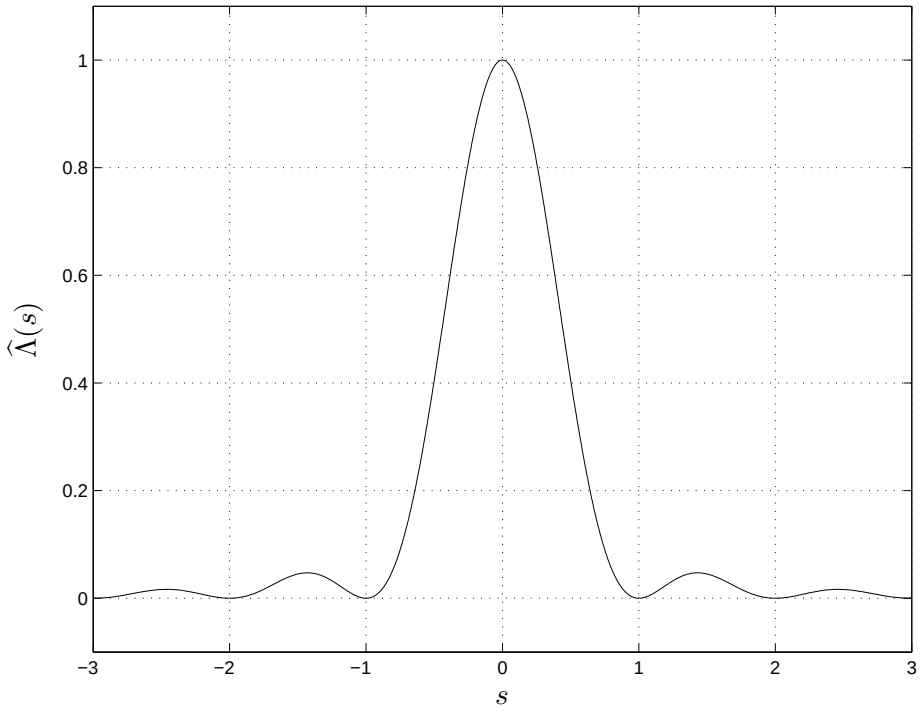


For the Fourier transform we compute (using integration by parts, and the factoring trick for the sine function):

$$\begin{aligned}
\mathcal{F}\Lambda(s) &= \int_{-\infty}^{\infty} \Lambda(x) e^{-2\pi i s x}\, dx = \int_{-1}^{0} (1+x) e^{-2\pi i s x}\, dx + \int_{0}^{1} (1-x) e^{-2\pi i s x}\, dx \\
&= \left(\frac{1 + 2i\pi s}{4\pi^2 s^2} - \frac{e^{2\pi i s}}{4\pi^2 s^2} \right) - \left(\frac{2i\pi s - 1}{4\pi^2 s^2} + \frac{e^{-2\pi i s}}{4\pi^2 s^2} \right) \\
&= -\frac{e^{-2\pi i s}(e^{2\pi i s} - 1)^2}{4\pi^2 s^2} = -\frac{e^{-2\pi i s}(e^{\pi i s}(e^{\pi i s} - e^{-\pi i s}))^2}{4\pi^2 s^2} \\
&= -\frac{e^{-2\pi i s} e^{2\pi i s} (2i)^2 \sin^2 \pi s}{4\pi^2 s^2} = \left(\frac{\sin \pi s}{\pi s} \right)^2 = \operatorname{sinc}^2 s.
\end{aligned}$$

It's no accident that the Fourier transform of the triangle function turns out to be the square of the Fourier transform of the rect function. It has to do with convolution, an operation we have seen for Fourier series and will see anew for Fourier transforms in the next chapter.

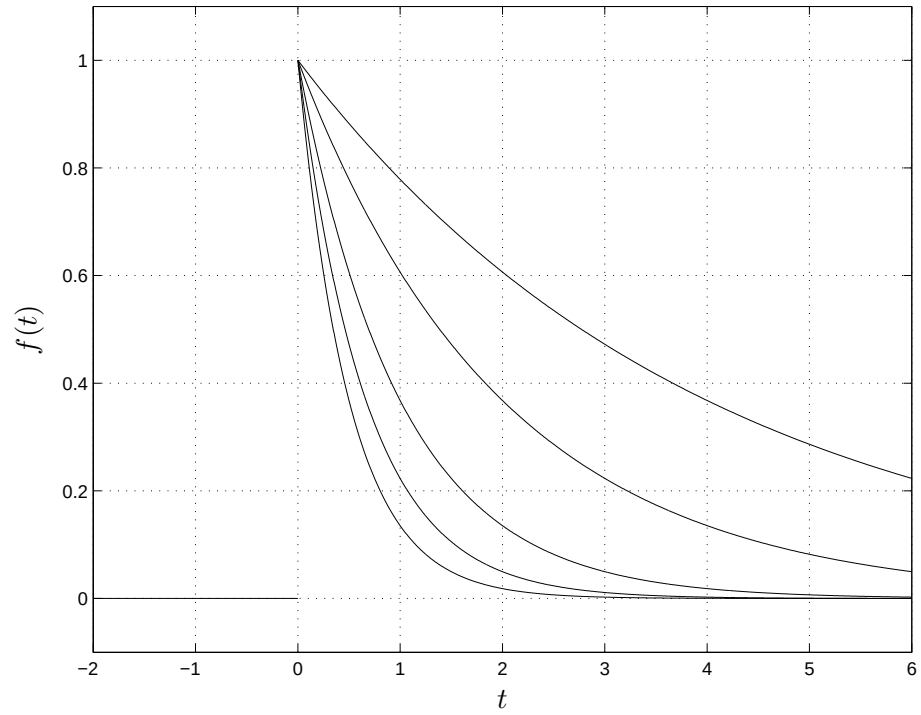
The graph of $\operatorname{sinc}^2 s$ looks like:



The exponential decay Another commonly occurring function is the (one-sided) exponential decay, defined by

$$f(t) = \begin{cases} 0 & t \leq 0 \\ e^{-at} & t > 0 \end{cases}$$

where a is a positive constant. This function models a signal that is zero, switched on, and then decays exponentially. Here are graphs for $a = 2$, 1.5, 1.0, 0.5, 0.25.



Which is which? If you can't say, see the discussion on scaling the independent variable at the end of this section.

Back to the exponential decay, we can calculate its Fourier transform directly.

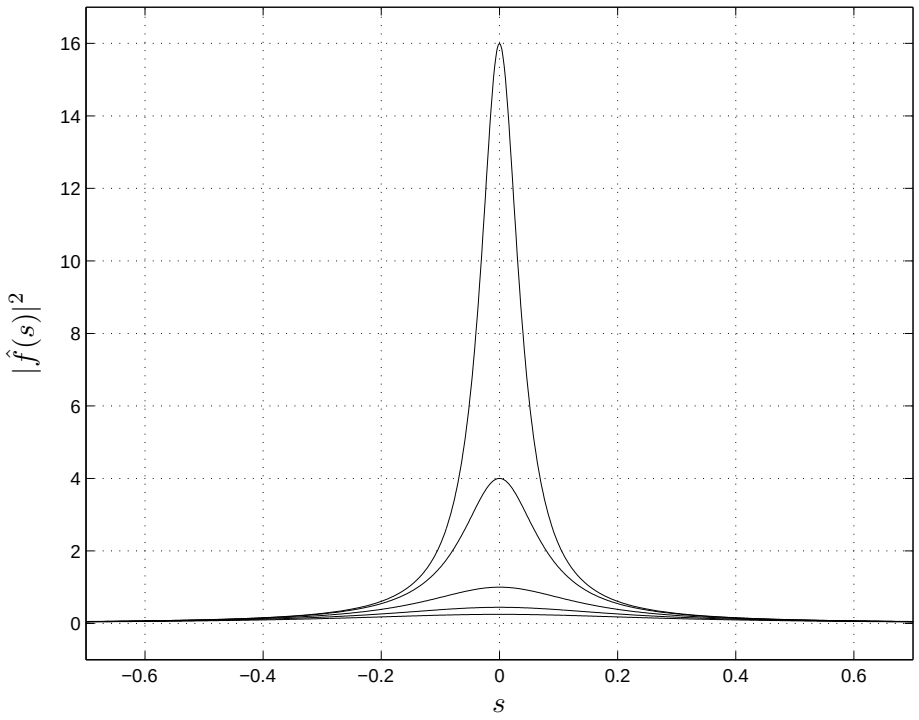
$$\begin{aligned}\mathcal{F}f(s) &= \int_0^\infty e^{-2\pi ist}e^{-at}\,dt = \int_0^\infty e^{-2\pi ist-at}\,dt \\ &= \int_0^\infty e^{(-2\pi is-a)t}\,dt = \left[\frac{e^{(-2\pi is-a)t}}{-2\pi is-a}\right]_{t=0}^{t=\infty} \\ &= \frac{e^{(-2\pi is)t}}{-2\pi is-a}e^{-at}\bigg|_{t=\infty} - \frac{e^{(-2\pi is-a)t}}{-2\pi is-a}\bigg|_{t=0} = \frac{1}{2\pi is+a}\end{aligned}$$

In this case, unlike the results for the rect function and the triangle function, the Fourier transform is complex. The fact that $\mathcal{F}\Pi(s)$ and $\mathcal{F}\Lambda(s)$ are real is because $\Pi(x)$ and $\Lambda(x)$ are even functions; we'll go over this shortly. There is no such symmetry for the exponential decay.

The power spectrum of the exponential decay is

$$|\mathcal{F}f(s)|^2 = \frac{1}{|2\pi is+a|^2} = \frac{1}{a^2+4\pi^2s^2}\,.$$

Here are graphs of this function for the same values of a as in the graphs of the exponential decay function.



Which is which? You'll soon learn to spot that immediately, relative to the pictures in the time domain, and it's an important issue. Also note that $|\mathcal{F}f(s)|^2$ *is* an even function of s even though $\mathcal{F}f(s)$ is not. We'll see why later. The shape of $|\mathcal{F}f(s)|^2$ is that of a "bell curve", though this is *not* Gaussian, a function we'll discuss just below. The curve is known as a *Lorenz profile* and comes up in analyzing the transition probabilities and lifetime of the excited state in atoms.

How does the graph of $f(ax)$ compare with the graph of $f(x)$? Let me remind you of some elementary lore on scaling the independent variable in a function and how scaling affects its graph. The

question is how the graph of $f(ax)$ compares with the graph of $f(x)$ when $0 < a < 1$ and when $a > 1$; I'm talking about any generic function $f(x)$ here. This is very simple, especially compared to what we've done and what we're going to do, but you'll want it at your fingertips and *everyone* has to think about it for a few seconds. Here's how to spend those few seconds.

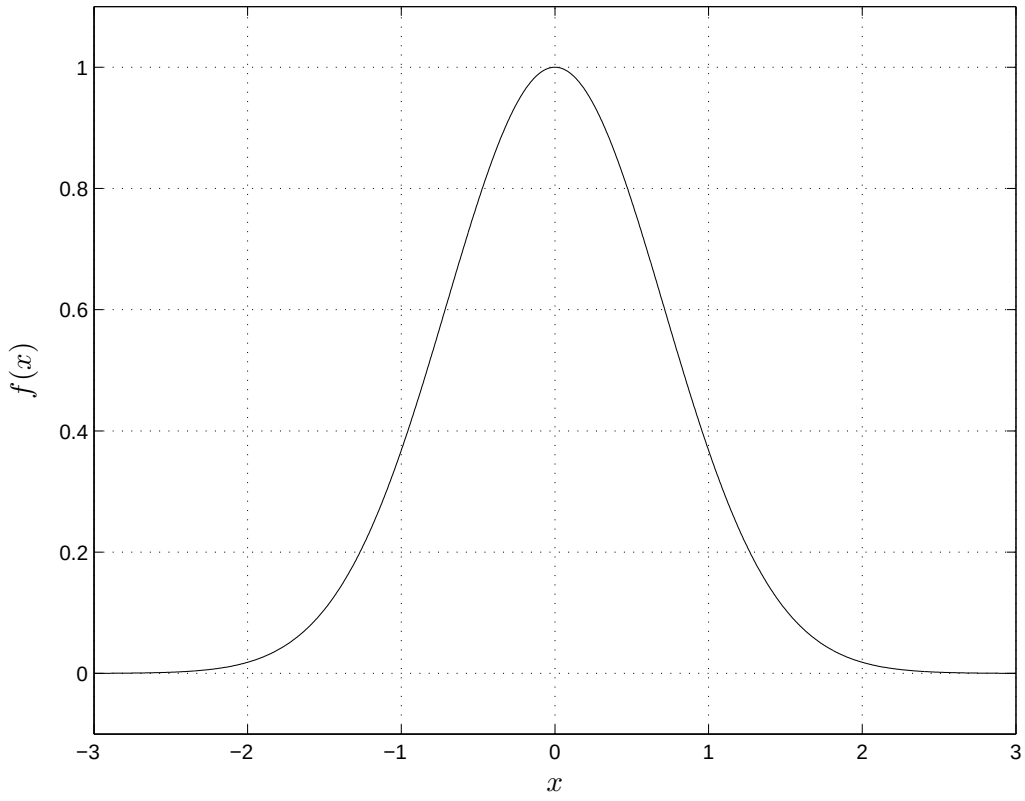
Consider, for example, the graph of $f(2x)$. The graph of $f(2x)$, compared with the graph of $f(x)$, is squeezed. Why? Think about what happens when you plot the graph of $f(2x)$ over, say, $-1 \leq x \leq 1$. When x goes from -1 to 1, $2x$ goes from -2 to 2, so while you're plotting $f(2x)$ over the interval from -1 to 1 you have to compute the values of $f(x)$ from -2 to 2. That's more of the function in less space, as it were, so the graph of $f(2x)$ is a squeezed version of the graph of $f(x)$. Clear?

Similar reasoning shows that the graph of $f(x/2)$ is stretched. If x goes from -1 to 1 then $x/2$ goes from $-1/2$ to $1/2$, so while you're plotting $f(x/2)$ over the interval -1 to 1 you have to compute the values of $f(x)$ from $-1/2$ to $1/2$. That's less of the function in more space, so the graph of $f(x/2)$ is a stretched version of the graph of $f(x)$.

2.2.2 For Whom the Bell Curve Tolls

Let's next consider the Gaussian function and its Fourier transform. We'll need this for many examples and problems. This function, the famous "bell shaped curve", was used by Gauss for various statistical problems. It has some striking properties with respect to the Fourier transform which, on the one hand, give it a special role within Fourier analysis, and on the other hand allow Fourier methods to be applied to other areas where the function comes up. We'll see an application to probability and statistics in Chapter 3.

The "basic Gaussian" is $f(x) = e^{-x^2}$. The shape of the graph is familiar to you.



For various applications one throws in extra factors to modify particular properties of the function. We'll

do this too, and there's not a complete agreement on what's best. There is an agreement that before anything else happens, one has to know the amazing equation[3]

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}.$$

Now, the function $f(x) = e^{-x^2}$ does not have an elementary antiderivative, so this integral cannot be found directly by an appeal to the Fundamental Theorem of Calculus. The fact that it *can* be evaluated exactly is one of the most famous tricks in mathematics. It's due to Euler, and you shouldn't go through life not having seen it. And even if you have seen it, it's worth seeing again; see the discussion following this section.

The Fourier transform of a Gaussian In whatever subject it's applied, it seems always to be useful to normalize the Gaussian so that the total area is 1. This can be done in several ways, but for Fourier analysis the best choice, as we shall see, is

$$f(x) = e^{-\pi x^2}\,.$$

You can check using the result for the integral of e^{-x^2} that

$$\int_{-\infty}^{\infty} e^{-\pi x^2}\,dx = 1\,.$$

Let's compute the Fourier transform

$$\mathcal{F}f(s) = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i s x}\,dx\,.$$

Differentiate with respect to s:

$$\frac{d}{ds}\mathcal{F}f(s) = \int_{-\infty}^{\infty} e^{-\pi x^2}(-2\pi i x) e^{-2\pi i s x}\,dx\,.$$

This is set up perfectly for an integration by parts, where $dv = -2\pi i x e^{-\pi x^2} dx$ and $u = e^{-2\pi i s x}$. Then $v = i e^{-\pi x^2}$, and evaluating the product uv at the limits $\pm\infty$ gives 0. Thus

$$\begin{aligned}\frac{d}{ds}\mathcal{F}f(s) &= -\int_{-\infty}^{\infty} i e^{-\pi x^2}(-2\pi i s) e^{-2\pi i s x}\,dx \\ &= -2\pi s \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i s x}\,dx \\ &= -2\pi s \mathcal{F}f(s)\end{aligned}$$

So $\mathcal{F}f(s)$ satisfies the simple differential equation

$$\frac{d}{ds}\mathcal{F}f(s) = -2\pi s \mathcal{F}f(s)$$

whose unique solution, incorporating the initial condition, is

$$\mathcal{F}f(s) = \mathcal{F}f(0) e^{-\pi s^2}\,.$$

[3] Speaking of this equation, William Thomson, after he became Lord Kelvin, said: "A mathematician is one to whom *that* is as obvious as that twice two makes four is to you." What a ridiculous statement.

But

$$\mathcal{F}f(0) = \int_{-\infty}^{\infty} e^{-\pi x^2}\, dx = 1\,.$$

Hence

$$\mathcal{F}f(s) = e^{-\pi s^2}\,.$$

We have found the remarkable fact that the Gaussian $f(x) = e^{-\pi x^2}$ is its own Fourier transform!

Evaluation of the Gaussian Integral We want to evaluate

$$I = \int_{-\infty}^{\infty} e^{-x^2}\, dx\,.$$

It doesn't matter what we call the variable of integration, so we can also write the integral as

$$I = \int_{-\infty}^{\infty} e^{-y^2}\, dy\,.$$

Therefore

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\, dy\right).$$

Because the variables aren't "coupled" here we can combine this into a double integral[4]

$$\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right) e^{-y^2} dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dx\, dy\,.$$

Now we make a change of variables, introducing polar coordinates, (r, θ). First, what about the limits of integration? To let both x and y range from $-\infty$ to ∞ is to describe the entire plane, and to describe the entire plane in polar coordinates is to let r go from 0 to ∞ and θ go from 0 to 2π. Next, $e^{-(x^2+y^2)}$ becomes e^{-r^2} and the area element $dx\, dy$ becomes $r\, dr\, d\theta$. It's the extra factor of r in the area element that makes all the difference. With the change to polar coordinates we have

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dx\, dy = \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\, r\, dr\, d\theta$$

Because of the factor r, the inner integral can be done directly:

$$\int_0^{\infty} e^{-r^2} r\, dr = -\tfrac{1}{2} e^{-r^2}\Big]_0^{\infty} = \tfrac{1}{2}\,.$$

The double integral then reduces to

$$I^2 = \int_0^{2\pi} \tfrac{1}{2}\, d\theta = \pi\,,$$

whence

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = I = \sqrt{\pi}\,.$$

Wonderful.

[4] We will see the same sort of thing when we work with the product of two Fourier transforms on our way to defining convolution in the next chapter.

2.2.3 General Properties and Formulas

We've started to build a storehouse of specific transforms. Let's now proceed along the other path awhile and develop some general properties. For this discussion — and indeed for much of our work over the next few lectures — we are going to abandon all worries about transforms existing, integrals converging, and whatever other worries you might be carrying. Relax and enjoy the ride.

2.2.4 Fourier transform pairs and duality

One striking feature of the Fourier transform and the inverse Fourier transform is the symmetry between the two formulas, something you *don't* see for Fourier series. For Fourier series the coefficients are given by an integral (a transform of $f(t)$ into $\hat{f}(n)$), but the "inverse transform" is the series itself. The Fourier transforms $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the same except for the minus sign in the exponential.[5] In words, we can say that if you replace s by $-s$ in the formula for the Fourier transform then you're taking the inverse Fourier transform. Likewise, if you replace t by $-t$ in the formula for the inverse Fourier transform then you're taking the Fourier transform. That is

$$\mathcal{F}f(-s) = \int_{-\infty}^{\infty} e^{-2\pi i(-s)t} f(t)\, dt = \int_{-\infty}^{\infty} e^{2\pi ist} f(t)\, dt = \mathcal{F}^{-1}f(s)$$
$$\mathcal{F}^{-1}f(-t) = \int_{-\infty}^{\infty} e^{2\pi is(-t)} f(s)\, ds = \int_{-\infty}^{\infty} e^{-2\pi ist} f(s)\, ds = \mathcal{F}f(t)$$

This might be a little confusing because you generally want to think of the two variables, s and t, as somehow associated with separate and different domains, one domain for the forward transform and one for the inverse transform, one for time and one for frequency, while in each of these formulas one variable is used in both domains. You have to get over this kind of confusion, because it's going to come up again. Think purely in terms of the math: The transform is an operation on a function that produces a new function. To write down the formula I have to evaluate the transform at a variable, but it's only a variable and it doesn't matter what I call it as long as I keep its role in the formula straight.

Also be observant what the notation in the formula says and, just as important, what it doesn't say. The first formula, for example, says what happens when you *first* take the Fourier transform of f and *then* evaluate it at $-s$, it's *not* a formula for $\mathcal{F}(f(-s))$ as in "first change s to $-s$ in the formula for f and then take the transform". I could have written the first displayed equation as $(\mathcal{F}f)(-s) = \mathcal{F}^{-1}f(s)$, with an extra parentheses around the $\mathcal{F}f$ to emphasize this, but I thought that looked too clumsy. Just be careful, please.

The equations

$$\mathcal{F}f(-s) = \mathcal{F}^{-1}f(s)$$
$$\mathcal{F}^{-1}f(-t) = \mathcal{F}f(t)$$

[5] Here's the reason that the formulas for the Fourier transform and its inverse appear so symmetric; it's quite a deep mathematical fact. As the general theory goes, if the original function is defined on a group then the transform (also defined in generality) is defined on the "dual group", which I won't define for you here. In the case of Fourier series the function is periodic, and so its natural domain is the circle (think of the circle as $[0, 1]$ with the endpoints identified). It turns out that the dual of the circle group is the integers, and that's why $\hat{f}$ is evaluated at integers n. It also turns out that when the group is $\mathbf{R}$ the dual group is again $\mathbf{R}$. Thus the Fourier transform of a function defined on $\mathbf{R}$ is itself defined on $\mathbf{R}$. Working through the general definitions of the Fourier transform and its inverse in this case produces the symmetric result that we have before us. Kick that one around over dinner some night.

are sometimes referred to as the "duality" property of the transforms. One also says that "the Fourier transform pair f and $\mathcal{F}f$ are related by duality", meaning exactly these relations. They look like different statements but you can get from one to the other. We'll set this up a little differently in the next section.

Here's an example of how duality is used. We know that

$$\mathcal{F}\Pi = \text{sinc}$$

and hence that

$$\mathcal{F}^{-1}\,\text{sinc} = \Pi\,.$$

By "duality" we can find $\mathcal{F}\,\text{sinc}$:

$$\mathcal{F}\,\text{sinc}(t) = \mathcal{F}^{-1}\,\text{sinc}(-t) = \Pi(-t)\,.$$

(Troubled by the variables? Remember, the left hand side is $(\mathcal{F}\,\text{sinc})(t)$.) Now with the *additional* knowledge that Π is an even function — $\Pi(-t) = \Pi(t)$ — we can conclude that

$$\mathcal{F}\,\text{sinc} = \Pi\,.$$

Let's apply the same argument to find $\mathcal{F}\,\text{sinc}^2$. Recall that Λ is the triangle function. We know that

$$\mathcal{F}\Lambda = \text{sinc}^2$$

and so

$$\mathcal{F}^{-1}\text{sinc}^2 = \Lambda\,.$$

But then

$$\mathcal{F}\text{sinc}^2(t) = (\mathcal{F}^{-1}\text{sinc}^2)(-t) = \Lambda(-t)$$

and since Λ is even,

$$\mathcal{F}\text{sinc}^2 = \Lambda\,.$$

Duality and reversed signals There's a slightly different take on duality that I prefer because it suppresses the variables and so I find it easier to remember. Starting with a signal $f(t)$ define the *reversed signal* f^- by

$$f^-(t) = f(-t)\,.$$

Note that a double reversal gives back the original signal,

$$(f^-)^- = f\,.$$

Note also that the conditions defining when a function is even or odd are easy to write in terms of the reversed signals:

$$\begin{aligned} &f \text{ is even if } f^- = f \\ &f \text{ is odd if } f^- = -f \end{aligned}$$

In words, a signal is even if reversing the signal doesn't change it, and a signal is odd if reversing the signal changes the sign. We'll pick up on this in the next section.

Simple enough — to reverse the signal is just to reverse the time. This is a general operation, of course, whatever the nature of the signal and whether or not the variable is time. Using this notation we can rewrite the first duality equation, $\mathcal{F}f(-s) = \mathcal{F}^{-1}f(s)$, as

$$(\mathcal{F}f)^- = \mathcal{F}^{-1}f$$

and we can rewrite the second duality equation, $\mathcal{F}^{-1}f(-t) = \mathcal{F}f(t)$, as

$$(\mathcal{F}^{-1}f)^- = \mathcal{F}f\,.$$

This makes it very clear that the two equations are saying the same thing. One is just the "reverse" of the other.

Furthermore, using this notation the result $\mathcal{F}\,\mathrm{sinc} = \Pi$, for example, goes a little more quickly:

$$\mathcal{F}\,\mathrm{sinc} = (\mathcal{F}^{-1}\,\mathrm{sinc})^- = \Pi^- = \Pi\,.$$

Likewise

$$\mathcal{F}\mathrm{sinc}^2 = (\mathcal{F}^{-1}\mathrm{sinc}^2)^- = \Lambda^- = \Lambda\,.$$

A natural variation on the preceding duality results is to ask what happens with $\mathcal{F}f^-$, the Fourier transform of the reversed signal. Let's work this out. By definition,

$$\mathcal{F}f^-(s) = \int_{-\infty}^{\infty} e^{-2\pi ist} f^-(t)\,dt = \int_{-\infty}^{\infty} e^{-2\pi ist} f(-t)\,dt\,.$$

There's only one thing to do at this point, and we'll be doing it a lot: make a change of variable in the integral. Let $u = -t$ so that $du = -dt$, or $dt = -du$. Then as t goes from $-\infty$ to ∞ the variable $u = -t$ goes from ∞ to $-\infty$ and we have

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-2\pi ist} f(-t)\,dt &= \int_{\infty}^{-\infty} e^{-2\pi is(-u)} f(u)\,(-du)\\ &= \int_{-\infty}^{\infty} e^{2\pi isu} f(u)\,du \quad \text{(the minus sign on the } du \text{ flips the limits back)}\\ &= \mathcal{F}^{-1}f(s)\end{aligned}$$

Thus, quite neatly,

$$\mathcal{F}f^- = \mathcal{F}^{-1}f$$

Even more neatly, if we now substitute $\mathcal{F}^{-1}f = (\mathcal{F}f)^-$ from earlier we have

$$\mathcal{F}f^- = (\mathcal{F}f)^-\,.$$

Note carefully where the parentheses are here. In words, the Fourier transform of the reversed signal is the reversed Fourier transform of the signal. That one I can remember.

To finish off these questions, we have to know what happens to $\mathcal{F}^{-1}f^-$. But we don't have to do a separate calculation here. Using our earlier duality result,

$$\mathcal{F}^{-1}f^- = (\mathcal{F}f^-)^- = (\mathcal{F}^{-1}f)^-\,.$$

In words, the inverse Fourier transform of the reversed signal is the reversed inverse Fourier transform of the signal. We can also take this one step farther and get back to $\mathcal{F}^{-1}f^- = \mathcal{F}f$

And so, the whole list of duality relations really boils down to just two:

$$\begin{aligned}\mathcal{F}f &= (\mathcal{F}^{-1}f)^-\\ \mathcal{F}f^- &= \mathcal{F}^{-1}f\end{aligned}$$

Learn these. Derive all others.

Here's one more:

$$\mathcal{F}(\mathcal{F}f)(s) = f(-s) \quad \text{or} \quad \mathcal{F}(\mathcal{F}f) = f^- \quad \text{without the variable.}$$

This identity is somewhat interesting in itself, as a variant of Fourier inversion. You can check it directly from the integral definitions, or from our earlier duality results.[6] Of course then also

$$\mathcal{F}(\mathcal{F}f^-) = f\,.$$

2.2.5 Even and odd symmetries and the Fourier transform

We've already had a number of occasions to use even and odd symmetries of functions. In the case of real-valued functions the conditions have obvious interpretations in terms of the symmetries of the graphs; the graph of an even function is symmetric about the y-axis and the graph of an odd function is symmetric through the origin. The (algebraic) definitions of even and odd apply to complex-valued as well as to real-valued functions, however, though the geometric picture is lacking when the function is complex-valued because we can't draw the graph. A function can be even, odd, or neither, but it can't be both unless it's identically zero.

How are symmetries of a function reflected in properties of its Fourier transform? I won't give a complete accounting, but here are a few important cases.

- If $f(x)$ is even or odd, respectively, then so is its Fourier transform.

Working with reversed signals, we have to show that $(\mathcal{F}f)^- = \mathcal{F}f$ if f is even and $(\mathcal{F}f)^- = -\mathcal{F}f$ if f is odd. It's lighting fast using the equations that we derived, above:

$$(\mathcal{F}f)^- = \mathcal{F}f^- = \begin{cases} \mathcal{F}f, & \text{if } f \text{ is even} \\ \mathcal{F}(-f) = -\mathcal{F}f & \text{if } f \text{ is odd} \end{cases}$$

Because the Fourier transform of a function is complex valued there are other symmetries we can consider for $\mathcal{F}f(s)$, namely what happens under complex conjugation.

- If $f(t)$ is real-valued then $(\mathcal{F}f)^- = \overline{\mathcal{F}f}$ and $\mathcal{F}(f^-) = \overline{\mathcal{F}f}$.

This is analogous to the conjugate symmetry property possessed by the Fourier coefficients for a real-valued periodic function. The derivation is essentially the same as it was for Fourier coefficients, but it may be helpful to repeat it for practice and to see the similarities.

$$\begin{aligned}
(\mathcal{F}f)^-(s) &= \mathcal{F}^{-1}f(s) \quad \text{(by duality)} \\
&= \int_{-\infty}^{\infty} e^{2\pi i s t} f(t)\,dt \\
&= \overline{\left\{ \int_{-\infty}^{\infty} e^{-2\pi i s t} f(t)\,dt \right\}} \quad (\overline{f(t)} = f(t) \text{ since } f(t) \text{ is real}) \\
&= \overline{\mathcal{F}f(s)}
\end{aligned}$$

[6] And you can then also then check that $\mathcal{F}(\mathcal{F}(\mathcal{F}(\mathcal{F}f)))(s) = f(s)$, i.e., $\mathcal{F}^4$ is the identity transformation. Some people attach mystical significance to this fact.

We can refine this if the function $f(t)$ itself has symmetry. For example, combining the last two results and remembering that a complex number is real if it's equal to its conjugate and is purely imaginary if it's equal to minus its conjugate, we have:

- If f is real valued and even then its Fourier transform is even and real valued.
- If f is real valued and odd function then its Fourier transform is odd and purely imaginary.

We saw this first point in action for Fourier transform of the rect function $\Pi(t)$ and for the triangle function $\Lambda(t)$. Both functions are even and their Fourier transforms, sinc and sinc^2, respectively, are even and real. Good thing it worked out that way.

2.2.6 Linearity

One of the simplest and most frequently invoked properties of the Fourier transform is that it is linear (operating on functions). This means:

$$\begin{aligned}\mathcal{F}(f+g)(s) &= \mathcal{F}f(s) + \mathcal{F}g(s)\\ \mathcal{F}(\alpha f)(s) &= \alpha\mathcal{F}f(s) \quad \text{for any number } \alpha \text{ (real or complex).}\end{aligned}$$

The linearity properties are easy to check from the corresponding properties for integrals, for example:

$$\begin{aligned}\mathcal{F}(f+g)(s) &= \int_{-\infty}^{\infty} (f(x)+g(x))e^{-2\pi isx}\,dx\\ &= \int_{-\infty}^{\infty} f(x)e^{-2\pi isx}\,dx + \int_{-\infty}^{\infty} g(x)e^{-2\pi isx}\,dx = \mathcal{F}f(s) + \mathcal{F}g(s)\,.\end{aligned}$$

We used (without comment) the property on multiples when we wrote $\mathcal{F}(-f) = -\mathcal{F}f$ in talking about odd functions and their transforms. I bet it didn't bother you that we hadn't yet stated the property formally.

2.2.7 The shift theorem

A shift of the variable t (a delay in time) has a simple effect on the Fourier transform. We would expect the magnitude of the Fourier transform $|\mathcal{F}f(s)|$ to stay the same, since shifting the original signal in time should not change the energy at any point in the spectrum. Hence the only change should be a phase shift in $\mathcal{F}f(s)$, and that's exactly what happens.

To compute the Fourier transform of $f(t+b)$ for any constant b, we have

$$\begin{aligned}\int_{-\infty}^{\infty} f(t+b)e^{-2\pi ist}\,dt &= \int_{-\infty}^{\infty} f(u)e^{-2\pi is(u-b)}\,du\\ &\qquad \text{(substituting } u = t+b\text{; the limits still go from } -\infty \text{ to } \infty)\\ &= \int_{-\infty}^{\infty} f(u)e^{-2\pi isu}e^{2\pi isb}\,du\\ &= e^{2\pi isb}\int_{-\infty}^{\infty} f(u)e^{-2\pi isu}\,du = e^{2\pi isb}\hat{f}(s).\end{aligned}$$

The best notation to capture this property is probably the pair notation, $f \rightleftharpoons F$.[7] Thus:

- If $f(t) \rightleftharpoons F(s)$ then $f(t+b) \rightleftharpoons e^{2\pi isb}F(s)$.
  - A little more generally, $f(t \pm b) \rightleftharpoons e^{\pm 2\pi isb}F(s)$.

Notice that, as promised, the magnitude of the Fourier transform has not changed under a time shift because the factor out front has magnitude 1:

$$\left|e^{\pm 2\pi isb}F(s)\right| = \left|e^{\pm 2\pi isb}\right| \, |F(s)| = |F(s)| \, .$$

2.2.8 The stretch (similarity) theorem

How does the Fourier transform change if we stretch or shrink the variable in the time domain? More precisely, we want to know if we scale t to at what happens to the Fourier transform of $f(at)$. First suppose $a > 0$. Then

$$\begin{aligned}\int_{-\infty}^{\infty} f(at)e^{-2\pi ist}\,dt &= \int_{-\infty}^{\infty} f(u)e^{-2\pi is(u/a)}\frac{1}{a}\,du \\ &\qquad \text{(substituting } u = at\text{; the limits go the same way because } a > 0) \\ &= \frac{1}{a}\int_{-\infty}^{\infty} f(u)e^{-2\pi i(s/a)u}\,du = \frac{1}{a}\mathcal{F}f\left(\frac{s}{a}\right)\end{aligned}$$

If $a < 0$ the limits of integration are reversed when we make the substitution $u = ax$, and so the resulting transform is $(-1/a)\mathcal{F}f(s/a)$. Since $-a$ is positive when a is negative, we can combine the two cases and present the Stretch Theorem in its full glory:

- If $f(t) \rightleftharpoons F(s)$ then $f(at) \rightleftharpoons \dfrac{1}{|a|}F\left(\dfrac{s}{a}\right)$.

This is also sometimes called the *Similarity Theorem* because changing the variable from x to ax is a change of scale, also known as a similarity.

There's an important observation that goes with the stretch theorem. Let's take a to be positive, just to be definite. If a is large (bigger than 1, at least) then the graph of $f(at)$ is squeezed horizontally compared to $f(t)$. Something different is happening in the frequency domain, in fact in two ways. The Fourier transform is $(1/a)F(s/a)$. If a is large then $F(s/a)$ is *stretched out* compared to $F(s)$, rather than squeezed in. Furthermore, multiplying by $1/a$, since the transform is $(1/a)F(a/s)$, also squashes down the values of the transform.

The opposite happens if a is small (less than 1). In that case the graph of $f(at)$ is stretched out horizontally compared to $f(t)$, while the Fourier transform is compressed *horizontally* and stretched *vertically*. The phrase that's often used to describe this phenomenon is that a signal cannot be *localized* (meaning

[7] This is, however, an excellent opportunity to complain about notational matters. Writing $\mathcal{F}f(t+b)$ invites the same anxieties that some of us had when changing signs. What's being transformed? What's being plugged in? There's no room to write an s. The hat notation is even worse — there's no place for the s, again, and do you really want to write $\widehat{f(t+b)}$ with such a wide hat?

concentrated at a point) in both the time domain and the frequency domain. We will see more precise formulations of this principle.[8]

To sum up, a function stretched out in the time domain is squeezed in the frequency domain, and vice versa. This is somewhat analogous to what happens to the spectrum of a periodic function for long or short periods. Say the period is T, and recall that the points in the spectrum are spaced $1/T$ apart, a fact we've used several times. If T is large then it's fair to think of the function as spread out in the time domain — it goes a long time before repeating. But then since $1/T$ is small, the spectrum is squeezed. On the other hand, if T is small then the function is squeezed in the time domain — it goes only a short time before repeating — while the spectrum is spread out, since $1/T$ is large.

> **Careful here** In the discussion just above I tried not to talk in terms of properties of the *graph of the transform* — though you may have reflexively thought in those terms and I slipped into it a little — because the transform is generally *complex valued*. You do see this squeezing and spreading phenomenon geometrically by looking at the graphs of $f(t)$ in the time domain and the *magnitude* of the Fourier transform in the frequency domain.[9]

Example: The stretched rect Hardly a felicitous phrase, "stretched rect", but the function comes up often in applications. Let $p > 0$ and define

$$\Pi_p(t) = \begin{cases} 1 & |t| < p/2 \\ 0 & |t| \geq p/2 \end{cases}$$

Thus Π_p is a rect function of width p. We can find its Fourier transform by direct integration, but we can also find it by means of the stretch theorem if we observe that

$$\Pi_p(t) = \Pi(t/p)\,.$$

To see this, write down the definition of Π and follow through:

$$\Pi(t/p) = \begin{cases} 1 & |t/p| < 1/2 \\ 0 & |t/p| \geq 1/2 \end{cases} = \begin{cases} 1 & |t| < p/2 \\ 0 & |t| \geq p/2 \end{cases} = \Pi_p(t)\,.$$

Now since $\Pi(t) \rightleftharpoons \operatorname{sinc} s$, by the stretch theorem

$$\Pi(t/p) \rightleftharpoons p \operatorname{sinc} ps\,,$$

and so

$$\mathcal{F}\Pi_p(s) = p \operatorname{sinc} ps\,.$$

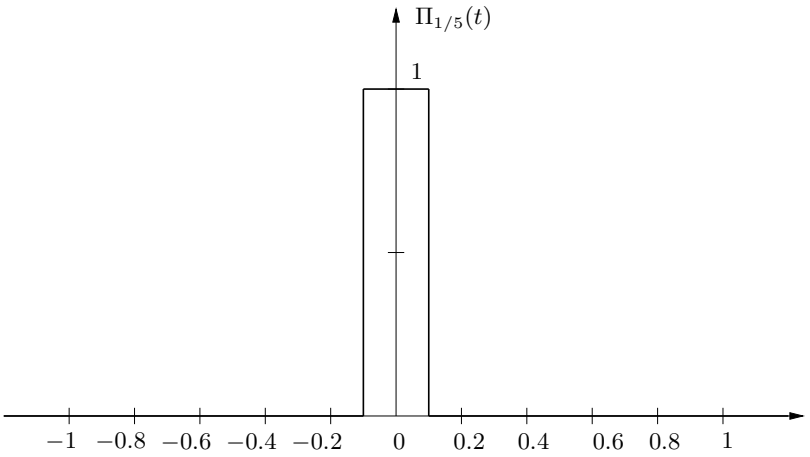
This is useful to know.

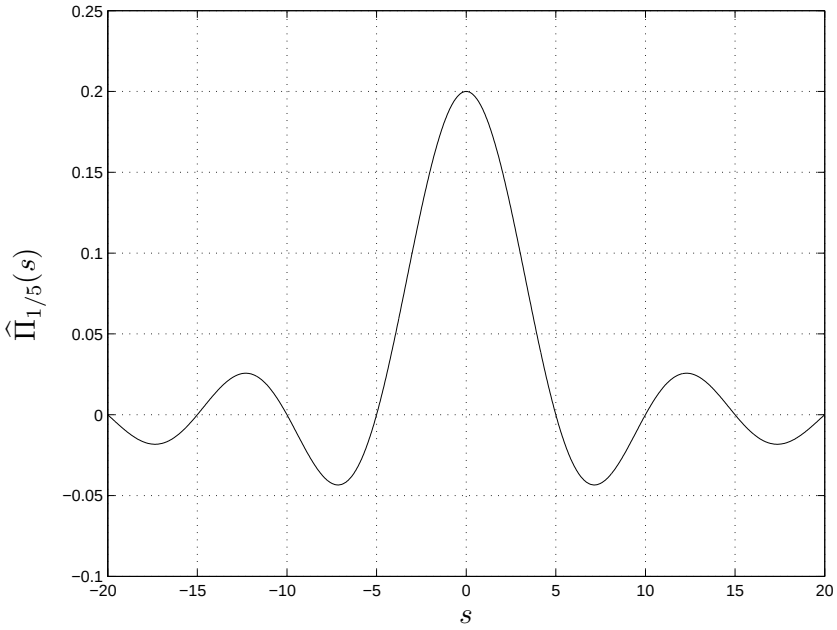
Here are plots of the Fourier transform pairs for $p = 1/5$ and $p = 5$, respectively. Note the scales on the axes.

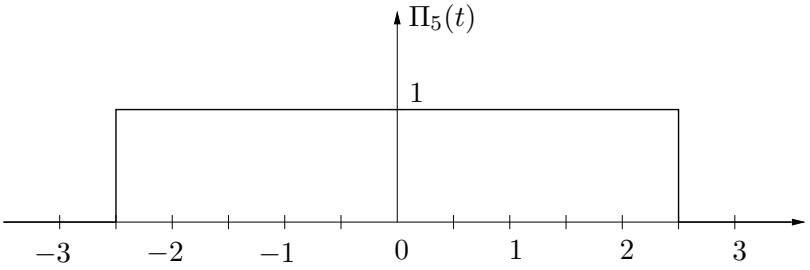
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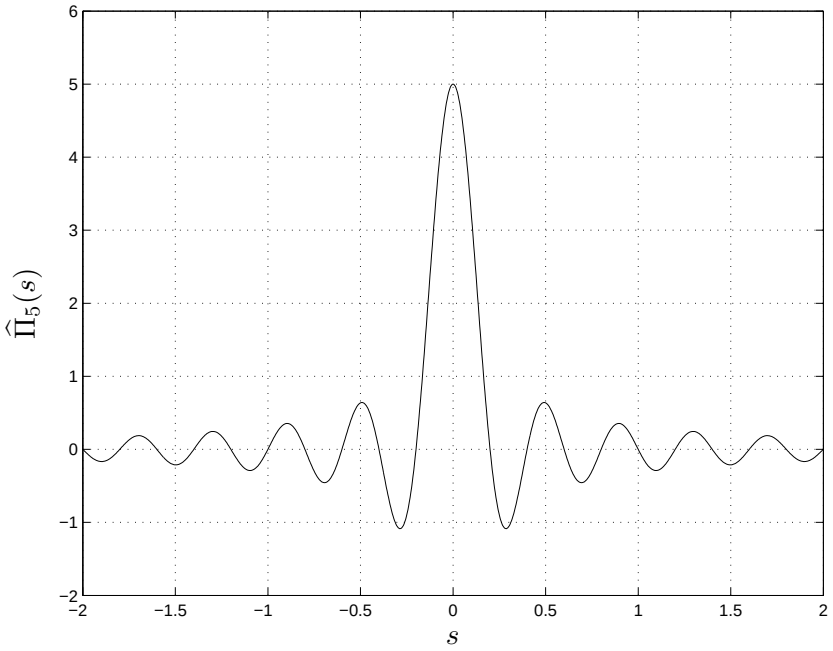
[8] In fact, the famous Heisenberg Uncertainty Principle in quantum mechanics is an example.

[9] We observed this for the one-sided exponential decay and its Fourier transform, and you should now go back to that example and match up the graphs of $|\mathcal{F}f|$ with the various values of the parameter.


$\Pi_{1/5}(t)$
1
−1 −0.8 −0.6 −0.4 −0.2 0 0.2 0.4 0.6 0.8 1


$\widehat{\Pi}_{1/5}(s)$
0.25
0.2
0.15
0.1
0.05
0
−0.05
−0.1
−20 −15 −10 −5 0 5 10 15 20
s


$\Pi_5(t)$
1
−3 −2 −1 0 1 2 3



2.2.9 Combining shifts and stretches

We can combine the shift theorem and the stretch theorem to find the Fourier transform of $f(ax+b)$, but it's a little involved.

Let's do an example first. It's easy to find the Fourier transform of $f(x) = \Pi((x-3)/2)$ by direct integration.

$$\begin{aligned} F(s) &= \int_2^4 e^{-2\pi isx}\, dx \\ &= -\frac{1}{2\pi is} e^{-2\pi isx}\Big]_{x=2}^{x=4} = -\frac{1}{2\pi is}(e^{-8\pi is} - e^{-4\pi is})\,. \end{aligned}$$

We can still bring the sinc function into this, but the factoring is a little trickier.

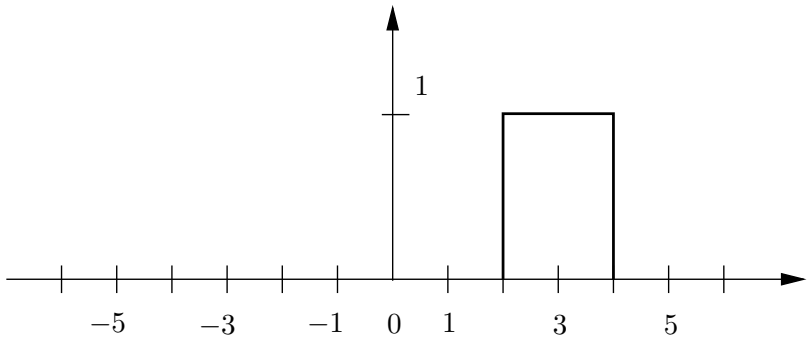
$$e^{-8\pi is} - e^{-4\pi is} = e^{-6\pi is}(e^{-2\pi is} - e^{2\pi is}) = e^{-6\pi is}(-2i)\sin 2\pi s\,.$$

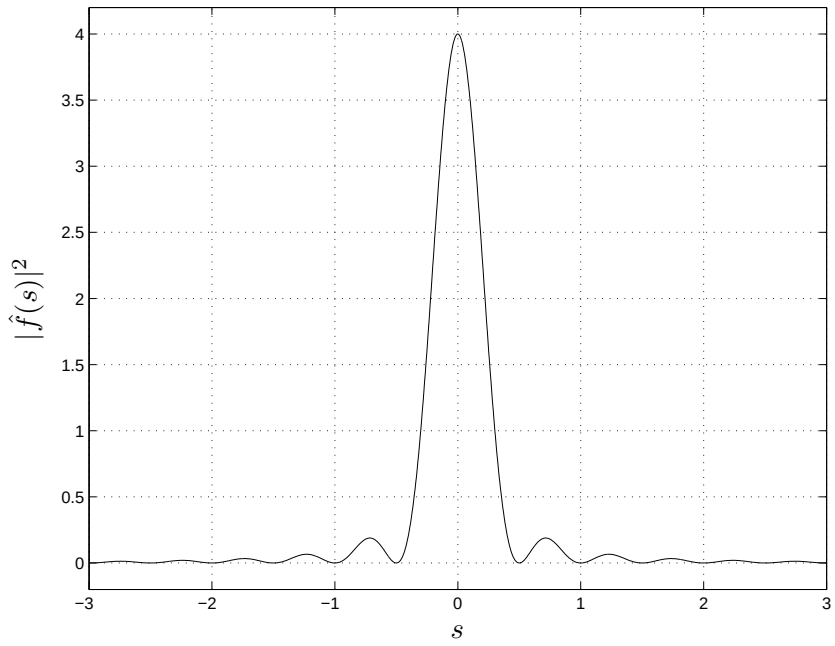
Plugging this into the above gives

$$F(s) = e^{-6\pi is}\frac{\sin 2\pi s}{\pi s} = 2e^{-6\pi is}\,\mathrm{sinc}\, 2s\,.$$

The Fourier transform has become complex — shifting the rect function has destroyed its symmetry.

Here's a plot of $\Pi((x-3)/2)$ and of $4\mathrm{sinc}^2 2s$, the square of the *magnitude* of its Fourier transform. Once again, looking at the latter gives you no information about the phases in the spectrum, only on the energies.





As an exercise you can establish the following general formula on how shifts and stretches combine:

- If $f(t) \rightleftharpoons F(s)$ then $f(at \pm b) = f\Big(a\Big(t \pm \dfrac{b}{a}\Big)\Big) \rightleftharpoons \dfrac{1}{|a|} e^{\pm 2\pi i s b/a} F\Big(\dfrac{s}{a}\Big)$.

Try this on $\Pi((x-3)/2) = \Pi\left(\frac{1}{2}x - \frac{3}{2}\right)$. With $a = 1/2$ and $b = -3/2$ we get

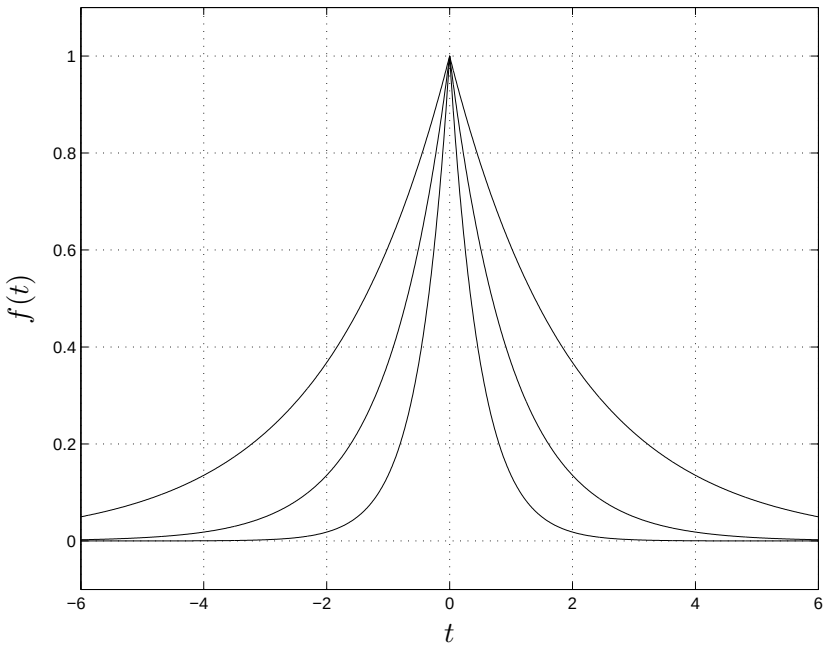
$$\mathcal{F}\left(\Pi\left(\tfrac{1}{2}x - \tfrac{3}{2}\right)\right) = 2e^{-6\pi i s}\widehat{\Pi}(2s) = 2e^{-6\pi i s}\,\mathrm{sinc}\,2s$$

just like before. Was there any doubt? (Note that I used the notation $\mathcal{F}$ here along with the hat notation. It's not ideal either, but it seemed like the best of a bad set of ways of writing the result.)

Example: two-sided exponential decay Here's an example of how you might combine the properties we've developed. Let's find the Fourier transform of the *two*-sided exponential decay

$$g(t) = e^{-a|t|}, \quad a \text{ a positive constant.}$$

Here are plots of $g(t)$ for $a = 0.5, 1, 2$. Match them!



We could find the transform directly — plugging into the formula for the Fourier transform would give us integrals we could do. However, we've already done half the work, so to speak, when we found the Fourier transform of the one-sided exponential decay. Recall that for

$$f(t) = \begin{cases} 0 & t < 0 \\ e^{-at} & t \geq 0 \end{cases} \quad \Rightarrow \quad F(s) = \hat{f}(s) = \frac{1}{2\pi i s + a}$$

and now realize

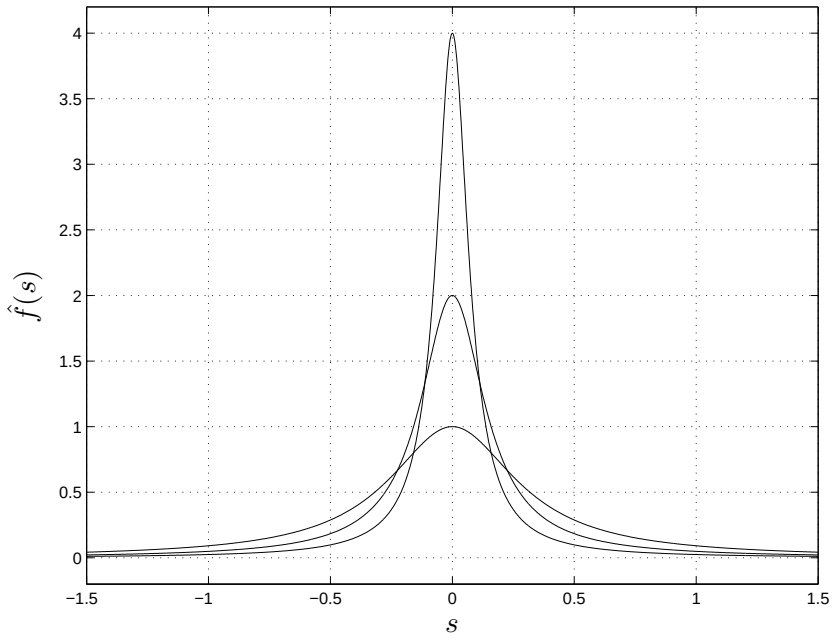
$$g(t) \quad \text{is almost equal to} \quad f(t) + f(-t)\,.$$

They agree except at the origin, where $g(0) = 1$ and $f(t)$ and $f(-t)$ are both one. But two functions which agree except for one point (or even finitely many points[10]) will clearly give the same result when integrated against $e^{-2\pi i s t}$. Therefore

$$\begin{aligned} G(s) = \mathcal{F}g(s) &= F(s) + F(-s) \\ &= \frac{1}{2\pi i s + a} + \frac{1}{-2\pi i s + a} = \frac{2a}{a^2 + 4\pi^2 s^2}\,. \end{aligned}$$

Note that $g(t)$ is even and $G(s)$ is real. These sorts of quick checks on correctness and consistency (evenness, oddness, real or purely imaginary, etc.) are useful when you're doing calculations. Here are plots of $G(s)$ for the $a = 0.5, 1, 2$. Match them!

[10] Or, more generally, sets of "measure zero"



In the future, we'll see an application of the two-sided exponential decay to solving a second order ordinary differential equation.

Example: Other Gaussians As mentioned, there are other ways of normalizing a Gaussian. For example, instead of $e^{-\pi x^2}$ we can take

$$g(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2} .$$

You might recognize this from applications to probability and statistics as the Gaussian with *mean* zero and *standard deviation* σ (or *variance* σ^2). The Gaussian with mean μ and standard deviation σ is the shifted version of this:

$$g(x, \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} .$$

Geometrically, σ is a measure of how peaked or spread out the curve is about the mean. In terms of the graph, the inflection points occur at $\mu \pm \sigma$; thus if σ is large the curve is spread out and if σ is small the curve is sharply peaked. The area under the graph is still 1.

The question for us is what happens to the Fourier transform when the Gaussian is modified in this way. This can be answered by our results on shifts and stretches, since that's all that's ever involved. Take the case of $\mu = 0$, for simplicity. To find the Fourier transform we can apply the similarity theorem: $f(ax) \rightleftharpoons (1/|a|)F(s/a)$. With $a = 1/\sigma\sqrt{2\pi}$ This gives

$$g(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2} \quad \Rightarrow \quad \hat{g}(s) = e^{-2\pi^2\sigma^2 s^2}$$

still a Gaussian, but not an exact replica of what we started with. Note that with $\mu = 0$ the Gaussian is even and the Fourier transform is real and even.

Chapter 3

Convolution

3.1 A $*$ is Born

How can we use one signal to modify another? Some of the properties of the Fourier transform that we have already derived can be thought of as addressing this question. The easiest is the result on additivity, according to which

$$\mathcal{F}(f+g) = \mathcal{F}f + \mathcal{F}g.$$

Adding the signal $g(t)$ to the signal $f(t)$ adds the amounts $\mathcal{F}g(s)$ to the frequency components $\mathcal{F}f(s)$. (Symmetrically, $f(t)$ modifies $g(t)$ in the same way.) The spectrum of $f+g$ may be more or less "complicated" than the spectrum of f and g alone, and it's an elementary operation in *both* the time domain and the frequency domain that produces or eliminates the complications. It's an operation that's also easily undone: See some frequencies you don't like in the spectrum (a bad buzz)? Then try adding something in or subtracting something out and see what the signal looks like.

We can view the question of using one signal to modify another in either the time domain or in the frequency domain, sometimes with equal ease and sometimes with one point of view preferred. We just looked at sums, what about products? The trivial case is multiplying by a constant, as in $\mathcal{F}(af)(s) = a\mathcal{F}f(s)$. The energies of the harmonics are all affected by the same amount, so, thinking of music for example, the signal sounds the same, only louder or softer. It's much less obvious how to scale the harmonics *separately.* That is, as a question "in the frequency domain", we ask:

> Is there some combination of the signals $f(t)$ and $g(t)$ so that in the frequency domain the Fourier transform is
>
> $$\mathcal{F}g(s)\mathcal{F}f(s)\,?$$
>
> In other words, in the time domain can we combine the signal $g(t)$ with the signal $f(t)$ so that the frequency components $\mathcal{F}f(s)$ of $f(t)$ are scaled by the frequency components $\mathcal{F}g(s)$ of $g(t)$? (Once again this is symmetric — we could say that the frequency components $\mathcal{F}g(s)$ are scaled by the frequency components $\mathcal{F}f(s)$.)

Let's check this out, and remember that the rigor police are off duty. No arrests will be made for unstated assumptions, divergent integrals, etc.

The product of the Fourier transforms of $f(t)$ and $g(t)$ is

$$\mathcal{F}g(s)\,\mathcal{F}f(s) = \int_{-\infty}^{\infty} e^{-2\pi ist} g(t)\,dt \int_{-\infty}^{\infty} e^{-2\pi isx} f(x)\,dx\,.$$

We used different variables of integration in the two integrals because we're going to combine the product into an iterated integral.[1]

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-2\pi ist} g(t)\,dt \int_{-\infty}^{\infty} e^{-2\pi isx} f(x)\,dx &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi ist} e^{-2\pi isx} g(t) f(x)\,dt\,dx \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi is(t+x)} g(t) f(x)\,dt\,dx \\ &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{-2\pi is(t+x)} g(t)\,dt\right) f(x)\,dx\end{aligned}$$

Now make the change of variable $u = t + x$ in the inner integral. Then $t = u - x$, $du = dt$, and the limits are the same. The result is

$$\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{-2\pi is(t+x)} g(t)\,dt\right) f(x)\,dx = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{-2\pi isu} g(u-x)\,du\right) f(x)\,dx$$

Next, switch the order of integration:

$$\begin{aligned}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{-2\pi isu} g(u-x)\,du\right) f(x)\,dx &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi isu} g(u-x) f(x)\,du\,dx \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi isu} g(u-x) f(x)\,dx\,du \\ &= \int_{-\infty}^{\infty} e^{-2\pi isu}\left(\int_{-\infty}^{\infty} g(u-x) f(x)\,dx\right) du\end{aligned}$$

Look at what's happened here. The inner integral is a function of u. Let's set it up on its own:

$$h(u) = \int_{-\infty}^{\infty} g(u-x) f(x)\,dx\,.$$

Then the outer integral produces the Fourier transform of h:

$$\int_{-\infty}^{\infty} e^{-2\pi isu}\left(\int_{-\infty}^{\infty} g(u-x) f(x)\,dx\right) du = \int_{-\infty}^{\infty} e^{-2\pi isu} h(u)\,du = \mathcal{F}h(s)$$

Switching the variable name for h from $h(u)$ to $h(t)$ (solely for psychological comfort), we have discovered that the signals $f(t)$ and $g(t)$ are combined into a signal

$$h(t) = \int_{-\infty}^{\infty} g(t-x) f(x)\,dx\,.$$

In other words,

$$\mathcal{F}h(s) = \mathcal{F}g(s)\mathcal{F}f(s)\,.$$

Remarkable.

We have solved our problem. The only thing to do is to realize what we've done and declare it to the world. We make the following definition:

[1] If you're uneasy with this (never mind issues of convergence) you might convince yourself that it's correct by working your way backwards from the double integral to the product of the two single integrals.

- **Convolution defined** The *convolution* of two functions $g(t)$ and $f(t)$ is the function

$$h(t) = \int_{-\infty}^{\infty} g(t-x)f(x)\,dx\,.$$

We use the notation

$$(g * f)(t) = \int_{-\infty}^{\infty} g(t-x)f(x)\,dx\,.$$

We can now proudly announce:

- **Convolution Theorem** $\mathcal{F}(g * f)(s) = \mathcal{F}g(s)\mathcal{F}f(s)$
  - In other notation: If $f(t) \rightleftharpoons F(s)$ and $g(t) \rightleftharpoons G(s)$ then $(g * f)(t) \rightleftharpoons G(s)F(s)$.
- In words: Convolution in the time domain corresponds to multiplication in the frequency domain.[2]

Recall that when we studied Fourier series, convolution came up in the form

$$(g * f)(t) = \int_{0}^{1} g(t-x)f(x)\,dx\,.$$

In that setting, for the integral to make sense, i.e., to be able to evaluate $g(t-x)$ at points outside the interval from 0 to 1, we had to assume that g was periodic. That's not an issue in the present setting, where we assume that $f(t)$ and $g(t)$ are defined for all t, so the factors in the integral

$$\int_{-\infty}^{\infty} g(t-x)f(x)\,dx$$

are defined everywhere. There may be questions to raise about whether the integral converges, and there are, but at least the setup makes sense.

Remark on notation, again It's common to see the people write the convolution as $g(t) * f(t)$, putting the variable t in each of g and f. There are times when that's OK, even sometimes preferable to introducing a lot of extra notation, but in general I think it's a bad idea because it can lead to all sorts of abuses and possible mistakes. For example, what's $g(2t) * f(t)$? If you plugged in too casually you might write this as the integral

$$\int_{-\infty}^{\infty} g(2t-x)f(x)\,dx\,.$$

That's wrong. The right answer in convolving $g(2t)$ and $f(t)$ is

$$\int_{-\infty}^{\infty} g(2(t-x))f(x)\,dx = \int_{-\infty}^{\infty} g(2t-2x)f(x)\,dx\,.$$

Make sure you understand why the first is wrong and second is right.[3]

[2] What we've just gone through is the same sort of thing we did when we "found" the formula for the Fourier coefficients for a periodic function. Remember the principle: First suppose the problem is solved and see what the answer must be. The second step, assuming the first one works, is to turn that solution into a definition and then announce to the world that you have solved your original problem based on your brilliant definition. Mathematicians, in particular, are very good at presenting their results and writing their books in this way — do step one in secret and tell the world only step two. It's extremely irritating.

[3] The way to be unambiguous about this is to say something like: "Let"s define $h(t) = g(2t)$, then $(h * f)(t) = \int_{-\infty}^{\infty} h(t-x)f(x)\,dx = \ldots$." I concede that this is too much of a hassle in most cases. Just be careful.

Let's see a quick application of our brilliant new discovery. As an exercise you can show (by hand) that

$$(\Pi * \Pi)(x) = \Lambda(x)$$

Recall that Λ is the triangle function. Applying the Convolution Theorem, we find that

$$\mathcal{F}\Lambda(s) = \mathcal{F}(\Pi * \Pi)(s) = \operatorname{sinc} s \cdot \operatorname{sinc} s = \operatorname{sinc}^2 s\,,$$

just like before. Was there any doubt?

Convolving in the frequency domain If you look at the argument for the convolution theorem $\mathcal{F}(g * f) = \mathcal{F}g \cdot \mathcal{F}f$, you'll see that we could have carried the whole thing out for the inverse Fourier transform, and given the symmetry between the Fourier transform and its inverse that's not surprising. That is, we also have

$$\mathcal{F}^{-1}(g * f) = \mathcal{F}^{-1}g \cdot \mathcal{F}^{-1}f\,.$$

What's more interesting, and doesn't follow without a little additional argument, is this:

$$\mathcal{F}(gf)(s) = (\mathcal{F}g * \mathcal{F}f)(s)\,.$$

In words:

- Multiplication in the time domain corresponds to convolution in the frequency domain.

Here's how the derivation goes. We'll need one of the duality formulas, the one that says

$$\mathcal{F}(\mathcal{F}f)(s) = f(-s) \quad \text{or} \quad \mathcal{F}(\mathcal{F}f) = f^- \quad \text{without the variable.}$$

To derive the identity $\mathcal{F}(gf) = \mathcal{F}g * \mathcal{F}f$, we write, for convenience, $h = \mathcal{F}f$ and $k = \mathcal{F}g$. Then we're to show

$$\mathcal{F}(gf) = k * h\,.$$

The one thing we know is how to take the Fourier transform of a convolution, so, in the present notation, $\mathcal{F}(k * h) = (\mathcal{F}k)(\mathcal{F}h)$. But now $\mathcal{F}k = \mathcal{F}\mathcal{F}g = g^-$, from the identity above, and likewise $\mathcal{F}h = \mathcal{F}\mathcal{F}f = f^-$. So $\mathcal{F}(k * h) = g^- f^- = (gf)^-$, or

$$gf = \mathcal{F}(k * h)^-\,.$$

Now, finally, take the Fourier transform of both sides of this last equation and appeal to the $\mathcal{F}\mathcal{F}$ identity again:

$$\mathcal{F}(gf) = \mathcal{F}(\mathcal{F}(k * h)^-) = k * h = \mathcal{F}g * \mathcal{F}f\,.$$

We're done.

Remark You may wonder why we didn't start by trying to prove $\mathcal{F}(gf)(s) = (\mathcal{F}g * \mathcal{F}f)(s)$ rather than $\mathcal{F}(g * f) = (\mathcal{F}f)(\mathcal{F}g)$ as we did. That is, it seems more "natural" to multiply signals in the time domain and see what effect this has in the frequency domain, so why not work with $\mathcal{F}(fg)$ directly? But write the integral for $\mathcal{F}(gf)$; there's nothing you can do with it to get toward $\mathcal{F}g * \mathcal{F}f$.

3.2 What is Convolution, Really?

There's not a single answer to that question. Those of you who have had a course in "Signals and Systems" probably saw convolution in connection with Linear Time Invariant Systems and the *impulse response* for such a system. (This already came up in connection with our solution of the heat equation.) That's a very natural setting for convolution and we'll consider it later, after we have the machinery of delta functions et al.

The fact is that convolution is used in many ways and for many reasons, and it can be a mistake to try to attach to it one particular meaning or interpretation. This multitude of interpretations and applications is somewhat like the situation with the definite integral. When you learned about the integral, chances are that it was introduced via an important motivating problem, typically recovering the distance traveled from the velocity, or finding the area under a curve. That's fine, but the integral is really a much more general and flexible concept than those two sample problems might suggest. You do yourself no service if every time you think to use an integral you think only of one of those problems. Likewise, you do yourself no service if you insist on one particular interpretation of convolution.

To pursue the analogy with the integral a little bit further, in pretty much *all* applications of the integral there is a general method at work: cut the problem into small pieces where it can be solved approximately, sum up the solution for the pieces, and pass to a limit.[4] There is also often a general method to working, or seeking to work with convolutions: usually there's something that has to do with *smoothing* and *averaging*, understood broadly. You see this in both the continuous case (which we're doing now) and the discrete case (which we'll do later).

For example, in using Fourier series to solve the heat equation on a circle, we saw that the solution was expressed as a convolution of the initial heat distribution with the Green's function (or fundamental solution). That's a smoothing and averaging interpretation (both!) of the convolution. It's also a linear systems interpretation of convolution, where the system is described by the heat equation.

In brief, we'll get to know the convolution by seeing it in action:

- Convolution *is* what convolution *does*.

That's probably the best answer to the question in the heading to this section.

3.2.1 But can I visualize convolution? or "Flip this, buddy"

I'm tempted to say don't bother. Again for those of you who have seen convolution in earlier courses, you've probably heard the expression "flip and drag". For

$$(g * f)(t) = \int_{-\infty}^{\infty} g(t-x)f(x)\,dx$$

here's what this means.

- Fix a value t. The graph of the function $g(x-t)$ has the same shape as $g(x)$ but shifted to the right by t. Then forming $g(t-x)$ flips the graph (left-right) about the line $x = t$. If the most interesting or important features of $g(x)$ are near $x = 0$, e.g., if it's sharply peaked there, then those features are shifted to $x = t$ for the function $g(t-x)$ (but there's the extra "flip" to keep in mind).

[4] This goes back to Archimedes, who called his paper on the subject "The Method".

- Multiply the two functions $f(x)$ and $g(t-x)$ and integrate with respect to x. Remember that the value of the convolution $(g*f)(t)$ is not just the product of the values of f and the flipped and shifted g, it's the *integral* of the product — much harder to visualize. Integrating the product sums up these values, that's the "dragging" part.

Smoothing and averaging I prefer to think of the convolution operation as using one function to smooth and average the other. (Say g is used to smooth f in $g*f$.) In many common applications $g(x)$ is a positive function, concentrated near 0, with total area 1,

$$\int_{-\infty}^{\infty} g(x)\,dx = 1,$$

like a sharply peaked Gaussian, for example (stay tuned). Then $g(t-x)$ is concentrated near t and still has area 1. For a fixed t, forming the integral

$$\int_{-\infty}^{\infty} g(t-x)f(x)\,dx$$

is like taking a weighted average of the values of $f(x)$ near $x=t$, weighted *by* the values of (the flipped and shifted) g. (It's a legitimate weighted average because $\int_{-\infty}^{\infty} g(x)\,dx = 1$.)

That's the averaging part of the description: Computing the convolution $g*f$ at t replaces the value $f(t)$ by a weighted average of the values of f near t. Where does the smoothing come in? Here's where.

- Changing t ("dragging" $g(t-x)$ through different values of t) repeats this operation.

Again take the case of an averaging-type function $g(t)$, as above. At a given value of t, $(g*f)(t)$ is a weighted average of values of f near t. Move t a little to a point t'. Then $(g*f)(t')$ is a weighted average of values of f near t', which will include values of f that entered into the average near t. Thus the values of the convolutions $(g*f)(t)$ and $(g*f)(t')$ will likely be closer to each other than are the values $f(t)$ and $f(t')$. That is, $(g*f)(t)$ is "smoothing" f as t varies — there's less of a change between values of the convolution than between values of f.

We'll study this in more detail later, but you've already seen at least one example of smoothing. The rect function $\Pi(x)$ is discontinuous — it has jumps at $\pm 1/2$. The convolution $\Pi * \Pi$ is the triangle function Λ, which is *continuous* — the jumps at the endpoints have been smoothed out. There's still a corner, but there's *no* discontinuity.

In fact, as an aphorism we can state

- The convolution $g*f$ is at least as smooth a function as g and f are separately.

A smear job, too Now, be a little careful in how you think about this averaging and smoothing process. Computing any value of $(g*f)(t)$ involves *all* of the values of g and *all* of the values of f, and adding the products of corresponding values of g and f with one of the functions flipped and dragged. If *both* $f(t)$ and $g(t)$ become identically zero after awhile then the convolution $g*f$ will also be identically zero outside of some interval. But if either $f(t)$ or $g(t)$ does not become identically zero then neither will the convolution. In addition to averaging and smoothing the convolution also "smears" out the factors — not a becoming description, but an accurate one.

Definitely keep the general description we've just gone through in mind, but as far as visualizing the convolution of any two old functions, I think it's of dubious value to beat yourself up trying to do that. It's hard geometrically, and it's hard computationally, in the sense that you have to calculate some tedious integrals. (You do have to do a few of these in your life — hence the homework assignment — but only a few.) For developing further intuition, I do recommend the Johns Hopkins web page on signals, systems and control:

`http://www.jhu.edu/~signals/`

There you'll find a Java applet called "Joy of Convolution" (and many other things). It will allow you to select sample curves $f(t)$ and $g(t)$, or to draw your own curves with a mouse, and then produce the convolution $(g * f)(t)$.

By the way, of course you can try to get some intuition for how the convolution looks by thinking of what's happening in the frequency domain. It's not so far fetched to try to imagine the Fourier transforms $\mathcal{F}f$, $\mathcal{F}g$, and their product, and then imagine the inverse transform to get you $g * f$.

3.3 Properties of Convolution: It's a Lot like Multiplication

Convolution behaves in many ways (not all ways) like multiplication. For example, it is commutative:

$$f * g = g * f\,.$$

So although it looks like the respective roles of f and g are different — one is "flipped and dragged", the other isn't — in fact they share equally in the end result.

Do we have to prove this? Not among friends. After all, we *defined* the convolution so that the convolution theorem holds, that is so that $\mathcal{F}(g * f) = \mathcal{F}g\mathcal{F}f$. But g and f enter symmetrically on the right hand side, so $g * f = f * g$ — $g(t)$ can be used to modify $f(t)$ or $f(t)$ can be used to modify $g(t)$.

Nevertheless, the commutativity property is easy to check from the definition:

$$\begin{aligned}(f * g)(t) &= \int_{-\infty}^{\infty} f(t-u)g(u)\,du \\ &= \int_{-\infty}^{\infty} g(t-v)f(v)\,dv \quad \text{(making the substitution } v = t-u\text{)} \\ &= (g * f)(t)\,.\end{aligned}$$

The same idea, a change of variable but with more bookkeeping, establishes that convolution is associative (an exercise for you in integrals):

$$(f * g) * h = f * (g * h)\,.$$

Much more easily one gets that

$$f * (g + h) = f * g + f * h\,.$$

The corresponding statements are easily verified in the frequency domain.

How about a "1"? Is there a function which is to convolution as 1 is to multiplication? Is there a function g such that

$$(g * f)(t) = f(t), \quad \text{for } \textit{all} \text{ functions } f?$$

What property would such a g have? Take Fourier transforms of both sides:

$$\mathcal{F}f(s)\mathcal{F}g(s) = \mathcal{F}f(s)\,.$$

Then $g(x)$ must be such that

$$\mathcal{F}g(s) = 1\,.$$

Is there such a g? Applying the inverse Fourier transform would lead to

$$\int_{-\infty}^{\infty} e^{2\pi i s x}\,dx\,,$$

and that integral does not exist — even I wouldn't try to slip that by the rigor police. Something is up here. Maybe Fourier inversion doesn't work in this case, or else there's no classical function whose Fourier transform is 1, or something. In fact, though the integral does not exist in any sense, the problem of a "1 for convolution" leads exactly to the delta function, or unit impulse — not a classical function, but a "generalized" function. We'll return to that shortly.

How about "division"? Suppose we know h and g in

$$h = f * g$$

and we want to solve for f. Again, taking Fourier transforms we would say

$$\mathcal{F}h = \mathcal{F}f \cdot \mathcal{F}g \;\Rightarrow\; \mathcal{F}f = \frac{\mathcal{F}h}{\mathcal{F}g}\,.$$

We'd like the convolution quotient to be the inverse Fourier transform of $\mathcal{F}h/\mathcal{F}g$. But there are problems caused by places where $\mathcal{F}g = 0$, along with the usual problems with the integral for the inverse Fourier transform to exist.

Solving for $f(t)$ is the *deconvolution* problem, which is extremely important in applications. Many times a noisy signal comes to you in the form $h = f * g$; the signal is f, the noise is g, you receive h. You make some assumptions about the nature of the noise, usually statistical assumptions, and you want to separate the signal from the noise. You want to deconvolve.

Other identities It's not hard to combine the various rules we have and develop an algebra of convolutions. Such identities can be of great use — it beats calculating integrals. Here's an assortment. (Lower and uppercase letters are Fourier pairs.)

$$\begin{aligned}
\big((f\cdot g) * (h\cdot k)\big)(t) &\rightleftharpoons \big((F * G)\cdot(H * K)\big)(s)\\
\big((f(t)+g(t))\cdot(h(t)+k(t)\big) &\rightleftharpoons \big(((F+G)*(H+K))\big)(s)\\
\big(f(t)\cdot(g*h)(t) &\rightleftharpoons \big(F*(G\cdot H)\big)(s)
\end{aligned}$$

You can write down others. Be confident — careful, but confident.

3.4 Convolution in Action I: A Little Bit on Filtering

"Filtering" is a generic term for just about any operation one might want to apply to a signal. We have to be reasonable, of course — there's usually some feature of the signal that one wants to enhance or eliminate, and one expects *something* of the original signal to be recognizable or recoverable after it's been

filtered. Most filters are described as somehow modifying the spectral content of a signal, and they are thus set up as an operation on the Fourier transform of a signal. We'll take up this topic in more detail when we discuss *linear time invariant* (LTI) systems, but it's worthwhile saying a little bit now because the most common filters operate through multiplication in the frequency domain, hence through convolution in the time domain.

The features are:

- An input signal $v(t)$
- An output signal $w(t)$
- The operation that produces $w(t)$ from $v(t)$ in the time domain is convolution with a function $h(t)$:

$$w(t) = (h * v)(t) = \int_{-\infty}^{\infty} h(t-x)v(x)\,dx$$

With this description the Fourier transforms of the input and output are related by multiplication in the frequency domain:

$$W(s) = H(s)V(s)\,,$$

where, following tradition, we denote the Fourier transforms by the corresponding capital letters. In this context $h(t)$ is usually called the *impulse response* [5] and $H(s)$ is called the *transfer function*. It seems to be a matter of course always to denote the impulse response by $h(t)$ and always to denote the transfer function by $H(s)$. Who am I to do otherwise?

Remember that $h(t)$, hence $H(s)$, is "fixed" in this discussion. It's wired into the circuit or coded into the software and it does what it does to *any* input you may give it. Filters based on convolution are usually designed to have a specific effect on the *spectrum* of an input, and so to design a filter is to design a transfer function. The operations, which you're invited to draw a block diagram for, are thus

$$\text{Input} \to \text{Fourier transform} \to \text{Multiply by } H \to \text{Inverse Fourier transform} = \text{output}$$

We want to see some examples of this today — filters that are in day-to-day use and the principles that go into their design.

One preliminary comment about how the spectra of the input and output are related. Write

$$V(s) = |V(s)|e^{i\phi_V(s)}, \quad \phi_V(s) = \tan^{-1}\left(\frac{\operatorname{Im} V(s)}{\operatorname{Re} V(s)}\right),$$

so the phase of $V(s)$ is $\phi_V(s)$, with similar notations for the phases of $W(s)$ and $H(s)$. Then

$$\begin{aligned} |W(s)|e^{i\phi_W(s)} &= |H(s)|\, e^{i\phi_H(s)}\, |V(s)\,|e^{i\phi_V(s)} \\ &= |H(s)|\, |V(s)|\, e^{i(\phi_H(s)+\phi_V(s))}. \end{aligned}$$

Thus the *magnitudes multiply* and the *phases add*:

$$\begin{aligned} |W(s)| &= |H(s)|\, |V(s)| \\ \phi_W(s) &= \phi_V(s) + \phi_H(s) \end{aligned}$$

[5] Because, as we'll see, it is how the system "responds" to a unit impulse.

Multiplying $V(s)$ by $H(s)$ can't make the spectrum of $V(s)$ any bigger[6], but it *can* make the spectrum smaller by zeroing out parts of it. Furthermore, there is no phase change when $\phi_H(s) = 0$, and this happens when $H(s)$ is real. In this case only the amplitude is changed when the signal goes through the filter. Common examples of filters that do both of these things — modify some part of the magnitude of the spectrum with no phase change — are lowpass, bandpass, highpass, and notch filters, to which we'll now turn.

3.4.1 Designing filters

Lowpass filters An ideal *lowpass filter* cuts off all frequencies *above* a certain amount ν_c ("c" for "cutoff") and lets all frequencies *below* ν_c pass through unchanged. (Hence the description "lowpass".) If we write the operation as

$$w(t) = (h * v)(t) \rightleftharpoons W(s) = H(s)V(s)\,,$$

then the transfer function we want is

$$H(s) = \begin{cases} 1 & |s| < \nu_c \\ 0 & |s| \geq \nu_c \end{cases}$$

Multiplying $V(s)$ by $H(s)$ leaves unchanged the spectrum of v for $|s| < \nu_c$ and eliminates the other frequencies. The transfer function is just a scaled rect function, and we can write it (to remind you) as

$$H(s) = \Pi_{2\nu_c}(s) = \Pi(s/2\nu_c) = \begin{cases} 1 & |s/2\nu_c| < \frac{1}{2} \\ 0 & |s/2\nu_c| \geq \frac{1}{2} \end{cases} = \begin{cases} 1 & |s| < \nu_c \\ 0 & |s| \geq \nu_c \end{cases}$$

In the time domain the impulse response is the inverse Fourier transform of $\Pi_{2\nu_c}$, and this is

$$h(t) = 2\nu_c \operatorname{sinc}(2\nu_c t)\,.$$

By the way, why is this called just a "lowpass filter"; aren't the frequencies below $-\nu_c$ also eliminated and so not "passed" by the filter? Yes, but remember that for real signals $v(t)$ (which is where this is applied) one has the symmetry relation $V(-s) = \overline{V(s)}$. The positive and negative frequencies combine in reconstructing the real signal in the inverse Fourier transform, much like what happens with Fourier series. Thus one wants to pass the frequencies with $-\nu_c < s < \nu_c$ and eliminate the frequencies with $s \geq \nu_c$ *and* $s \leq -\nu_c$.

And, by the way, why is this called an *ideal* lowpass filter? Because the cutoff is a sharp one — right at a particular frequency ν_c. In practice this cannot be achieved, and much of the original art of filter design is concerned with useful approximations to a sharp cutoff.

Bandpass filters Another very common filter passes a particular band of frequencies through unchanged and eliminates all others. This is the ideal *bandpass filter*. Its transfer function, $B(s)$, can be constructed by shifting and combining the transfer function $H(s)$ for the lowpass filter.

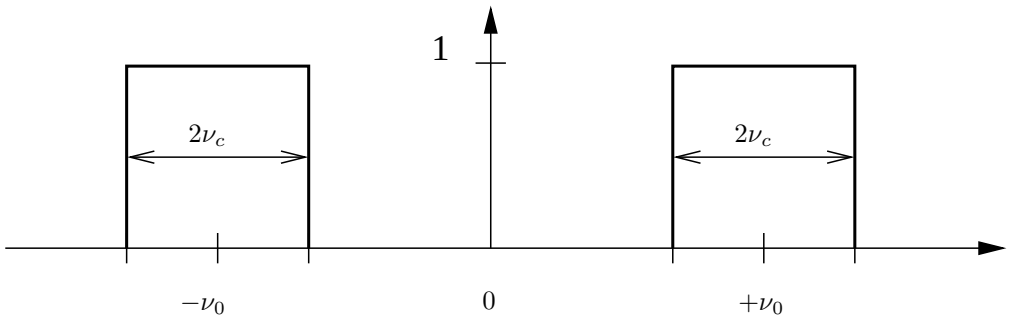
We center our bandpass filter at $\pm\nu_0$ and cut off frequencies more than ν_c above and below ν_0; just as for the lowpass filter we pass symmetric bands of positive frequencies and negative frequencies, and eliminate

[6] In s, that is; the spectrum of the output takes up no more of $\mathbf{R}$ than the spectrum of the input. One says that no new frequencies are added to the spectrum

everything else. That is, we define the transfer function of a bandpass filter to be

$$\begin{aligned} B(s) &= \begin{cases} 1 & \nu_0 - \nu_c < |s| < \nu_0 + \nu_c \\ 0 & \text{otherwise} \end{cases} \\ &= H(s - \nu_0) + H(s + \nu_0) \end{aligned}$$

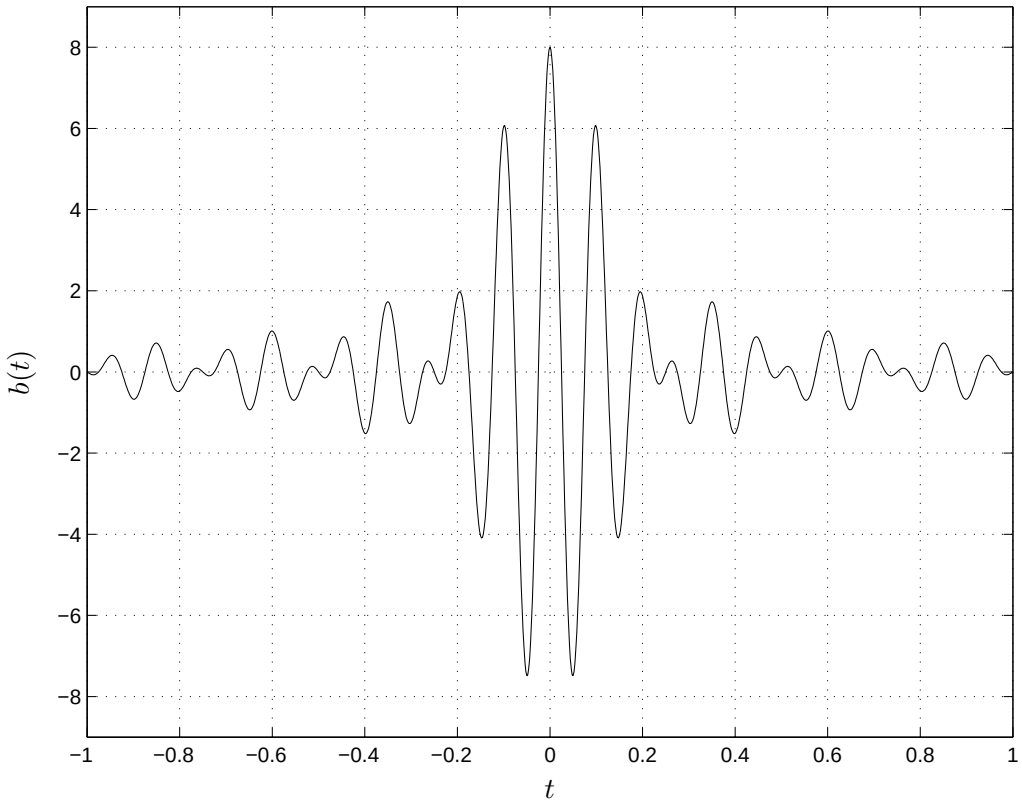
Here's the graph.



From the representation of $B(s)$ in terms of $H(s)$ it's easy to find the impulse response, $b(t)$. That's given by

$$\begin{aligned} b(t) &= h(t)e^{2\pi i\nu_0 t} + h(t)e^{-2\pi i\nu_0 t} \quad \text{(using the shift theorem or the modulation theorem)} \\ &= 4\nu_c \cos(2\pi\nu_0 t)\,\mathrm{sinc}(2\nu_c t). \end{aligned}$$

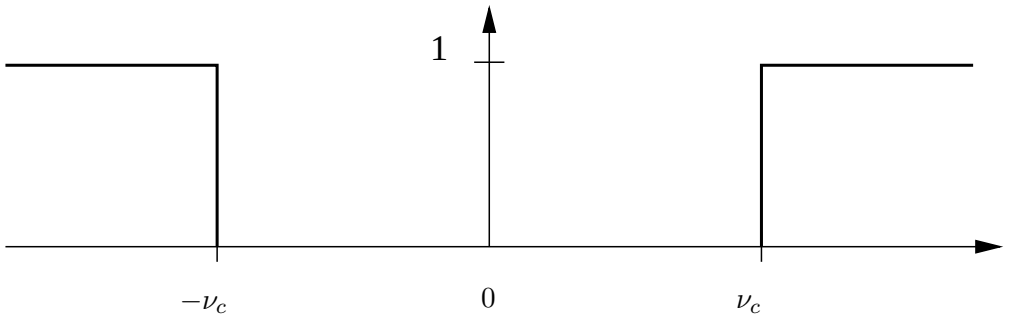
Here's a plot of $b(t)$ for $\nu_0 = 10$ and $\nu_c = 2$:



Now, tell the truth, do you really think you could just flip and drag and figure out what the convolution looks like of that thing with some other thing?

Highpass filters The twin to an ideal lowpass filter is an ideal high pass filter, where all frequencies above a cutoff frequency ν_c are passed through unchanged and all frequencies below are eliminated. You might use this, for example, if there's a slow "drift" in your data that suggests a low frequency disturbance or noise that you may want to eliminate. Highpass filters are used on images to sharpen edges and details (associated with high spatial frequencies) and eliminate blurring (associated with low spatial frequencies).

The graph of the transfer function for an ideal highpass filter looks like:



It's easy to write a formula for this; it's just

$$\text{High}(s) = 1 - \Pi_{2\nu_c}(s)\,,$$

where ν_c is the cutoff frequency.[7] At this point we're stuck. We can't find the impulse response because we haven't yet gained the knowledge that the inverse Fourier transform of 1 is the δ function. Think of the highpass filter as the evil twin of the lowpass filter.

Notch filters The evil twin of a bandpass filter is a *notch filter*. The effect of a notch filter is to *eliminate* frequencies within a given band (the "notch") and to pass frequencies outside that band. To get the transfer function we just subtract a bandpass transfer function from 1. Using the one we already have:

$$\text{Notch}(s) = 1 - B(s) = 1 - (H(s-\nu_0) + H(s+\nu_0))\,.$$

This will eliminate the positive frequencies between $\nu_0 - \nu_c$ and $\nu_0 + \nu_c$, and the symmetric corresponding negative frequencies between $-\nu_0 - \nu_c$ and $-\nu_0 + \nu_c$, and pass all frequencies outside of these two bands. You can draw your own graph of that.

Unfortunately, for the impulse response we're in the same position here as we were for the highpass filter. We cannot write down the impulse response without recourse to δ's, so this will have to wait.

3.5 Convolution in Action II: Differential Equations

One of the most common uses of convolution and the Fourier transform is in solving differential equations. Solving differential equations was Fourier's original motivation for Fourier *series* and the use of the Fourier transform to this end has continued to exercise a strong influence on the theory and the applications. We'll consider several illustrations, from a simple ordinary differential equation to problems associated with the heat equation. We'll also revisit the problem of a signal propagating along a cable.

[7] OK, this High(s) is 1 at the endpoints $\pm\nu_c$ instead of 0, but that makes no practical difference. On the other hand, this is a further argument for defining Π to have value 1/2 at the endpoints, for then the transfer functions for the low and highpass filters agree in how they cut.

The derivative formula To put the Fourier transform to work, we need a formula for the Fourier transform of the derivative, and as you found in homework:

$$(\mathcal{F}f')(s) = 2\pi i s\,\mathcal{F}f(s)\,.$$

We see that differentiation has been transformed into multiplication, another remarkable feature of the Fourier transform and another reason for its usefulness. Formulas for higher derivatives also hold, and the result is:

$$(\mathcal{F}f^{(n)})(s) = (2\pi i s)^n\,\mathcal{F}f(s)\,.$$

We'll come back to these formulas in another context a little later.

In general, a *differential operator* can be thought of as a polynomial in d/dx, say of the form

$$P\Big(\frac{d}{dx}\Big) = a_n\Big(\frac{d}{dx}\Big)^n + a_{n-1}\Big(\frac{d}{dx}\Big)^{n-1} + \cdots + a_1\frac{d}{dx} + a_0\,,$$

and when applied to a function $f(x)$ the result is

$$a_n f^{(n)} + a_{n-1}f^{(n-1)} + \cdots + a_1 f' + a_0 f\,.$$

If we now take the Fourier transform of this expression, we wind up with the Fourier transform of f multiplied by the corresponding n-th degree polynomial evaluated at $2\pi i s$.

$$\begin{aligned}\left(\mathcal{F}\Big(P\Big(\frac{d}{dx}\Big)f\Big)\right)(s) &= P(2\pi i s)\,\mathcal{F}f(s)\\ &= \big(a_n(2\pi i s)^n + a_{n-1}(2\pi i s)^{n-1} + \cdots + a_1(2\pi i s) + a_0\big)\,\mathcal{F}f(s)\,.\end{aligned}$$

Don't underestimate how important this is.

A simple ordinary differential equation and how to solve it You might like starting off with the classic second order, ordinary differential equation

$$u'' - u = -f$$

Maybe you've looked at a different form of this equation, but I'm writing it this way to make the subsequent calculations a little easier. $f(t)$ is a given function and you want to find $u(t)$.

Take the Fourier transform of both sides:

$$\begin{aligned}(2\pi i s)^2\mathcal{F}u - \mathcal{F}u &= -\mathcal{F}f\\ -4\pi^2 s^2\mathcal{F}u - \mathcal{F}u &= -\mathcal{F}f\\ (1+4\pi^2 s^2)\mathcal{F}u &= \mathcal{F}f\end{aligned}$$

So we can solve for $\mathcal{F}u$ as

$$\mathcal{F}u = \frac{1}{1+4\pi^2 s^2}\mathcal{F}f\,,$$

and — with a little struggle — we recognize $1/(1+4\pi^2 s^2)$ as the Fourier transform of $\frac{1}{2}e^{-|t|}$, that is,

$$\mathcal{F}u = \mathcal{F}\left(\tfrac{1}{2}e^{-|t|}\right)\cdot\mathcal{F}f\,.$$

The right hand side is the product of two Fourier transforms. Therefore, according to the convolution theorem,

$$u(t) = \tfrac{1}{2}e^{-|t|} * f(t)\,.$$

Written out in full this is

$$u(t) = \tfrac{1}{2}\int_{-\infty}^{\infty} e^{-|t-\tau|}f(\tau)\,d\tau\,.$$

And there you have the two-sided exponential decay in action, as well as convolution.

The heat equation Remember the heat equation? In one spatial dimension, the equation that describes the rates of change of the temperature $u(x,t)$ of the body at a point x and time t (with some normalization of the constants associated with the material) is the partial differential equation

$$u_t = \tfrac{1}{2}u_{xx}\,.$$

In our earlier work on Fourier series we considered heat flow on a circle, and we looked for solutions that are periodic function of x on the interval $[0,1]$, so u was to satisfy $u(x+1,t) = u(x,t)$. This time we want to consider the problem of heat flow on the "infinite rod". A rod of great length (effectively of infinite length) is provided with an initial temperature distribution $f(x)$ and we want to find a solution $u(x,t)$ of the heat equation with

$$u(x,0) = f(x)\,.$$

Both $f(x)$ and $u(x,t)$ are defined for $-\infty < x < \infty$, and there is no assumption of periodicity. Knowing the Fourier transform of the Gaussian is essential for the treatment we're about to give.

The idea is to take the Fourier transform of both sides of the heat equation, "with respect to x". The Fourier transform of the right hand side of the equation, $\frac{1}{2}u_{xx}(x,t)$, is

$$\tfrac{1}{2}\mathcal{F}u_{xx}(s,t) = \tfrac{1}{2}(2\pi is)^2\mathcal{F}u(s,t) = -2\pi^2 s^2\mathcal{F}u(s,t)\,,$$

from the derivative formula. Observe that the "frequency variable" s is now in the first slot of the transformed function and that the time variable t is just going along for the ride. For the left hand side, $u_t(x,t)$, we do something different. We have

$$\begin{aligned}\mathcal{F}u_t(s,t) &= \int_{-\infty}^{\infty} u_t(x,t)e^{-2\pi isx}\,dx \quad \text{(Fourier transform in } x\text{)}\\ &= \int_{-\infty}^{\infty} \frac{\partial}{\partial t}u(x,t)e^{-2\pi isx}\,dx\\ &= \frac{\partial}{\partial t}\int_{-\infty}^{\infty} u(x,t)e^{-2\pi isx}\,dx = \frac{\partial}{\partial t}\hat{u}(s,t).\end{aligned}$$

Thus taking the Fourier transform (with respect to x) of both sides of the equation

$$u_t = \tfrac{1}{2}u_{xx}$$

leads to

$$\frac{\partial \mathcal{F}u(s,t)}{\partial t} = -2\pi^2 s^2\mathcal{F}u(s,t)\,.$$

This is a differential equation in t — an ordinary differential equation, despite the partial derivative symbol — and we can solve it:

$$\mathcal{F}u(s,t) = \mathcal{F}u(s,0)e^{-2\pi^2 s^2 t}\,.$$

What is the initial condition, $\mathcal{F}u(s,0)$?

$$\begin{aligned}\mathcal{F}u(s,0) &= \int_{-\infty}^{\infty} u(x,0)e^{-2\pi isx}\,dx\\ &= \int_{-\infty}^{\infty} f(x)e^{-2\pi isx}\,dx = \mathcal{F}f(s)\end{aligned}$$

Putting it all together,

$$\mathcal{F}u(s,t) = \mathcal{F}f(s)e^{-2\pi^2 s^2 t}\,.$$

We recognize (we are good) that the exponential factor on the right hand side is the Fourier transform of the Gaussian,

$$g(x,t) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}\,.$$

We then have a product of two Fourier transforms,

$$\mathcal{F}u(s,t) = \mathcal{F}g(s,t)\,\mathcal{F}f(s)$$

and we invert this to obtain a convolution in the spatial domain:

$$u(x,t) = g(x,t) * f(x) = \left(\frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}\right) * f(x) \quad \text{(convolution in } x\text{)}$$

or, written out,

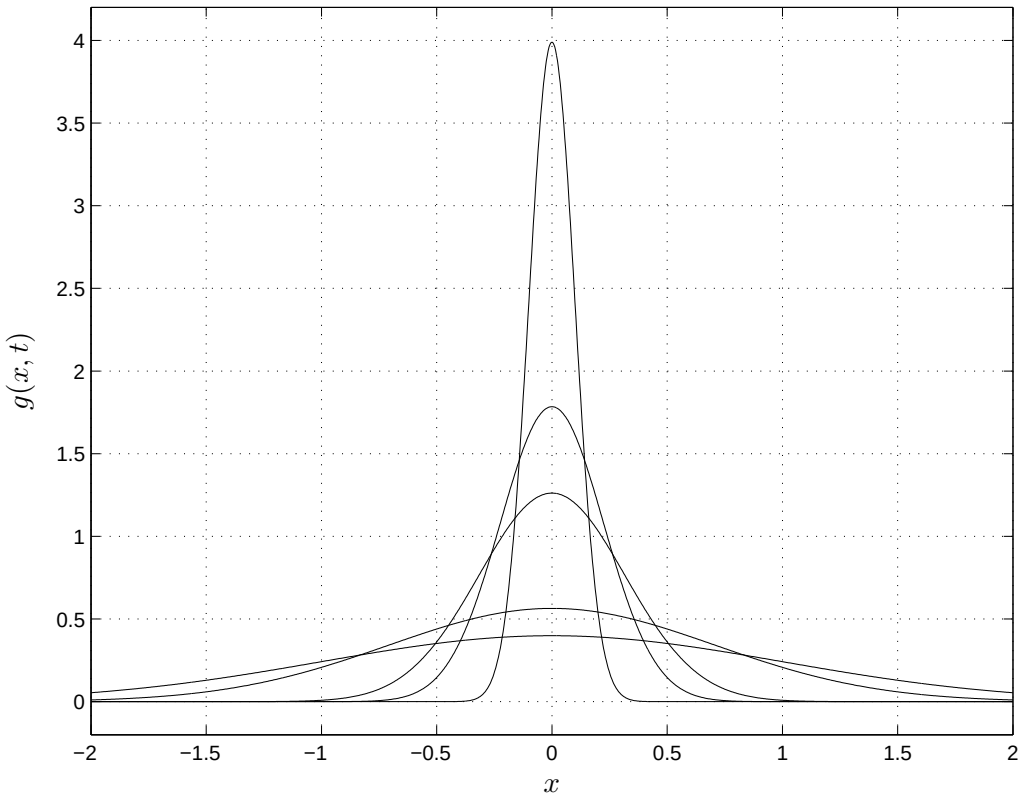
$$u(x,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-(x-y)^2/2t} f(y)\,dy\,.$$

It's reasonable to believe that the temperature $u(x,t)$ of the rod at a point x at a time $t > 0$ is some kind of averaged, smoothed version of the initial temperature $f(x) = u(x,0)$. That's convolution at work.

The function

$$g(x,t) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}\,.$$

is called the *heat kernel* (or Green's function, or fundamental solution) for the heat equation for the infinite rod. Here are plots of $g(x,t)$, as a function of x, for $t = 1,\ 0.5,\ 0.1,\ 0.05,\ 0.01$.



You can see that the curves are becoming more concentrated near $x = 0$. Nevertheless, they are doing so in a way that keeps the area under each curve 1. For

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}\, dx = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{-\pi u^2} \sqrt{2\pi t}\, du \quad \text{(making the substitution } u = x/\sqrt{2\pi t}.\text{)}$$
$$= \int_{-\infty}^{\infty} e^{-\pi u^2}\, du = 1$$

We'll see later that the $g(x,t)$ serve as an approximation to the δ function as $t \to 0$.

You might ask at this point: Didn't we already solve the heat equation? Is what we did then related to what we just did now? Indeed we did and indeed they are: see Section 3.5.

More on diffusion — back to the cable Recall from our earlier discussion that William Thomson appealed to the heat equation to study the delay in a signal sent along a long, undersea telegraph cable. The physical intuition, as of the mid 19th century, was that charge "diffused" along the cable. To reconstruct part of Thomson's solution (essentially) we must begin with a slightly different setup. The equation is the same

$$u_t = \tfrac{1}{2} u_{xx}\,,$$

so we're choosing constants as above and not explicitly incorporating physical parameters such as resistance per length, capacitance per length, etc., but the initial and boundary conditions are different.

We consider a *semi-infinite* rod, having one end (at $x = 0$) but effectively extending infinitely in the positive x-direction. Instead of an initial distribution of temperature along the entire rod, we consider a source of heat (or voltage) $f(t)$ at the end $x = 0$. Thus we have the initial condition

$$u(0,t) = f(t)\,.$$

We suppose that

$$u(x,0) = 0\,,$$

meaning that at $t = 0$ there's no temperature (or charge) in the rod. We also assume that $u(x,t)$ and its derivatives tend to zero as $x \to \infty$. Finally, we set

$$u(x,t) = 0 \quad \text{for} \quad x < 0$$

so that we can regard $u(x,t)$ as defined for all x. We want a solution that expresses $u(x,t)$, the temperature (or voltage) at a position $x > 0$ and time $t > 0$ in terms of the initial temperature (or voltage) $f(t)$ at the endpoint $x = 0$.

The analysis of this is *really* involved. It's quite a striking formula that works out in the end, but, be warned, the end is a way off. Proceed only if interested.

First take the Fourier transform of $u(x,t)$ with respect to x (the notation $\hat{u}$ seems more natural here):

$$\hat{u}(s,t) = \int_{-\infty}^{\infty} e^{-2\pi i s x} u(x,t)\, dx\,.$$

Then, using the heat equation,

$$\frac{\partial}{\partial t}\hat{u}(s,t) = \int_{-\infty}^{\infty} e^{-2\pi i s x} \frac{\partial}{\partial t} u(x,t)\, dx = \int_{-\infty}^{\infty} e^{-2\pi i s x} \frac{\partial^2}{\partial x^2} \tfrac{1}{2} u(x,t)\, dx\,.$$

We need integrate only from 0 to ∞ since $u(x,t)$ is identically 0 for $x<0$. We integrate by parts once:

$$\begin{aligned}\int_0^\infty e^{-2\pi isx}\frac{1}{2}\frac{\partial^2}{\partial x^2}u(x,t)\,dx &= \frac{1}{2}\left(\left[e^{-2\pi isx}\,\frac{\partial}{\partial x}u(x,t)\right]_{x=0}^{x=\infty} + 2\pi is\int_0^\infty \frac{\partial}{\partial x}u(x,t)\,e^{-2\pi isx}\,dx\right)\\ &= -\tfrac{1}{2}u_x(0,t) + \pi is\int_0^\infty \frac{\partial}{\partial x}u(x,t)\,e^{-2\pi isx}\,dx\,,\end{aligned}$$

taking the boundary conditions on $u(x,t)$ into account. Now integrate by parts a second time:

$$\begin{aligned}\int_0^\infty \frac{\partial}{\partial x}u(x,t)\,e^{-2\pi isx}\,dx &= \left[e^{-2\pi isx}\,u(x,t)\right]_{x=0}^{x=\infty} + 2\pi is\int_0^\infty e^{-2\pi ist}\,u(x,t)\,dx\\ &= -u(0,t) + 2\pi is\int_0^\infty e^{-2\pi ist}\,u(x,t)\,dx\\ &= -f(t) + 2\pi is\int_{-\infty}^\infty e^{-2\pi ist}\,u(x,t)\,dx\\ &\qquad \text{(we drop the bottom limit back to } -\infty \text{ to bring back the}\\ &\qquad \text{Fourier transform)}\\ &= -f(t) + 2\pi is\,\hat{u}(s,t).\end{aligned}$$

Putting these calculations together yields

$$\frac{\partial}{\partial t}\hat{u}(s,t) = -\tfrac{1}{2}u_x(0,t) - \pi isf(t) - 2\pi^2s^2\hat{u}(s,t)\,.$$

Now, this is a linear, first order, ordinary differential equation (in t) for $\hat{u}$. It's of the general type

$$y'(t) + P(t)y(t) = Q(t)\,,$$

and if you cast your mind back and search for knowledge from the dim past you will recall that to solve such an equation you multiply both sides by the integrating factor

$$e^{\int_0^t P(\tau)\,d\tau}$$

which produces

$$\left(y(t)e^{\int_0^t P(\tau)\,d\tau}\right)' = e^{\int_0^t P(\tau)\,d\tau}Q(t)\,.$$

From here you get $y(t)$ by direct integration. For our particular application we have

$$\begin{aligned}&P(t) = 2\pi^2s^2 \quad \text{(that's a constant as far as we're concerned because there's no } t)\\ &Q(t) = -\tfrac{1}{2}u_x(0,t) - \pi isf(t).\end{aligned}$$

The integrating factor is $e^{2\pi^2s^2t}$ and we're to solve[8]

$$(e^{2\pi^2s^2t}\hat{u}(t))' = e^{2\pi^2s^2t}\left(-\tfrac{1}{2}u_x(0,t) - \pi isf(t)\right)\,.$$

Write τ for t and integrate both sides from 0 to t with respect to τ:

$$e^{2\pi^2s^2t}\hat{u}(s,t) - \hat{u}(s,0) = \int_0^t e^{2\pi s^2\tau}\left(-\tfrac{1}{2}u_x(0,\tau) - \pi isf(\tau)\right)\,d\tau\,.$$

[8] I want to carry this out so you don't miss anything

But $\hat{u}(s,0) = 0$ since $u(x,0)$ is identically 0, so

$$\begin{aligned}\hat{u}(s,t) &= e^{-2\pi^2 s^2 t}\int_0^t e^{2\pi s^2\tau}\left(-\tfrac{1}{2}u_x(0,\tau) - \pi i s f(\tau)\right)\,d\tau \\ &= \int_0^t e^{-2\pi^2 s^2(t-\tau)}\left(-\tfrac{1}{2}u_x(0,\tau) - \pi i s f(\tau)\right)\,d\tau.\end{aligned}$$

We need to take the inverse transform of this to get $u(x,t)$. Be not afraid:

$$\begin{aligned}u(x,t) &= \int_{-\infty}^{\infty} e^{2\pi i s x}\hat{u}(s,t)\,ds \\ &= \int_{-\infty}^{\infty} e^{2\pi i s x}\Big(\int_0^t e^{-2\pi^2 s^2(t-\tau)}\left(-\tfrac{1}{2}u_x(0,\tau) - \pi i s f(\tau)\right)\,d\tau\Big)\,ds \\ &= \int_0^t\int_{-\infty}^{\infty} e^{2\pi i s x}e^{-2\pi^2 s^2(t-\tau)}\left(-\tfrac{1}{2}u_x(0,\tau) - \pi i s f(\tau)\right)\,ds\,d\tau\,.\end{aligned}$$

Appearances to the contrary, this is not hopeless. Let's pull out the inner integral for further examination:

$$\begin{aligned}&\int_{-\infty}^{\infty} e^{2\pi i s x}(e^{-2\pi^2 s^2(t-\tau)}\left(-\tfrac{1}{2}u_x(0,\tau) - \pi i s f(\tau)\right))\,ds = \\ &\quad -\tfrac{1}{2}u_x(0,\tau)\int_{-\infty}^{\infty} e^{2\pi i s x}e^{-2\pi^2 s^2(t-\tau)}\,ds - \pi i f(\tau)\int_{-\infty}^{\infty} e^{2\pi i s x}\,s\,e^{-2\pi^2 s^2(t-\tau)}\,ds\end{aligned}$$

The first integral is the inverse Fourier transform of a Gaussian; we want to find $\mathcal{F}^{-1}\big(e^{-2\pi s^2(t-\tau)}\big)$. Recall the formulas

$$\mathcal{F}\Big(\frac{1}{\sigma\sqrt{2\pi}}e^{-x^2/2\sigma^2}\Big) = e^{-2\pi^2\sigma^2 s^2}\,, \quad \mathcal{F}(e^{-x^2/2\sigma^2}) = \sigma\sqrt{2\pi}\,e^{-2\pi^2\sigma^2 s^2}.$$

Apply this with

$$\sigma = \frac{1}{2\pi\sqrt{(t-\tau)}}\,.$$

Then, using duality and evenness of the Gaussian, we have

$$\int_{-\infty}^{\infty} e^{2\pi i s x}e^{-2\pi s^2(t-\tau)}\,ds = \mathcal{F}^{-1}\big(e^{-2\pi s^2(t-\tau)}\big) = \frac{e^{-x^2/2(t-\tau)}}{\sqrt{2\pi(t-\tau)}}\,.$$

In the second integral we want to find $\mathcal{F}^{-1}(s\,e^{-2\pi^2 s^2(t-\tau)})$. For this, note that

$$s\,e^{-2\pi^2 s^2(t-\tau)} = -\frac{1}{4\pi^2(t-\tau)}\frac{d}{ds}e^{-2\pi^2 s^2(t-\tau)}$$

and hence

$$\int_{-\infty}^{\infty} e^{2\pi i s x}\,s\,e^{-2\pi^2 s^2(t-\tau)}\,ds = \mathcal{F}^{-1}\Big(-\frac{1}{4\pi^2(t-\tau)}\frac{d}{ds}e^{-2\pi^2 s^2(t-\tau)}\Big) = -\frac{1}{4\pi^2(t-\tau)}\mathcal{F}^{-1}\Big(\frac{d}{ds}e^{-2\pi^2 s^2(t-\tau)}\Big)\,.$$

We know how to take the inverse Fourier transform of a derivative, or rather we know how to take the (forward) Fourier transform, and that's all we need by another application of duality. We use, for a general function f,

$$\mathcal{F}^{-1}f' = (\mathcal{F}f')^- = (2\pi i x\mathcal{F}f)^- = -2\pi i x(\mathcal{F}f)^- = -2\pi i x\mathcal{F}^{-1}f\,.$$

Apply this to

$$\mathcal{F}^{-1}\left(\frac{d}{ds}e^{-2\pi^2 s^2(t-\tau)}\right) = -2\pi i x \mathcal{F}^{-1}\big(e^{-2\pi^2 s^2(t-\tau)}\big)$$
$$= -2\pi i x \frac{1}{\sqrt{2\pi(t-\tau)}}\, e^{-x^2/2(t-\tau)} \quad \text{(from our earlier calculation, fortunately)}$$

Then

$$-\frac{1}{4\pi^2(t-\tau)}\mathcal{F}^{-1}\Big(\frac{d}{ds}e^{-2\pi^2 s^2(t-\tau)}\Big) = \frac{2\pi i x}{4\pi^2(t-\tau)}\frac{e^{-x^2/2(t-\tau)}}{\sqrt{2\pi(t-\tau)}} = \frac{i}{2\pi}\frac{x\, e^{-x^2/2(t-\tau)}}{\sqrt{2\pi(t-\tau)^3}}\,.$$

That is,

$$\mathcal{F}^{-1}\Big(s\, e^{-2\pi^2 s^2(t-\tau)}\Big) = \frac{i}{2\pi}\frac{x\, e^{-x^2/2(t-\tau)}}{\sqrt{2\pi(t-\tau)^3}}\,.$$

Finally getting back to the expression for $u(x,t)$, we can combine what we've calculated for the inverse Fourier transforms and write

$$u(x,t) = -\tfrac{1}{2}\int_0^t u_x(0,\tau)\mathcal{F}^{-1}\left(e^{-2\pi s^2(t-\tau)}\right)\,d\tau \;-\; \pi i\int_0^t f(\tau)\mathcal{F}^{-1}\left(s\, e^{-2\pi^2 s^2(t-\tau)}\right)\,d\tau$$
$$= -\tfrac{1}{2}\int_0^t u_x(0,\tau)\frac{e^{-x^2/2(t-\tau)}}{\sqrt{2\pi(t-\tau)}}\,d\tau \;+\; \tfrac{1}{2}\int_0^t f(\tau)\,\frac{x\, e^{-x^2/2(t-\tau)}}{\sqrt{2\pi(t-\tau)^3}}\,d\tau.$$

We're almost there. We'd like to eliminate $u_x(0,\tau)$ from this formula and express $u(x,t)$ in terms of $f(t)$ only. This can be accomplished by a very clever, and I'd say highly nonobvious observation. We know that $u(x,t)$ is zero for $x<0$; we have defined it to be so. Hence the integral expression for $u(x,t)$ is zero for $x<0$. Because of the evenness and oddness *in* x of the two integrands this has a consequence for the values of the integrals when x is positive. (The first integrand is even in x and the second is odd in x.) In fact, the integrals are equal!

Let me explain what happens in a general situation, stripped down, so you can see the idea. Suppose we have

$$\Phi(x,t) = \int_0^t \phi(x,\tau)\,d\tau + \int_0^t \psi(x,\tau)\,d\tau$$

where we know that: $\Phi(x,t)$ is zero for $x<0$; $\phi(x,\tau)$ is even in x; $\psi(x,\tau)$ is odd in x. Take $a>0$. Then $\Phi(-a,\tau)=0$, hence using the evenness of $\phi(x,\tau)$ and the oddness of $\psi(x,\tau)$,

$$0 = \int_0^t \phi(-a,\tau)\,d\tau + \int_0^t \psi(-a,\tau)\,d\tau = \int_0^t \phi(a,\tau)\,d\tau - \int_0^t \psi(a,\tau)\,d\tau\,.$$

We conclude that for all $a>0$,

$$\int_0^t \phi(a,\tau) = \int_0^t \psi(a,\tau)\,d\tau\,,$$

and hence for $x>0$ (writing x for a)

$$\Phi(x,t) = \int_0^t \phi(x,\tau)\,d\tau + \int_0^t \psi(x,\tau)\,d\tau$$
$$= 2\int_0^t \psi(x,\tau)\,d\tau = 2\int_0^t \phi(x,\tau)\,d\tau \quad \text{(either } \phi \text{ or } \psi \text{ could be used).}$$

We apply this in our situation with

$$\phi(x,\tau) = -\tfrac{1}{2}u_x(0,\tau)\frac{e^{-x^2/2(t-\tau)}}{\sqrt{2\pi(t-\tau)}}\,, \quad \psi(x,\tau) = \tfrac{1}{2}f(\tau)\,\frac{x\, e^{-x^2/2(t-\tau)}}{\sqrt{2\pi(t-\tau)^3}}\,.$$

The result is that we can eliminate the integral with the $u_x(0,\tau)$ and write the solution — the final solution — as

$$u(x,t) = \int_0^t f(\tau)\,\frac{x\,e^{-x^2/2(t-\tau)}}{\sqrt{2\pi(t-\tau)^3}}\,d\tau\,.$$

This form of the solution was the one given by Stokes. He wrote to Thomson:

> In working out myself various forms of the solution of the equation $dv/dt = d^2v/dx^2$ [Note: He puts a 1 on the right hand side instead of a 1/2] under the condition $v = 0$ when $t = 0$ from $x = 0$ to $x = \infty$; $v = f(t)$ when $x = 0$ from $t = 0$ to $t = \infty$ I found the solution ... was ...
>
> $$v(x,t) = \frac{x}{2\sqrt{\pi}}\int_0^t (t-t')^{-3/2}e^{-x^2/4(t-t')}f(t')\,dt'\,.$$

Didn't We Already Solve the Heat Equation? Our first application of Fourier series (*the* first application of Fourier series) was to solve the heat equation. Let's recall the setup and the form of the solution. We heat a circle, which we consider to be the interval $0 \le x \le 1$ with the endpoints identified. If the initial distribution of temperature is the function $f(x)$ then the temperature $u(x,t)$ at a point x at time $t > 0$ is given by

$$u(x,t) = \int_0^1 g(x-y)f(y)\,dy\,,$$

where

$$g(u) = \sum_{n=-\infty}^{\infty} e^{-2\pi^2 n^2 t}e^{2\pi i n u}\,.$$

That was our first encounter with convolution. Now, analogous to what we did, above, we might write instead

$$g(x,t) = \sum_{n=-\infty}^{\infty} e^{-2\pi^2 n^2 t}e^{2\pi i n x}$$

and the solution as

$$u(x,t) = g(x,t) * f(x) = \int_0^1 \sum_{n=-\infty}^{\infty} e^{-2\pi^2 n^2 t}e^{2\pi i n(x-y)}f(y)\,dy\,,$$

a convolution in the spatial variable, but with limits of integration just from 0 to 1. Here $f(x)$, $g(x,t)$, and $u(x,t)$ are periodic of period 1 in x.

How does this compare to what we did for the rod? If we imagine initially heating up a circle as heating up an infinite rod by a *periodic* function $f(x)$ then shouldn't we be able to express the temperature $u(x,t)$ for the circle as we did for the rod? We will show that the solution for a circle does have the *same form* as the solution for the infinite rod by means of the remarkable identity:

$$\sum_{n=-\infty}^{\infty} e^{-(x-n)^2/2t} = \sqrt{2\pi t}\sum_{n=-\infty}^{\infty} e^{-2\pi^2 n^2 t}e^{2\pi i n x}$$

Needless to say, this is *not* obvious.

> As an aside, for general interest, a special case of this identity is particularly famous. The *Jacobi theta function* is defined by
>
> $$\vartheta(t) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t}\,,$$

for $t > 0$. It comes up in surprisingly diverse pure and applied fields, including number theory, and statistical mechanics (where it is used to study "partition functions"). Jacobi's identity is

$$\vartheta(t) = \frac{1}{\sqrt{t}}\vartheta\left(\frac{1}{t}\right) .$$

It follows from the identity above, with $x = 0$ and replacing t by $1/2\pi t$.

We'll show later why the general identity holds. But first, assuming that it does, let's work with the solution of the heat equation for a circle and see what we get. Applying the identity to Green's function $g(x,t)$ for heat flow on the circle we have

$$g(x,t) = \sum_{n=-\infty}^{\infty} e^{-2\pi^2 n^2 t} e^{2\pi i n x} = \frac{1}{\sqrt{2\pi t}} \sum_{n=-\infty}^{\infty} e^{-(x-n)^2/2t}$$

Regard the initial distribution of heat $f(x)$ as being defined on all of $\mathbf{R}$ and having period 1. Then

$$\begin{aligned}
u(x,t) &= \int_0^1 \sum_{n=-\infty}^{\infty} e^{-2\pi^2 n^2 t} e^{2\pi i n(x-y)} f(y)\, dy \\
&= \frac{1}{\sqrt{2\pi t}} \int_0^1 \sum_{n=-\infty}^{\infty} e^{-(x-y-n)^2/2t} f(y)\, dy \quad \text{(using the Green's function identity)} \\
&= \frac{1}{\sqrt{2\pi t}} \sum_{n=-\infty}^{\infty} \int_0^1 e^{-(x-y-n)^2/2t} f(y)\, dy \\
&= \frac{1}{\sqrt{2\pi t}} \sum_{n=-\infty}^{\infty} \int_n^{n+1} e^{-(x-u)^2/2t} f(u-n)\, du \quad \text{(substituting } u = y+n) \\
&= \frac{1}{\sqrt{2\pi t}} \sum_{n=-\infty}^{\infty} \int_n^{n+1} e^{-(x-u)^2/2t} f(u)\, du \quad \text{(using that } f \text{ has period 1)} \\
&= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{-(x-u)^2/2t} f(u)\, du .
\end{aligned}$$

Voilà, we are back to the solution of the heat equation on the line.

Incidentally, since the problem was originally formulated for heating a circle, the function $u(x,t)$ is periodic in x. Can we see that from this form of the solution? Yes, for

$$\begin{aligned}
u(x+1,t) &= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{-(x+1-u)^2/2t} f(u)\, du \\
&= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{-(x-w)^2/2t} f(w+1)\, dw \quad \text{(substituting } w = u-1) \\
&= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{-(x-w)^2/2t} f(w)\, dw \quad \text{(using the periodicity of } f(x)) \\
&= u(x,t) .
\end{aligned}$$

Now let's derive the identity

$$\sum_{n=-\infty}^{\infty} e^{-(x-n)^2/2t} = \sqrt{2\pi t} \sum_{n=-\infty}^{\infty} e^{-2\pi^2 n^2 t} e^{2\pi i n x}$$

This is a great combination of many of the things we've developed to this point, and it will come up again.[9] Consider the left hand side as a function of x, say

$$h(x) = \sum_{n=-\infty}^{\infty} e^{-(x-n)^2/2t}\,.$$

This is a periodic function of period 1 — it's the *periodization* of the Gaussian $e^{-x^2/2t}$. (It's even not hard to show that the series converges, *etc.*, but we won't go through that.) What are its Fourier coefficients? We can calculate them:

$$\begin{aligned}
\hat{h}(k) &= \int_0^1 h(x)e^{-2\pi ikx}\,dx \\
&= \int_0^1 \left(\sum_{n=-\infty}^{\infty} e^{-(x-n)^2/2t}\right) e^{-2\pi ikx}\,dx \\
&= \sum_{n=-\infty}^{\infty} \int_0^1 e^{-(x-n)^2/2t} e^{-2\pi ikx}\,dx \\
&= \sum_{n=-\infty}^{\infty} \int_{-n}^{-n+1} e^{-u^2/2t} e^{-2\pi iku}\,du \\
&\qquad \text{(substituting } u = x-n \text{ and using periodicity of } e^{-2\pi ikx}\text{)} \\
&= \int_{-\infty}^{\infty} e^{-u^2/2t} e^{-2\pi iku}\,du
\end{aligned}$$

But this last integral is exactly the Fourier transform of the Gaussian $e^{-x^2/2t}$ at $s = k$. We know how to do that — the answer is $\sqrt{2\pi t}\,e^{-2\pi^2 k^2 t}$.

We have shown that the Fourier coefficients of $h(x)$ are

$$\hat{h}(k) = \sqrt{2\pi t}\,e^{-2\pi^2 k^2 t}\,.$$

Since the function is equal to its Fourier series (really equal here because all the series converge and all that) we conclude that

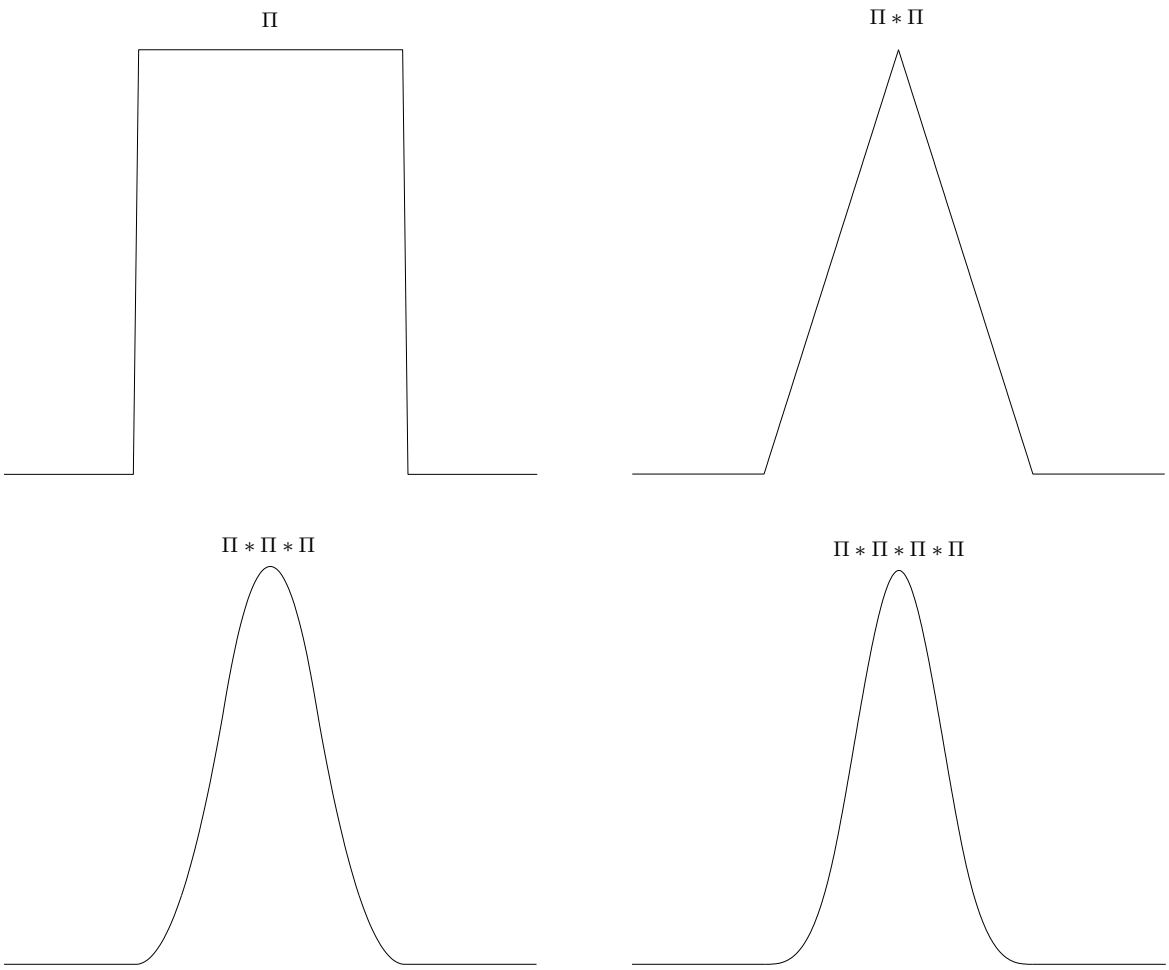
$$\begin{aligned}
h(x) &= \sum_{n=-\infty}^{\infty} e^{-(x-n)^2/2t} \\
&= \sum_{n=-\infty}^{\infty} \hat{h}(n)e^{2\pi inx} = \sqrt{2\pi t} \sum_{n=-\infty}^{\infty} e^{-2\pi^2 n^2 t} e^{2\pi inx}\,,
\end{aligned}$$

and there's the identity we wanted to prove.

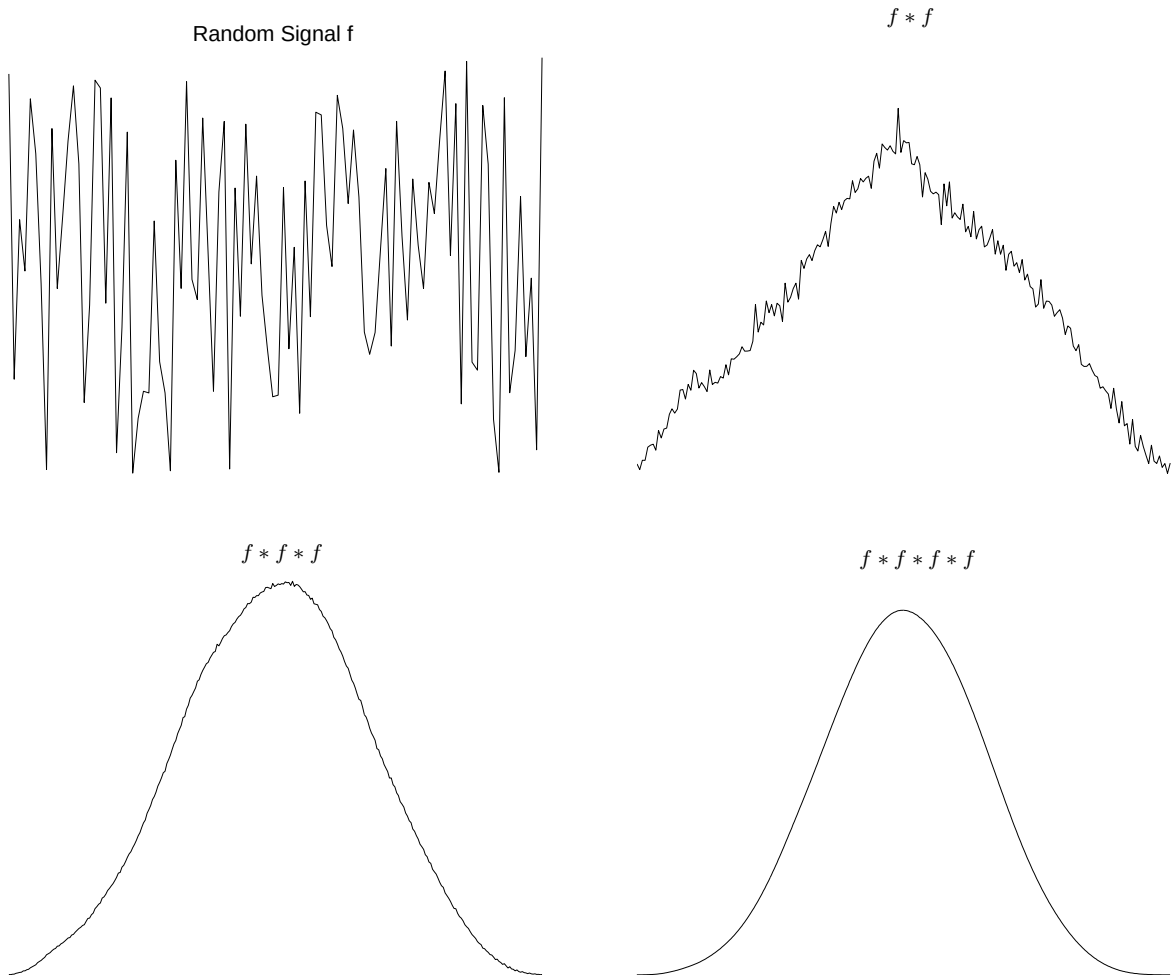
3.6 Convolution in Action III: The Central Limit Theorem

Several times we've met the idea that convolution is a smoothing operation. Let me begin with some graphical examples of this, convolving a discontinuous or rough function repeatedly with itself. For homework you computed, by hand, the convolution of the rectangle function Π with itself a few times. Here are plots of this, up to $\Pi * \Pi * \Pi * \Pi$.

[9] It's worth your effort to go through this. The calculations in this special case will come up more generally when we do the Poisson Summation Formula. That formula is the basis of the sampling theorem.



Not only are the convolutions becoming smoother, but the unmistakable shape of a Gaussian is emerging. Is this a coincidence, based on the particularly simple nature of the function Π, or is something more going on? Here is a plot of, literally, a random function $f(x)$ — the values $f(x)$ are just randomly chosen numbers between 0 and 1 — and its self-convolution up to the four-fold convolution $f * f * f * f$.



From seeming chaos, again we see a Gaussian emerging. The object of this section is to explain this phenomenon, to give substance to the following famous quotation:

> Everyone believes in the normal approximation, the experimenters because they think it is a mathematical theorem, the mathematicians because they think it is an experimental fact.
>
> G. Lippman, French Physicist, 1845–1921

The "normal approximation" (or normal distribution) is the Gaussian. The "mathematical theorem" here is the *Central Limit Theorem*. To understand the theorem and to appreciate the "experimental fact", we have to develop some ideas from probability.

3.6.1 Random variables

In whatever field of science or engineering you pursue you *will* use probabilistic ideas. You *will* use the Gaussian. I'm going under the assumption that you probably know some probability, and probably some statistics, too, even if only in an informal way. For our present work, where complete generality based on exquisitely precise terminology is *not* the goal, we only need a light dose of some of the fundamental notions.

The fundamental notion is the *random variable*. A random variable is a number you don't know yet.[10] By that I mean that it, or rather *its value*, is the numerical result of some process, like a measurement

[10] I think this phrase to describe a random variable is due to Sam Savage in Management Science & Engineering.

or the result of an experiment. The assumption is that you can make the measurement, you can perform the experiment, but until you do you don't know the value of the random variable. It's called "random" because a particular object to be measured is thought of as being drawn "at random" from a collection of all such objects. For example:

Random Variable	Value of random variable
Height of people in US population	Height of particular person
Length of pins produced	Length of particular pin
Momentum of atoms in a gas	Momentum of particular atom
Resistance of resistors off a production line	Resistance of a particular resistor
Toss of coin	0 or 1 (head or tail)
Roll of dice	Sum of numbers that come up

A common notation is to write X for the name of the random variable and x for its value. If you then think that a random variable X is just a function, you're right, but deciding what the *domain* of such a function should be, and what mathematical structure to require of both the domain and the function, demands the kind of precision that we don't want to get into. This was a long time in coming. Consider, for example, Mark Kac's comment: "independent random variables were to me (and others, including my teacher Steinhaus) shadowy and not really well-defined objects." Kac was one of the most eminent probabilists of the 20th century.

3.6.2 Probability distributions and probability density functions

"Random variable" is the fundamental notion, but not the fundamental object of study. For a given random variable what we're most interested in is how its values are distributed. For this it's helpful to distinguish between two types of random variables.

- A random variable is *discrete* if its values are among only a finite number of possibilities.
    - For example "Roll the die" is a discrete random variable with values 1, 2, 3, 4, 5 or 6. "Toss the coin" is a discrete random variable with values 0 and 1. (A random variable with values 0 and 1 is the basic random variable in coding and information theory.)
- A random variable is *continuous* if its values do not form a discrete set, typically filling up one or more intervals of real numbers.
    - For example "length of a pin" is a continuous random variable since, in theory, the length of a pin can vary continuously.

For a discrete random variable we are used to the idea of displaying the distribution of values as a *histogram*. We set up bins, one corresponding to each of the possible values, we run the random process however many times we please, and for each bin we draw a bar with height indicating the *percentage* that value occurs among all actual outcomes of the runs.[11] Since we plot percentages, or fractions, the total area of the histogram is 100%, or just 1.

A series of runs of the same experiment or the same measurement will produce histograms of varying shapes.[12] We often expect some kind of limiting shape as we increase the number of runs, or we may

[11] I have gotten into heated arguments with physicist friends who insist on plotting frequencies of values rather than percentages. Idiots.

[12] A run is like: "Do the experiment 10 times and make a histogram of your results for those 10 trials." A series of runs is like: "Do your run of 10 times, again. And again."

suppose that the ideal distribution has some shape, and then compare the actual data from a series of runs to the ideal, theoretical answer.

- The theoretical histogram is called the *probability distribution*.
- The function that describes the histogram (the shape of the distribution) is called the *probability density function* or *pdf*, of the random variable.

Is there a difference between the probability distribution and the probability density function? No, not really — it's like distinguishing between the graph of a function and the function. Both terms are in common use, more or less interchangeably.

- The *probability* that any particular value comes up is the area of its bin in the probability distribution, which is therefore a number between 0 and 1.

If the random variable is called X and the value we're interested in is x we write this as

$$\text{Prob}(X = x) = \text{area of the bin over } x\,.$$

Also

$$\text{Prob}(a \le X \le b) = \text{areas of the bins from } a \text{ to } b\,.$$

Thus probability is the percentage of the occurrence of a particular outcome, or range of outcomes, among all possible outcomes. We *must* base the definition of probability on what we presume or assume is the distribution function for a given random variable. A statement about probabilities for a run of experiments is then a statement about long term trends, thought of as an approximation to the ideal distribution.

One can also introduce probability distributions and probability density functions for continuous random variables. You can think of this — in fact you probably should think of this — as a continuous version of a probability histogram. It's a tricky business, however, to "take a limit" of the distribution for a discrete random variable, which have bins of a definite size, to produce a distribution for a continuous random variable, imagining the latter as having infinitely many infinitesimal bins.

It's easiest, and best, to define the distribution for a continuous random variable directly.

- A *probability density function* is a nonnegative function $p(x)$ with area 1, i.e.,

$$\int_{-\infty}^{\infty} p(x)\,dx = 1\,.$$

Remember, x is the measured value of some experiment. By convention, we take x to go from $-\infty$ to ∞ so we don't constantly have to say how far the values extend.

Here's one quick and important property of pdfs:

- If $p(x)$ is a pdf and $a > 0$ then $ap(ax)$ is also a pdf.

To show this we have to check that the integral of $ap(ax)$ is 1. But

$$\int_{-\infty}^{\infty} ap(ax)\,dx = \int_{-\infty}^{\infty} ap(u)\frac{1}{a}\,du = \int_{-\infty}^{\infty} p(u)\,du = 1\,,$$

making the change of variable $u = ax$. We'll soon see this property in action.

- We think of a pdf as being associated with a random variable X whose values are x and we write p_X if we want to emphasize this. The (probability) distribution of X is the graph of p_X, but, again, the terms probability density function and probability distribution are used interchangeably.
- *Probability* is defined by

$$\begin{aligned}\text{Prob}(X \le a) &= \text{Area under the curve for } x \le a \\ &= \int_{-\infty}^{a} p_X(x)\,dx,\end{aligned}$$

Also

$$\text{Prob}(a \le X \le b) = \int_a^b p_X(x)\,dx\,.$$

For continuous random variables it really only makes sense to talk about the probability of a range of values occurring, not the probability of the occurrence of a single value. Think of the pdf as describing a limit of a (discrete) histogram: If the bins are becoming infinitely thin, what kind of event could land in an infinitely thin bin?[13]

Finally, for variable t, say, we can view

$$P(t) = \int_{-\infty}^{t} p(x)\,dx$$

as the "probability function". It's also called the *cumulative probability* or the *cumulative density function.*[14] We then have

$$\text{Prob}(X \le t) = P(t)$$

and

$$\text{Prob}(a \le X \le b) = P(b) - P(a)\,.$$

According to the fundamental theorem of calculus we can recover the probability density function from $P(t)$ by differentiation:

$$\frac{d}{dt}P(t) = p(t)\,.$$

In short, to know $p(t)$ is to know $P(t)$ and vice versa. You might not think this news is of any particular practical importance, but you're about to see that it is.

3.6.3 Mean, variance, and standard deviation

Suppose X is a random variable with pdf $p(x)$. The x's are the values assumed by X, so the *mean* μ of X is the weighted average of these values, weighted according to p. That is,

$$\mu(X) = \int_{-\infty}^{\infty} xp(x)\,dx\,.$$

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[13] There's also the familiar integral identity

$$\int_a^a p_X(x)\,dx = 0$$

to contend with. In this context we would interpret this as saying that $\text{Prob}(X = a) = 0$.

[14] Cumulative density function is the preferred term because it allows for a three letter acronym: *cdf.*

Be careful here — the mean of X, defined to be the integral of $xp(x)$, is *not* the average value of the function $p(x)$. It might be that $\mu(X) = \infty$, of course, i.e., that the integral of $xp_X(x)$ does not converge. This has to be checked for any particular example.

If $\mu(X) < \infty$ then we can always "subtract off the mean" to assume that X has mean zero. Here's what this means, no pun intended; in fact let's do something slightly more general. What do we mean by $X - a$, when X is a random variable and a is a constant? Nothing deep — you "do the experiment" to get a value of X (X is a number you don't know yet) then you subtract a from it. What is the pdf of $X - a$? To figure that out, we have

$$\begin{aligned}\text{Prob}(X - a \le t) &= \text{Prob}(X \le t + a) \\ &= \int_{-\infty}^{t+a} p(x)\,dx \\ &= \int_{-\infty}^{t} p(u + a)\,du \quad (\text{substituting } u = x - a).\end{aligned}$$

This identifies the pdf of $X - a$ as $p(x + a)$, the shifted pdf of X.[15]

Next, what is the mean of $X - a$. It must be $\mu(X) - a$ (common sense, please). Let's check this now knowing what pdf to integrate.

$$\begin{aligned}\mu(X - a) &= \int_{-\infty}^{\infty} xp(x + a)\,dx \\ &= \int_{-\infty}^{\infty} (u - a)p(u)\,du \quad (\text{substituting } u = x + a) \\ &= \int_{-\infty}^{\infty} up(u)\,du - a\int_{-\infty}^{\infty} p(u)\,du = \mu(X) - a\,.\end{aligned}$$

Note that translating the pdf $p(x)$ to $p(x + a)$ does nothing to the shape, or areas, of the distribution, hence does nothing to calculating any probabilities based on $p(x)$. As promised, the mean is $\mu(X) - a$. We are also happy to be certain now that "subtracting off the mean", as in $X - \mu(X)$, really does result in a random variable with mean 0. This normalization is often a convenient one to make in deriving formulas.

Suppose that the mean $\mu(X)$ is finite. The *variance* σ^2 is a measure of the amount that the values of the random variable deviate from the mean, *on average*, i.e., as weighted by the pdf $p(x)$. Since some values are above the mean and some are below we weight the *square* of the differences, $(x - \mu(X))^2$, by $p(x)$ and define

$$\sigma^2(X) = \int_{-\infty}^{\infty} (x - \mu(X))^2 p(x)\,dx\,.$$

If we have normalized so that the mean is zero this becomes simply

$$\sigma^2(X) = \int_{-\infty}^{\infty} x^2 p(x)\,dx\,.$$

The *standard deviation* is $\sigma(X)$, the square root of the variance. Even if the mean is finite it might be that $\sigma^2(X)$ is infinite; this, too, has to be checked for any particular example.

---

[15] This is an illustration of the practical importance of going *from* the probability function *to* the pdf. We identified the pdf by knowing the probability function. This won't be the last time we do this.

We've just seen that we can normalize the mean of a random variable to be 0. Assuming that the variance is finite, can we normalize it in some helpful way? Suppose X has pdf p and let a be a positive constant. Then

$$\begin{aligned}\text{Prob}\left(\frac{1}{a}X \le t\right) &= \text{Prob}(X \le at)\\ &= \int_{-\infty}^{at} p(x)\,dx\\ &= \int_{-\infty}^{t} ap(au)\,du \quad \text{(making the substitution } u = \frac{1}{a}x\text{)}\end{aligned}$$

This says that the random variable $\frac{1}{a}X$ has pdf $ap(ax)$. (Here in action is the scaled pdf $ap(ax)$, which we had as an example of operations on pdf's.) Suppose that we've normalized the mean of X to be 0. Then the variance of $\frac{1}{a}X$ is

$$\begin{aligned}\sigma^2\left(\frac{1}{a}X\right) &= \int_{-\infty}^{\infty} x^2 ap(ax)\,dx\\ &= a\int_{-\infty}^{\infty} \frac{1}{a^2}u^2 p(u)\frac{1}{a}\,du \quad \text{(making the substitution } u = ax\text{)}\\ &= \frac{1}{a^2}\int_{-\infty}^{\infty} u^2 p(u)\,du = \frac{1}{a^2}\sigma^2(X)\end{aligned}$$

In particular, if we choose $a = \sigma(X)$ then the variance of $\frac{1}{a}X$ is one. This is also a convenient normalization for many formulas.

In summary:

- Given a random variable X with finite $\mu(X)$ and $\sigma(X) < \infty$, it is possible to normalize and assume that $\mu(X) = 0$ and $\sigma^2(X) = 1$.

You see these assumptions a lot.

3.6.4 Two examples

Let's be sure we have two leading examples of pdfs to refer to.

The uniform distribution "Uniform" refers to a random process where all possible outcomes are equally likely. In the discrete case tossing a coin or throwing a die are examples. All bins in the ideal histogram have the same height, two bins of height 1/2 for the toss of a coin, six bins of height 1/6 for the throw of a single die, and N bins of height $1/N$ for a discrete random variable with N values.

For a continuous random variable the uniform distribution is identically 1 on an interval of length 1 and zero elsewhere. We've seen such a graph before. If we shift to the interval from $-1/2$ to $1/2$, it's the graph of the ever versatile rect function. $\Pi(x)$ is now starring in yet another role, that of the uniform distribution.

The mean is 0, obviously,[16] but to verify this formally:

$$\mu = \int_{-\infty}^{\infty} x\Pi(x)\,dx = \int_{-1/2}^{1/2} x\,dx = \tfrac{1}{2}x^2\Big]_{-1/2}^{+1/2} = 0\,.$$

[16] ... the mean of the random variable with pdf $p(x)$ is *not* the average value of $p(x)$ ...

The variance is then

$$\sigma^2 = \int_{-\infty}^{\infty} x^2 \Pi(x)\, dx = \int_{-1/2}^{1/2} x^2\, dx = \tfrac{1}{3}x^3\Big]_{-1/2}^{+1/2} = \tfrac{1}{12}\,,$$

perhaps not quite so obvious.

The normal distribution This whole lecture is about getting to Gaussians, so it seems appropriate that at some point I mention:

The Gaussian is a pdf.

Indeed, to borrow information from earlier work in this chapter, the Gaussian

$$g(x, \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}\,.$$

is a pdf with mean μ and variance σ^2. The distribution associated with such a Gaussian is called a *normal distribution*. There, it's official. But why is it "normal"? You're soon to find out.

3.6.5 Independence

An important extra property that random variables may have is *independence*. The plain English description of independence is that one event or measurement doesn't influence another event or measurement. Each flip of a coin, roll of a die, or measurement of a resistor is a new event, not influenced by previous events.

Operationally, independence implies that the probabilities multiply: If two random variables X_1 and X_2 are independent then

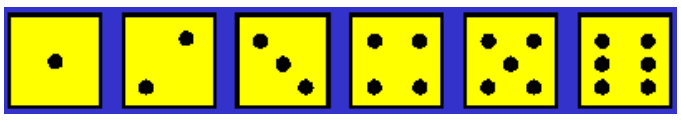
$$\text{Prob}(X_1 \le a \text{ and } X_2 \le b) = \text{Prob}(X_1 \le a) \cdot \text{Prob}(X_2 \le b)\,.$$

In words, if $X_1 \le a$ occurs r percent and $X_2 \le b$ occurs s percent then, if the events are independent, the percent that $X_1 \le a$ occurs *and* $X_2 \le b$ occurs is r percent of s percent, or rs percent.

3.6.6 Convolution appears

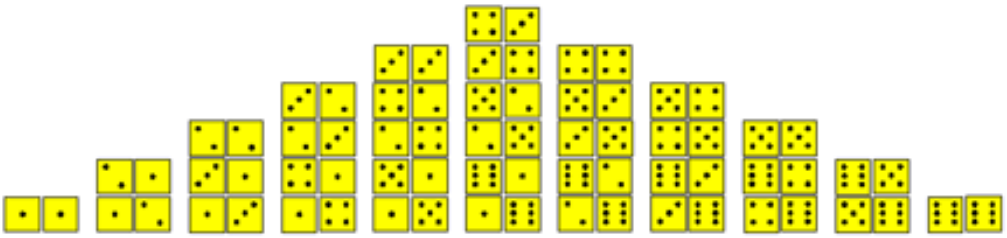
Using the terminology we've developed, we can begin to be more precise about the content of the Central Limit Theorem. That result — the ubiquity of the bell-shaped curve — has to do with *sums* of independent random variables and with the *distributions* of those sums.

While we'll work with continuous random variables, let's look at the discrete random variable X = "roll the dice" as an example. The ideal histogram for the toss of a single die is uniform — each number 1 through 6 comes up with equal probability. We might represent it pictorially like this:



I don't mean to think just of a picture of dice here — I mean to think of the distribution as six bins of equal height 1/6, each bin corresponding to one of the six possible tosses.

What about the *sum* of the tosses of two dice? What is the distribution, theoretically, of the sums? The possible values of the sum are 2 through 12, but the values do not occur with equal probability. There's only one way of making 2 and one way of making 12, but there are more ways of making the other possible sums. In fact, 7 is the most probable sum, with six ways it can be achieved. We might represent the distribution for the sum of two dice pictorially like this:



It's triangular. Now let's see ... For the single random variable $X =$ "roll one die" we have a distribution like a rect function. For the sum, say random variables $X_1 + X_2 =$ "roll of die 1 plus roll of die 2", the distribution looks like the triangle function ....

The key discovery is this:

> **Convolution and probability density functions** The probability density function of the sum of two independent random variables is the convolution of the probability density functions of each.

What a beautiful, elegant, and useful statement! Let's see why it works.

We can get a good intuitive sense of why this result might hold by looking again at the discrete case and at the example of tossing two dice. To ask about the distribution of the sum of two dice is to ask about the probabilities of particular numbers coming up, and these we can compute directly using the rules of probability. Take, for example, the probability that the sum is 7. Count the ways, distinguishing which throw is first:

$$\begin{aligned}
\text{Prob}(\text{Sum} = 7) &= \text{Prob}(\{1 \text{ and } 6\} \text{ or } \{2 \text{ and } 5\} \text{ or } \{3 \text{ and } 4\} \text{ or } \{4 \text{ and } 3\} \text{ or } \{5 \text{ and } 2\} \text{ or } \{6 \text{ and } 1\}) \\
&= \text{Prob}(1 \text{ and } 6) + \text{Prob}(2 \text{ and } 5) + \text{Prob}(3 \text{ and } 4) + \\
&\qquad \text{Prob}(4 \text{ and } 3) + \text{Prob}(5 \text{ and } 2) + \text{Prob}(6 \text{ and } 1) \\
&\qquad\qquad \text{(probabilities add when events are mutually exclusive)} \\
&= \text{Prob}(1)\,\text{Prob}(6) + \text{Prob}(2)\,\text{Prob}(5) + \text{Prob}(3)\,\text{Prob}(4) + \\
&\qquad \text{Prob}(4)\,\text{Prob}(3) + \text{Prob}(5)\,\text{Prob}(2) + \text{Prob}(6)\,\text{Prob}(1) \\
&\qquad\qquad \text{(probabilities multiply when events are independent)} \\
&= 6\left(\frac{1}{6}\right)^2 = \frac{1}{6}\,.
\end{aligned}$$

The particular answer, $\text{Prob}(\text{Sum} = 7) = 1/6$, is not important here[17] — it's the form of the expression

[17] But do note that it agrees with what we can observe from the graphic of the sum of two dice. We see that the total number of possibilities for two throws is 36 and that 7 comes up $6/36 = 1/6$ of the time.

for the solution that should catch your eye. We can write it as

$$\text{Prob}(\text{Sum} = 7) = \sum_{k=1}^{6} \text{Prob}(k)\,\text{Prob}(7-k)$$

which is visibly a discrete convolution of Prob with itself — it has the same form as an integral convolution with the sum replacing the integral.

We can extend this observation by introducing

$$p(n) = \begin{cases} \frac{1}{6} & n = 1, 2, \ldots, 6 \\ 0 & \text{otherwise} \end{cases}$$

This is the discrete uniform density for the random variable "Throw one die". Then, by the same reasoning as above,

$$\text{Prob}(\text{Sum of two dice} = n) = \sum_{k=-\infty}^{\infty} p(k)p(n-k)\,.$$

You can check that this gives the right answers, including the answer 0 for n bigger than 12 or n less than 2:

n	Prob(Sum = n)
2	1/36
3	2/36
4	3/36
5	4/36
6	5/36
7	6/36
8	5/36
9	4/36
10	3/36
11	2/36
12	1/36

Now let's turn to the case of continuous random variables, and in the following argument look for similarities to the example we just treated. Let X_1 and X_2 be independent random variables with probability density functions $p_1(x_1)$ and $p_2(x_2)$. Because X_1 and X_2 are independent,

$$\text{Prob}(a_1 \le X_1 \le b_1 \text{ and } a_2 \le X_2 \le b_2) = \left(\int_{a_1}^{b_1} p_1(x_1)\,dx_1\right)\left(\int_{a_2}^{b_2} p_2(x_2)\,dx_2\right)$$

Using what has now become a familiar trick, we write this as a double integral.

$$\left(\int_{a_1}^{b_1} p_1(x_1)\,dx_1\right)\left(\int_{a_2}^{b_2} p_2(x_2)\,dx_2\right) = \int_{a_2}^{b_2}\int_{a_1}^{b_1} p_1(x_1)p_2(x_2)\,dx_1\,dx_2\,,$$

that is,

$$\text{Prob}(a_1 \le X_1 \le b_1 \text{ and } a_2 \le X_2 \le b_2) = \int_{a_2}^{b_2}\int_{a_1}^{b_1} p_1(x_1)p_2(x_2)\,dx_1\,dx_2\,.$$

If we let a_1 and a_2 drop to $-\infty$ then

$$\text{Prob}(X_1 \le b_1 \text{ and } X_2 \le b_2) = \int_{-\infty}^{b_2}\int_{-\infty}^{b_1} p_1(x_1)p_2(x_2)\,dx_1\,dx_2\,.$$

Since this holds for any b_1 and b_2, we can conclude that

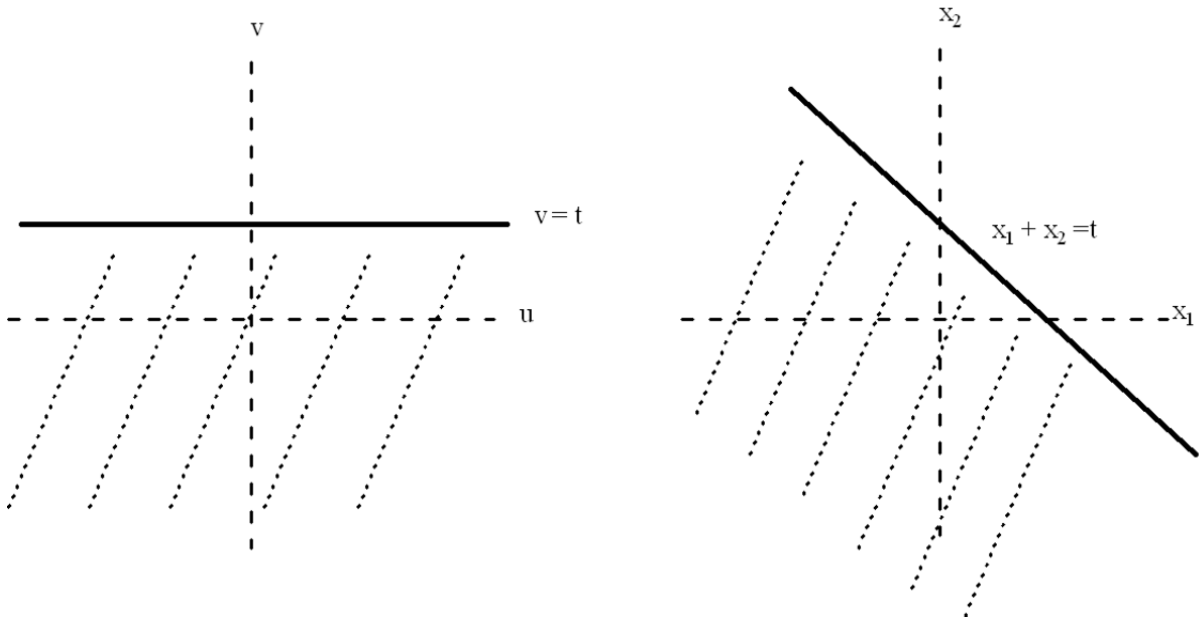
$$\text{Prob}(X_1 + X_2 \le t) = \iint_{x_1+x_2\le t} p_1(x_1)p_2(x_2)\,dx_1\,dx_2$$

for every t. In words, the probability that $X_1 + X_2 \le t$ is computed by integrating the *joint probability density* $p_1(x_1)p_2(x_2)$ over the region in the (x_1, x_2)-plane where $x_1 + x_2 \le t$.

We're going to make a change of variable in this double integral. We let

$$x_1 = u$$
$$x_2 = v - u$$

Notice that $x_1 + x_2 = v$. Thus under this transformation the (oblique) line $x_1 + x_2 = t$ becomes the horizontal line $v = t$, and the region $x_1 + x_2 \le t$ in the (x_1, x_2)-plane becomes the half-plane $v \le t$ in the (u, v)-plane.



The integral then becomes

$$\begin{aligned}\iint_{x_1+x_2\le t} p_1(x_1)p_2(x_2)\,dx_1\,dx_2 &= \int_{-\infty}^{t}\int_{-\infty}^{\infty} p_1(u)p_2(v-u)\,du\,dv \\ &\qquad \text{(the convolution of } p_1 \text{ and } p_2 \text{ is inside!)} \\ &= \int_{-\infty}^{t} (p_2 * p_1)(v)\,dv\,.\end{aligned}$$

To summarize, we now see that the probability $\text{Prob}(X_1 + X_2 \le t)$ for any t is given by

$$\text{Prob}(X_1 + X_2 \le t) = \int_{-\infty}^{t} (p_2 * p_1)(v)\,dv\,.$$

Therefore the probability density function of $X_1 + X_2$ is $(p_2 * p_1)(t)$.

This extends to the sum of any finite number of random variables: If $X_1, X_2, \ldots, X_n$ are independent random variables with probability density functions $p_1, p_2, \ldots, p_n$, respectively, then the probability density function of $X_1 + X_2 + \cdots + X_n$ is $p_1 * p_2 * \cdots * p_n$. Cool. Cool. ... Cool.

For a single probability density $p(x)$ we'll write

$$p^{*n}(x) = (p * p * \cdots * p)(x) \quad (n \text{ factors of } p, \text{ i.e., } n-1 \text{ convolutions of } p \text{ with itself}).$$

3.7 The Central Limit Theorem: The Bell Curve Tolls for Thee

The Central Limit Theorem says something like the sum of n independent random variables is well approximated by a Gaussian if n is large. That means the sum is *distributed* like a Gaussian. To make a true statement, we have to make a few assumptions — but not many — on how the random variables themselves are distributed. Call the random variables X_1, X_2,..., X_n. We assume first of all that the X's are independent. We also assume that all of X's have the same probability density function.[18] There's some terminology and an acronym that goes along with this, naturally. One says that the X's are *independent and identically distributed*, or *iid*. In particular the X's all have the same mean, say μ, and they all have the same standard deviation, say σ.

Consider the sum

$$S_n = X_1 + X_2 + \cdots X_n \,.$$

We want to say that S_n is distributed like a Gaussian as n increases, but which Gaussian? The mean and standard deviation for the X's are all the same, but for S_n they are changing with n. It's not hard to show, though, that for S_n the mean scales by n and thus the standard deviation scales by $\sqrt{n}$:

$$\mu(S_n) = n\mu$$
$$\sigma(S_n) = \sqrt{n}\,\sigma$$

For the derivations see Section 3.9.

So to make sense of S_n approaching a particular Gaussian we should therefore recenter and rescale the sum, say fix the mean to be zero, and fix the standard deviation to be 1. That is, we should work with

$$\frac{S_n - n\mu}{\sqrt{n}\,\sigma}$$

and ask what happens as $n \to \infty$. One form of the Central Limit Theorem[19] says that

$$\lim_{n\to\infty} \text{Prob}\left(a < \frac{S_n - n\mu}{\sqrt{n}\sigma} < b\right) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-x^2/2}\, dx \,.$$

On the right hand side is the Gaussian $(1/\sqrt{2\pi})e^{-x^2/2}$ with mean 0 and standard deviation 1. The theorem says that probabilities for the *normalized* sum of the random variables approach those based on this Gaussian.

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[18] So you can sort of think of the random variables as being the same for practical purposes, like making the same measurements in different trials, or throwing a die hundreds of times, recording the results, and then doing it again.

[19] Abbreviated, of course, as CLT.

We'll focus on the convergence of the pdf's for S_n — sort of an unintegrated form of the way the CLT is stated, above. Let $p(x)$ be the common probability density function for the $X_1, X_2, \ldots, X_n$. (The pdf for the iid X's, for those who like to compress their terminology.) We'll start by assuming already that $\mu = 0$ and $\sigma = 1$ for the X's. This means that

$$\int_{-\infty}^{\infty} xp(x)\,dx = 0 \quad \text{and} \quad \int_{-\infty}^{\infty} x^2 p(x)\,dx = 1\,,$$

in addition to

$$\int_{-\infty}^{\infty} p(x)\,dx = 1\,,$$

which is true for every pdf.

Now, the mean of S_n is zero, but the standard deviation is $\sqrt{n}$, so we want to work $S_n/\sqrt{n}$. What is the pdf of this? We've shown that the pdf for $S_n = X_1 + \cdots + X_n$ is

$$p^{*n}(x) = (p * p * \cdots * p)(x)\,.$$

Hence the probability density function for $S_n/\sqrt{n}$ is

$$p_n(x) = \sqrt{n}\,p^{*n}(\sqrt{n}\,x)\,.$$

(Careful here: It's $(p * p * \cdots * p)(\sqrt{n}\,x)$, not $p(\sqrt{n}\,x) * p(\sqrt{n}\,x) * \cdots p(\sqrt{n}\,x)$.)

We're all set to show:

> **Central Limit Theorem** Let $X_1, X_2, \ldots, X_n$ be independent, identically distributed random variables with mean 0 and standard deviation 1. Let $p_n(x)$ be the probability density function for $S_n/\sqrt{n} = (X_1 + X_2 + \cdots + X_n)/\sqrt{n}$. Then
>
> $$p_n(x) \to \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \quad \text{as } n \to \infty\,.$$

The idea is to take the Fourier transform of p_n, which, by the Convolution Theorem, will essentially be the *product* of the Fourier transforms of p. Products are easier than convolutions, and the hope is to use the assumptions on p to get some information on the form of this product as $n \to \infty$.

Begin with the Fourier transform of

$$p_n(x) = \sqrt{n}\,p^{*n}(\sqrt{n}\,x)\,.$$

We'll use the capital letter notation and write $P(s) = \mathcal{F}p(s)$. Then the Fourier transform of $p_n(x)$ is

$$P^n\left(\frac{s}{\sqrt{n}}\right) \quad \text{(ordinary } n\text{th power here).}$$

The normalization of mean zero and standard deviation 1 allows us to do something with $P(s/\sqrt{n})$. Using

a Taylor series approximation for the exponential function, we have

$$\begin{aligned}
P\left(\frac{s}{\sqrt{n}}\right) &= \int_{-\infty}^{\infty} e^{-2\pi isx/\sqrt{n}}\, p(x)\, dx \\
&= \int_{-\infty}^{\infty} \left(1 - \frac{2\pi isx}{\sqrt{n}} + \frac{1}{2}\left(\frac{2\pi isx}{\sqrt{n}}\right)^2 + \text{small}\right) p(x)\, dx \\
&= \int_{-\infty}^{\infty} \left(1 - \frac{2\pi isx}{\sqrt{n}} - \frac{2\pi^2 s^2 x^2}{n} + \text{small}\right) p(x)\, dx \\
&= \int_{-\infty}^{\infty} p(x)\, dx - \frac{2\pi is}{\sqrt{n}} \int_{-\infty}^{\infty} xp(x)\, dx - \frac{2\pi^2 s^2}{n} \int_{-\infty}^{\infty} x^2 p(x)\, dx + \int_{-\infty}^{\infty} (\text{small})\, p(x)\, dx \\
&= 1 - \frac{2\pi^2 s^2}{n} + \text{small}\,.
\end{aligned}$$

In the last step we used the normalizations

$$\int_{-\infty}^{\infty} p(x)\, dx = 1\,, \quad \int_{-\infty}^{\infty} xp(x)\, dx = 0\,, \quad \int_{-\infty}^{\infty} x^2 p(x)\, dx = 1\,.$$

That "small" term tends to 0 faster than $1/n$ as $n \to \infty$ — see Section 3.10 for more details.

Using the well known fact that $\big(1 + x/n\big)^n \to e^x$, we have for large n

$$P^n\left(\frac{s}{\sqrt{n}}\right) \approx \left(1 - \frac{2\pi^2 s^2}{n}\right)^n \approx e^{-2\pi^2 s^2}\,.$$

Taking the inverse Fourier transform of $e^{-2\pi^2 s^2}$ and knowing what happens to the Gaussian, taking the limit as $n \to \infty$, taking the rest of the day off for a job well done, we conclude that

$$p_n(x) \to \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,.$$

Catch your breath and relax.

3.8 Fourier transform formulas under different normalizations

With convolution now part of our working lives we've seen the major operations and formulas involving Fourier transforms. At the end of Section 2.1 we cautioned that there are different conventions for defining the Fourier transform, and different conventions result in different formulas. Here is a summary of what you'll find out there.

To be as general as possible let's write, as we did back in Section 2.1,

$$\mathcal{F}f(s) = \frac{1}{A} \int_{-\infty}^{\infty} e^{iBst} f(t)\, dt\,.$$

We use $A = 1$ and $B = -2\pi$ but different Fourier practitioners may well use any of the following pairs of values:

$$\begin{array}{ll}
A = \sqrt{2\pi} & B = \pm 1 \\
A = 1 & B = \pm 2\pi \\
A = 1 & B = \pm 1
\end{array}$$

Whatever you choose, here's what you'll then have to live with:

$$\begin{aligned} \mathcal{F}\mathcal{F}f &= \frac{2\pi}{A^2|B|} f^- \mathcal{F}(f') && = -iB\mathcal{F}f \\ (\mathcal{F}f)' &= iB\mathcal{F}(tf(t)) \\ \mathcal{F}(f*g) &= A(\mathcal{F}f)(\mathcal{F}g) \end{aligned}$$

3.9 Appendix: The Mean and Standard Deviation for the Sum of Random Variables

The setup for the Central Limit Theorem involves the sum

$$S_n = X_1 + X_2 + \cdots + X_n$$

of n independent random variables, all having the same pdf $p(x)$. Thus all of the X's have the same mean and the same variance

$$\mu = \int_{-\infty}^{\infty} xp(x)\,dx\,, \quad \sigma^2 = \int_{-\infty}^{\infty} x^2 p(x)\,dx\,.$$

We needed to know that the mean and the standard deviation of S_n were

$$\mu(S_n) = n\mu\,, \quad \sigma(S_n) = \sqrt{n}\,\sigma\,.$$

Take the first of these. The pdf for $S_2 = X_1 + X_2$ is $p * p$, and hence

$$\begin{aligned} \mu(S_2) &= \int_{-\infty}^{\infty} x(p*p)(x)\,dx \\ &= \int_{-\infty}^{\infty} x\left(\int_{-\infty}^{\infty} p(x-y)p(y)\,dy\right)dx \\ &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} xp(x-y)\,dx\right)p(y)\,dy \\ &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} (u+y)p(u)\,du\right)p(y)\,dy \quad (\text{using } u = x-y) \\ &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} up(u)\,du + y\int_{-\infty}^{\infty} p(u)\,du\right)p(y)\,dy \\ &= \int_{-\infty}^{\infty} (\mu+y)p(y)\,dy \quad (\text{using } \textstyle\int_{-\infty}^{\infty} up(u)\,du = \mu \text{ and } \int_{-\infty}^{\infty} p(u)du = 1) \\ &= \mu\int_{-\infty}^{\infty} p(u)\,du + \int_{-\infty}^{\infty} yp(y)\,dy \\ &= \mu + \mu\,. \end{aligned}$$

By induction we get $\mu(S_n) = n\mu$.

How about the variance, or standard deviation? Again let's do this for $S_2 = X_1 + X_2$. We first assume that the mean of the X's is 0 and therefore the mean S_2 is 0, so that

$$\int_{-\infty}^{\infty} xp(x)\,dx = 0 \quad \text{and} \quad \int_{-\infty}^{\infty} x(p*p)(x)\,dx = 0\,.$$

Then the variance of S_2 is

$$\begin{aligned}
\sigma^2(S_2) &= \int_{-\infty}^{\infty} x^2 (p*p)(x)\,dx \\
&= \int_{-\infty}^{\infty} x^2 \left(\int_{-\infty}^{\infty} p(x-y)p(y)\,dy \right) dx \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x^2 p(x-y)\,dx \right) p(y)\,dy \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} (u+y)^2 p(u)\,du \right) p(y)\,dy \quad (\text{using } u = x-y) \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} (u^2 + 2uy + y^2) p(u)\,du \right) p(y)\,dy \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} u^2 p(u)\,du + 2y \int_{-\infty}^{\infty} up(u)\,du + y^2 \int_{-\infty}^{\infty} p(u)\,du \right) p(y)\,dy \\
&= \int_{-\infty}^{\infty} (\sigma^2 + y^2) p(y)\,dy \quad (\text{using } \textstyle\int_{-\infty}^{\infty} u^2 p(u)du = \sigma^2 \text{ and } \int_{-\infty}^{\infty} up(u)du = 0) \\
&= \sigma^2 \int_{-\infty}^{\infty} p(y)\,dy + \int_{-\infty}^{\infty} y^2 p(y)\,dy \\
&= \sigma^2 + \sigma^2 \;=\; 2\sigma^2 .
\end{aligned}$$

So the variance is of (S_2) is $2\sigma^2$ and the standard deviation is $\sigma(S_2) = \sqrt{2}\sigma$. Once again we see by induction that

$$\sigma(S_n) = \sqrt{n}\,\sigma .$$

Pretty nice, really. I'll let you decide what to do if the mean is not zero at the start.

3.10 Appendix: More Details on the Derivation of the Central Limit Theorem

In the proof of the Central Limit Theorem we had the following chain of equalities:

$$\begin{aligned}
P\left(\frac{s}{\sqrt{n}}\right) &= \int_{-\infty}^{\infty} e^{-2\pi i s x/\sqrt{n}}\, p(x)\,dx \\
&= \int_{-\infty}^{\infty} \left(1 - \frac{2\pi i s x}{\sqrt{n}} + \frac{1}{2}\left(\frac{2\pi i s x}{\sqrt{n}}\right)^2 + \text{small}\right) p(x)\,dx \\
&= \int_{-\infty}^{\infty} \left(1 - \frac{2\pi i s x}{\sqrt{n}} - \frac{2\pi^2 s^2 x^2}{n} + \text{small}\right) p(x)\,dx \\
&= \int_{-\infty}^{\infty} p(x)\,dx - \frac{2\pi i s}{\sqrt{n}} \int_{-\infty}^{\infty} xp(x)\,dx - \frac{2\pi^2 s^2}{n} \int_{-\infty}^{\infty} x^2 p(x)\,dx + \int_{-\infty}^{\infty} (\text{small})\, p(x)\,dx \\
&= 1 - \frac{2\pi^2 s^2}{n} + \text{small} .
\end{aligned}$$

To see more carefully what's going on with the "small" part, here's a different way of writing this.

$$\begin{aligned}
P\left(\frac{s}{\sqrt{n}}\right) &= \int_{-\infty}^{\infty} e^{-2\pi i s x/\sqrt{n}}\, p(x)\,dx \\
&= \int_{-\infty}^{\infty} \left(1 - \frac{2\pi i s x}{\sqrt{n}} + \frac{1}{2}\Big(\frac{2\pi i s x}{\sqrt{n}}\Big)^2 \big(1 + \epsilon_n(x)\big)\right) p(x)\,dx .
\end{aligned}$$

Here, $\epsilon_n(x)$ is bounded and tends to zero pointwise as $n \to \infty$. Therefore

$$P\left(\frac{s}{\sqrt{n}}\right) = 1 - \frac{2\pi^2 s^2}{n}\left(1 + \int_{-\infty}^{\infty} x^2 \epsilon_n(x) p(x)\, dx\right)$$

But since, by assumption,

$$\int_{-\infty}^{\infty} x^2 p(x)\, dx = 1$$

it's clear that

$$\int_{-\infty}^{\infty} x^2 \epsilon_n(x) p(x)\, dx = o(1)$$

as $n \to \infty$, i.e.,

$$\hat{p}\left(\frac{s}{\sqrt{n}}\right) = 1 - \frac{2\pi^2 s^2}{n}(1 + o(1)) = 1 - \frac{2\pi^2 s^2}{n} + \frac{o(1)}{n}\,.$$

Here we use the symbol $o(1)$ to denote a quantity that tends to zero as $n \to \infty$.

3.11 Appendix: Heisenberg's Inequality

Since we've gone to the trouble of introducing some of the terminology from probability and statistics (mean, variance, etc.), I thought you might appreciate seeing another application.

Consider the stretch theorem, which reads

- If $f(t) \rightleftharpoons F(s)$ then $f(at) \rightleftharpoons \frac{1}{|a|} F\left(\frac{s}{a}\right)$.

If a is large then $f(at)$ is squeezed and $(1/|a|)F(s/a)$ is stretched. Conversely if a is small then $f(at)$ is stretched and $(1/|a|)F(s/a)$ is squeezed.

A more quantitative statement of the trade-off between the spread of a signal and the spread of its Fourier transform is related to (equivalent to) that most famous inequality in quantum mechanics, the Heisenberg Uncertainty Principle.

Suppose $f(x)$ is a signal with finite energy,

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx < \infty\,.$$

We can normalize f by dividing f by the square root of its energy and thus assume that

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = 1\,.$$

We can then regard $|f(x)|^2$ as defining a probability density function, and it has a mean and a variance. Now, by Parseval's identity (which I've stated and will derive later),

$$\int_{-\infty}^{\infty} |\hat{f}(s)|^2\, ds = \int_{-\infty}^{\infty} |f(x)|^2\, dx = 1\,.$$

Thus $|\hat{f}(s)|^2$ also defines a probability distribution, and it too has a mean and variance. How do they compare to those of $|f(x)|^2$?

As earlier, we shift $f(x)$, or rather $|f(x)|^2$, to assume that the mean is 0. The effect on $\hat{f}(s)$ of shifting $f(x)$ is to multiply by a complex exponential, which has absolute value 1 and hence does not affect $|\hat{f}(s)|^2$. In the same manner we can shift $\hat{f}(s)$ so it has zero mean, and again there will be no effect on $|f(x)|^2$.

To summarize, we assume that the probability distributions $|f(x)|^2$ and $|\hat{f}(s)|^2$ each have mean 0, and we are interested in comparing their variances;

$$\sigma^2(f) = \int_{-\infty}^{\infty} x^2 |f(x)|^2 \, dx \quad \text{and} \quad \sigma^2(\hat{f}) = \int_{-\infty}^{\infty} s^2 |\hat{f}(s)|^2 \, ds \, .$$

The Heisenberg uncertainty principle states that

$$\sigma(f)\sigma(\hat{f}) \geq \frac{1}{4\pi} \, .$$

In words, this says that not both of $\sigma(f)$ and $\sigma(\hat{f})$ can be small — if one is tiny, the other has to be big enough so that their product is at least $1/4\pi$.

After all the setup, the argument to deduce the lower bound is pretty easy, except for a little trick right in the middle. It's also helpful to assume that we're working with complex-valued functions — the trick that comes up is a little easier to verify in that case. Finally, we're going to assume that $|f(x)|$ decreases rapidly enough at $\pm\infty$. You'll see what's needed. The result can be proved for more general functions via approximation arguments. Here we go.

$$\begin{aligned}
4\pi^2\sigma(f)^2\sigma(\hat{f})^2 &= 4\pi^2 \int_{-\infty}^{\infty} x^2 |f(x)|^2 \, dx \int_{-\infty}^{\infty} s^2 |\hat{f}(s)|^2 \, ds \\
&= \int_{-\infty}^{\infty} x^2 |f(x)|^2 \, dx \int_{-\infty}^{\infty} |2\pi i s|^2 |\hat{f}(s)|^2 \, ds \\
&= \int_{-\infty}^{\infty} |xf(x)|^2 \, dx \int_{-\infty}^{\infty} |\widehat{f'}(s)|^2 \, ds \\
&= \int_{-\infty}^{\infty} |xf(x)|^2 \, dx \int_{-\infty}^{\infty} |f'(x)|^2 \, dx \quad \text{(by Parseval's identity applied to } f'(x)) \\
&\geq \Big(\int_{-\infty}^{\infty} |x\overline{f(x)} f'(x)| \, dx \Big)^2 \quad \text{(by the Cauchy-Schwarz inequality)}
\end{aligned}$$

Here comes the trick. In the integrand we have $|\overline{f(x)}f'(x)|$. The magnitude of any complex number is always greater than its real part; draw a picture — the complex number is a vector, which is always longer than its x-component. Hence

$$\begin{aligned}
|x\overline{f(x)}f'(x)| &\geq x \, \mathrm{Re}\{\overline{f(x)}f'(x)\} \\
&= x \, \tfrac{1}{2}\big(\overline{f(x)}f'(x) + f(x)f'(x)^* \big) = x \frac{d}{dx} \tfrac{1}{2} \big(\overline{f(x)} f(x) \big) = x \frac{d}{dx} \tfrac{1}{2} |f(x)|^2 \, .
\end{aligned}$$

Use this in the last line, above:

$$\Big(\int_{-\infty}^{\infty} |x\overline{f(x)}f'(x)| \, dx \Big)^2 \geq \Big(\int_{-\infty}^{\infty} x \frac{d}{dx} \tfrac{1}{2} |f(x)|^2 \, dx \Big)^2$$

Now integrate by parts with $u = x$, $dv = \frac{1}{2}\frac{d}{dx}|f(x)|^2 \, dx$. The term $uv]_{-\infty}^{\infty}$ drops out because *we assume it does*, i.e., we assume that $x|f(x)|$ goes to zero as $x \to \pm\infty$. Therefore we're left with the integral of $v \, du$ (and the whole thing is squared). That is,

$$\Big(\int_{-\infty}^{\infty} x \frac{d}{dx} \tfrac{1}{2} |f(x)|^2 \, dx \Big)^2 \, dx = \tfrac{1}{4} \Big(\int_{-\infty}^{\infty} |f(x)|^2 \, dx \Big)^2 = \tfrac{1}{4} \, .$$

To summarize, we have shown that

$$4\pi^2\sigma(f)^2\sigma(\hat{f})^2 \geq \tfrac{1}{4} \quad \text{or} \quad \sigma(f)\sigma(\hat{f}) \geq \frac{1}{4\pi} \, .$$

Remark One can show, using the case of equality in the Cauchy-Schwarz inequality, that equality holds in Heisenberg's inequality exactly for constant multiples of $f(x) = e^{-kx^2}$ — yet another spooky appearance of the Gaussian.

Is this quantum mechanics? The quantum mechanics of a particle moving in one dimension that goes along with this inequality runs as follows — in skeletal form, with no attempt at motivation:

The *state* of a particle moving in one dimension is given by a complex-valued function ψ in $L^2(\mathbf{R})$, the square integrable functions on the real line. (L^2 plays a big role in quantum mechanics — you need a space to work in, and L^2 is the space. Really.) Probabilities are done with complex quantities in this business, and the first notion is that the probability of finding the particle in the interval $a \le x \le b$ is given by

$$\int_a^b \psi(x)^* \psi(x)\, dx\,,$$

where in this field it's customary to write the complex conjugate of a quantity using an asterisk instead of an overline.

An *observable* is a symmetric linear operator A, operating on some subset of functions (states) in $L^2(\mathbf{R})$. The *average* of A in the state ψ is defined to be

$$\int_{-\infty}^{\infty} \psi(x)^* (A\psi)(x)\, dx$$

One important observable is the "position of the particle", and this, as it turns out, is associated to the operator "multiplication by x". Thus the average position is

$$\int_{-\infty}^{\infty} \psi(x)^* (A\psi)(x)\, dx = \int_{-\infty}^{\infty} \psi(x)^* x\psi(x)\, dx = \int_{-\infty}^{\infty} x|\psi(x)|^2\, dx\,.$$

Another important observable is *momentum*, and this is associated with the operator

$$B = \frac{1}{2\pi i}\frac{d}{dx}\,.$$

The average momentum is then

$$\begin{aligned}\int_{-\infty}^{\infty} \psi(x)^* (B\psi)(x)\, dx &= \int_{-\infty}^{\infty} \psi(x)^* \frac{1}{2\pi i}\psi'(x)\, dx \\ &= \int_{-\infty}^{\infty} \hat{\psi}(s)^* s\hat{\psi}(s)\, ds \quad \text{(using the Parseval identity for products of functions)} \\ &= \int_{-\infty}^{\infty} s|\psi(s)|^2\, ds\,.\end{aligned}$$

The position and momentum operators do *not* commute:

$$(AB - BA)(\psi) = \frac{1}{2\pi i}\left(x\frac{d}{dx} - \frac{d}{dx}x\right)(\psi) = -\frac{1}{2\pi i}\psi\,.$$

In quantum mechanics this means that the position and momentum cannot *simultaneously* be measured with arbitrary accuracy. The Heisenberg inequality, as a lower bound for the product of the two variances, is a quantitative way of stating this.

Chapter 4

Distributions and Their Fourier Transforms

4.1 The Day of Reckoning

We've been playing a little fast and loose with the Fourier transform — applying Fourier inversion, appealing to duality, and all that. "Fast and loose" is an understatement if ever there was one, but it's also true that we haven't done anything "wrong". All of our formulas and all of our applications have been correct, if not fully justified. Nevertheless, we have to come to terms with some fundamental questions. It will take us some time, but in the end we will have settled on a very wide class of signals with these properties:

- The allowed signals include δ's, unit steps, ramps, sines, cosines, and all other standard signals that the world's economy depends on.
- The Fourier transform and its inverse are defined for all of these signals.
- Fourier inversion works.

These are the three most important features of the development to come, but we'll also reestablish some of our specific results and as an added benefit we'll even finish off differential calculus!

4.1.1 A too simple criterion and an example

It's not hard to write down an assumption on a function that guarantees the existence of its Fourier transform and even implies a little more than existence.

- If $\displaystyle\int_{-\infty}^{\infty} |f(t)|\,dt < \infty$ then $\mathcal{F}f$ and $\mathcal{F}^{-1}f$ exist and are continuous.

Existence follows from

$$\begin{aligned} |\mathcal{F}f(s)| &= \left| \int_{-\infty}^{\infty} e^{-2\pi i s t} f(t)\,dt \right| \\ &\leq \int_{-\infty}^{\infty} |e^{-2\pi i s t}|\,|f(t)|\,dt = \int_{-\infty}^{\infty} |f(t)|\,dt < \infty\,. \end{aligned}$$

Here we've used that the magnitude of the integral is less that the integral of the magnitude.[1] There's actually something to say here, but while it's not complicated, I'd just as soon defer this and other comments on "general facts on integrals" to Section 4.3; read it if only lightly — it provides some additional orientation.

Continuity is the little extra information we get beyond existence. Continuity follows as follows. For any s and s' we have

$$\begin{aligned}|\mathcal{F}f(s) - \mathcal{F}f(s')| &= \left|\int_{-\infty}^{\infty} e^{-2\pi ist} f(t)\,dt - \int_{-\infty}^{\infty} e^{-2\pi is't} f(t)\,dt\right| \\ &= \left|\int_{-\infty}^{\infty} (e^{-2\pi ist} - e^{-2\pi is't}) f(t)\,dt\right| \le \int_{-\infty}^{\infty} |e^{-2\pi ist} - e^{-2\pi is't}|\,|f(t)|\,dt\end{aligned}$$

As a consequence of $\int_{-\infty}^{\infty} |f(t)|\,dt < \infty$ we can take the limit as $s' \to s$ inside the integral. If we do that then $|e^{-2\pi ist} - e^{-2\pi is't}| \to 0$, that is,

$$|\mathcal{F}f(s) - \mathcal{F}f(s')| \to 0 \quad \text{as} \quad s' \to s$$

which says that $\mathcal{F}f(s)$ is continuous. The same argument works to show that $\mathcal{F}^{-1}f$ is continuous.[2]

We haven't said anything here about Fourier inversion — no such statement appears in the criterion. Let's look right away at an example.

The very first example we computed, and still an important one, is the Fourier transform of Π. We found directly that

$$\mathcal{F}\Pi(s) = \int_{-\infty}^{\infty} e^{-2\pi ist}\Pi(t)\,dt = \int_{-1/2}^{1/2} e^{-2\pi ist}\,dt = \operatorname{sinc} s\,.$$

No problem there, no problem whatsoever. The criterion even applies; Π is in $L^1(\mathbf{R})$ since

$$\int_{-\infty}^{\infty} |\Pi(t)|\,dt = \int_{-1/2}^{1/2} 1\,dt = 1\,.$$

Furthermore, the transform $\mathcal{F}\Pi(s) = \operatorname{sinc} s$ is continuous. That's worth remarking on: Although the signal jumps (Π has a discontinuity) the *Fourier transform* does not, just as guaranteed by the preceding result — make this part of your intuition on the Fourier transform vis à vis the signal.

Appealing to the Fourier inversion theorem and what we called duality, we then said

$$\mathcal{F}\operatorname{sinc}(t) = \int_{-\infty}^{\infty} e^{-2\pi ist} \operatorname{sinc} t\,dt = \Pi(s)\,.$$

Here we have a problem. The sinc function *does not* satisfy the integrability criterion. It is my sad duty to inform you that

$$\int_{-\infty}^{\infty} |\operatorname{sinc} t|\,dt = \infty\,.$$

I'll give you two ways of seeing the failure of $|\operatorname{sinc} t|$ to be integrable. First, if sinc did satisfy the criterion $\int_{-\infty}^{\infty} |\operatorname{sinc} t|\,dt < \infty$ then its Fourier transform would be continuous. But its Fourier transform, which *has*

[1] Magnitude, not absolute value, because the integral is complex number.

[2] So another general fact we've used here is that we can take the limit inside the integral. Save yourself for other things and let some of these "general facts" ride without insisting on complete justifications — they're everywhere once you let the rigor police back on the beat.

to come out to be Π, is *not* continuous. Or, if you don't like that, here's a direct argument. We can find infinitely many intervals where $|\sin \pi t| \geq 1/2$; this happens when t is between $1/6$ and $5/6$, and that repeats for infinitely many intervals, for example on $I_n = [\frac{1}{6} + 2n, \frac{5}{6} + 2n]$, $n = 0, 1, 2, \ldots$, because $\sin \pi t$ is periodic of period 2. The I_n all have length $2/3$. On I_n we have $|t| \leq \frac{5}{6} + 2n$, so

$$\frac{1}{|t|} \geq \frac{1}{5/6 + 2n}$$

and

$$\int_{I_n} \frac{|\sin \pi t|}{\pi |t|}\, dt \geq \frac{1}{2\pi} \frac{1}{5/6 + 2n} \int_{I_n} dt = \frac{1}{2\pi} \frac{2}{3} \frac{1}{5/6 + 2n}\,.$$

Then

$$\int_{-\infty}^{\infty} \frac{|\sin \pi t|}{\pi |t|}\, dt \geq \sum_n \int_{I_n} \frac{|\sin \pi t|}{\pi |t|}\, dt = \frac{1}{3\pi} \sum_{n=1}^{\infty} \frac{1}{5/6 + 2n} = \infty\,.$$

It's true that $|\operatorname{sinc} t| = |\sin \pi t/\pi t|$ tends to 0 as $t \to \pm\infty$ — the $1/t$ factor makes that happen — but not "fast enough" to make the integral of $|\operatorname{sinc} t|$ converge.

This is the most basic example in the theory! It's not clear that the integral defining the Fourier transform of sinc exists, at least it doesn't follow from the criterion. Doesn't this bother you? Isn't it a little embarrassing that multibillion dollar industries seem to depend on integrals that don't converge?

In fact, there isn't so much of a problem with either Π or sinc. It *is* true that

$$\int_{-\infty}^{\infty} e^{-2\pi i s t} \operatorname{sinc} s\, ds = \begin{cases} 1 & |t| < \frac{1}{2} \\ 0 & |t| > \frac{1}{2} \end{cases}$$

However showing this — evaluating the improper integral that defines the Fourier transform — requires special arguments and techniques. The sinc function oscillates, as do the real and imaginary parts of the complex exponential, and integrating $e^{-2\pi i s t} \operatorname{sinc} s$ involves enough cancellation for the limit

$$\lim_{\substack{a \to -\infty \\ b \to \infty}} \int_a^b e^{-2\pi i s t} \operatorname{sinc} s\, ds$$

to exist.

Thus Fourier inversion, and duality, can be pushed through in this case. At least almost. You'll notice that I didn't say anything about the points $t = \pm 1/2$, where there's a jump in Π in the time domain. In those cases the improper integral does not exist, but with some additional interpretations one might be able to convince a sympathetic friend that

$$\int_{-\infty}^{\infty} e^{-2\pi i (\pm 1/2) s} \operatorname{sinc} s\, ds = \tfrac{1}{2}$$

in the appropriate sense (invoking "principle value integrals" — more on this in a later lecture). At best this is *post hoc* and needs some fast talking.[3]

The truth is that cancellations that occur in the sinc integral or in its Fourier transform are a very subtle and dicey thing. Such risky encounters are to be avoided. We'd like a more robust, trustworthy theory.

[3] One might also then argue that defining $\Pi(\pm 1/2) = 1/2$ is the best choice. I don't want to get into it.

The news so far Here's a quick summary of the situation. The Fourier transform of $f(t)$ is defined when

$$\int_{-\infty}^{\infty} |f(t)|\,dt < \infty\,.$$

We allow f to be complex valued in this definition. The collection of all functions on $\mathbf{R}$ satisfying this condition is denoted by $L^1(\mathbf{R})$, the superscript 1 indicating that we integrate $|f(t)|$ to the first power.[4] The L^1-norm of F is defined by

$$\|f\|_1 = \int_{-\infty}^{\infty} |f(t)|\,dt\,.$$

Many of the examples we worked with are L^1-functions — the rect function, the triangle function, the exponential decay (one or two-sided), Gaussians — so our computations of the Fourier transforms in those cases were perfectly justifiable (and correct). Note that L^1-functions can have discontinuities, as in the rect function.

The criterion says that if $f \in L^1(\mathbf{R})$ then $\mathcal{F}f$ exists. We can also say

$$|\mathcal{F}f(s)| = \left|\int_{-\infty}^{\infty} e^{-2\pi ist} f(t)\,dt\right| \leq \int_{-\infty}^{\infty} |f(t)|\,dt = \|f\|_1\,.$$

That is:

- The magnitude of the Fourier transform is bounded by the L^1-norm of the function.

This is a handy estimate to be able to write down — we'll use it shortly. However, to issue a warning:

> Fourier transforms of $L^1(\mathbf{R})$ functions may themselves *not* be in L^1, like for the sinc function, so we don't know without further work what more can be done, if anything.

The conclusion is that L^1-integrability of a signal is just too simple a criterion on which to build a really helpful theory. This is a serious issue for us to understand. Its resolution will greatly extend the usefulness of the methods we have come to rely on.

There are other problems, too. Take, for example, the signal $f(t) = \cos 2\pi t$. As it stands now, this signal does not even *have* a Fourier transform — does not have a spectrum! — for the integral

$$\int_{-\infty}^{\infty} e^{-2\pi ist} \cos 2\pi t\,dt$$

does not converge, no way, no how. This is no good.

Before we bury $L^1(\mathbf{R})$ as too restrictive for our needs, here's one more good thing about it. There's actually a stronger consequence for $\mathcal{F}f$ than just continuity.

- If $\displaystyle\int_{-\infty}^{\infty} |f(t)|\,dt < \infty$ then $\mathcal{F}f(s) \to 0$ as $s \to \pm\infty$.

[4] And the letter "L" indicating that it's really the Lebesgue integral that should be employed.

This is called the Riemann-Lebesgue lemma and it's more difficult to prove than showing simply that $\mathcal{F}f$ is continuous. I'll comment on it later; see Section 4.19. One might view the result as saying that $\mathcal{F}f(s)$ is at least *trying* to be integrable. It's continuous and it tends to zero as $s \to \pm\infty$. Unfortunately, the fact that $\mathcal{F}f(s) \to 0$ does not imply that it's integrable (think of sinc, again).[5] If we knew something, or could insist on something about the *rate* at which a signal or its transform tends to zero at $\pm\infty$ then perhaps we could push on further.

4.1.2 The path, the way

To repeat, we want our theory to encompass the following three points:

- The allowed signals include δ's, unit steps, ramps, sines, cosines, and all other standard signals that the world's economy depends on.
- The Fourier transform and its inverse are defined for all of these signals.
- Fourier inversion works.

Fiddling around with $L^1(\mathbf{R})$ or substitutes, putting extra conditions on jumps — all have been used. The path to success lies elsewhere. It is well marked and firmly established, but it involves a break with the classical point of view. The outline of how all this is settled goes like this:

1. We single out a collection of functions $\mathcal{S}$ for which convergence of the Fourier integrals is assured, for which a function *and* its Fourier transform are both in $\mathcal{S}$, and for which Fourier inversion works. Furthermore, Parseval's identity holds:

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-\infty}^{\infty} |\mathcal{F}f(s)|^2\,ds\,.$$

   This much *is* classical; new ideas with new *intentions*, yes, but not new *objects*. Perhaps surprisingly it's not so hard to find a suitable collection $\mathcal{S}$, at least if one knows what one is looking for. But what comes next is definitely not "classical". It had been first anticipated and used effectively in an early form by O. Heaviside, developed, somewhat, and dismissed, mostly, soon after by less talented people, then cultivated by and often associated with the work of P. Dirac, and finally refined by L. Schwartz.

2. $\mathcal{S}$ forms a class of *test functions* which, in turn, serve to define a larger class of *generalized functions* or *distributions*, called, for this class of test functions the *tempered distributions*, $\mathcal{T}$. Precisely because $\mathcal{S}$ was chosen to be the ideal Fourier friendly space of classical signals, the tempered distributions are likewise well suited for Fourier methods. The collection of tempered distributions includes, for example, L^1 and L^2-functions (which can be wildly discontinuous), the sinc function, and complex exponentials (hence periodic functions). But it includes much more, like the delta functions and related objects.

3. The Fourier transform and its inverse will be defined so as to operate on these tempered distributions, and they operate to produce distributions of the same type. Thus the inverse Fourier transform can be applied, and the Fourier inversion theorem holds in this setting.

4. In the case when a tempered distributions "comes from a function" — in a way we'll make precise — the Fourier transform reduces to the usual definition as an integral, when the integral makes sense. However, tempered distributions are more general than functions, so we really will have done something new and we won't have lost anything in the process.

[5] For that matter, a function in $L^1(\mathbf{R})$ need not tend to zero at $\pm\infty$; that's also discussed in Appendix 1.

Our goal is to hit the relatively few main ideas in the outline above, suppressing the considerable mass of details. In practical terms this will enable us to introduce delta functions and the like as tools for computation, and to feel a greater measure of confidence in the range of applicability of the formulas. We're taking this path because it works, it's very interesting, and it's easy to compute with. I especially want you to believe the last point.

We'll touch on some other approaches to defining distributions and generalized Fourier transforms, but as far as I'm concerned they are the equivalent of vacuum tube technology. You can do distributions in other ways, and some people really love building things with vacuum tubes, but wouldn't you rather learn something a little more up to date?

4.2 The Right Functions for Fourier Transforms: Rapidly Decreasing Functions

Mathematics progresses more by making intelligent definitions than by proving theorems. The hardest work is often in formulating the fundamental concepts in the right way, a way that will then make the deductions from those definitions (relatively) easy and natural. This can take awhile to sort out, and a subject might be reworked several times as it matures; when new discoveries are made and one sees where things end up, there's a tendency to go back and change the starting point so that the trip becomes easier. Mathematicians may be more self-conscious about this process, but there are certainly examples in engineering where close attention to the basic definitions has shaped a field — think of Shannon's work on Information Theory, for a particularly striking example.

Nevertheless, engineers, in particular, often find this tiresome, wanting to *do something* and not "just talk about it": "Devices don't have hypotheses", as one of my colleagues put it. One can also have too much of a good thing — too many trips back to the starting point to rewrite the rules can make it hard to follow the game, especially if one has already played by the earlier rules. I'm sympathetic to both of these criticisms, and for our present work on the Fourier transform I'll try to steer a course that makes the definitions reasonable and lets us make steady forward progress.

4.2.1 Smoothness and decay

To ask "how fast" $\mathcal{F}f(s)$ might tend to zero, depending on what additional assumptions we might make about the function $f(x)$ *beyond* integrability, will lead to our defining "rapidly decreasing functions", and this is the key. Integrability is too weak a condition on the signal f, but it does imply that $\mathcal{F}f(s)$ is continuous and tends to 0 at $\pm\infty$. What we're going to do is study the relationship between the *smoothness* of a function — not just continuity, but how many times it can be differentiated — and the rate at which its Fourier transform decays at infinity.

We'll always assume that $f(x)$ is absolutely integrable, and so has a Fourier transform. Let's suppose, more stringently, that

- $xf(x)$ is integrable, i.e., $\displaystyle\int_{-\infty}^{\infty} |xf(x)|\,dx < \infty$.

Then $xf(x)$ has a Fourier transform, and so does $-2\pi ixf(x)$ and its Fourier transform is

$$\begin{aligned}\mathcal{F}(-2\pi ixf(x)) &= \int_{-\infty}^{\infty}(-2\pi ix)e^{-2\pi isx}f(x)\,dx \\ &= \int_{-\infty}^{\infty}\left(\frac{d}{ds}e^{-2\pi isx}\right)f(x)\,dx = \frac{d}{ds}\int_{-\infty}^{\infty}e^{-2\pi isx}f(x)\,dx \\ &\qquad \text{(switching } d/ds \text{ and the integral is justified by the integrability of } |xf(x)|) \\ &= \frac{d}{ds}(\mathcal{F}f)(s)\end{aligned}$$

This says that the Fourier transform $\mathcal{F}f(s)$ is differentiable and that its derivative is $\mathcal{F}(-2\pi ixf(x))$. When $f(x)$ is merely integrable we know that $\mathcal{F}f(s)$ is merely continuous, but with the extra assumption on the integrability of $xf(x)$ we conclude that $\mathcal{F}f(s)$ is actually differentiable. (And its derivative is continuous. Why?)

For one more go-round in this direction, what if $x^2f(x)$ is integrable? Then, by the same argument,

$$\begin{aligned}\mathcal{F}((-2\pi ix)^2f(x)) &= \int_{-\infty}^{\infty}(-2\pi ix)^2e^{-2\pi isx}f(x)\,dx \\ &= \int_{-\infty}^{\infty}\left(\frac{d^2}{ds^2}e^{-2\pi isx}\right)f(x)\,dx = \frac{d^2}{ds^2}\int_{-\infty}^{\infty}e^{-2\pi isx}f(x)\,dx = \frac{d^2}{ds^2}(\mathcal{F}f)(s)\,,\end{aligned}$$

and we see that $\mathcal{F}f$ is twice differentiable. (And its second derivative is continuous.)

Clearly we can proceed like this, and as a somewhat imprecise headline we might then announce:

- Faster decay of $f(x)$ at infinity leads to a greater smoothness of the Fourier transform.

Now let's take this in another direction, with an assumption on the smoothness of the signal. Suppose $f(x)$ is differentiable, that its derivative is integrable, and that $f(x) \to 0$ as $x \to \pm\infty$. I've thrown in all the assumptions I need to justify the following calculation:

$$\begin{aligned}\mathcal{F}f(s) &= \int_{-\infty}^{\infty}e^{-2\pi isx}f(x)\,dx \\ &= \left[f(x)\frac{e^{-2\pi isx}}{-2\pi is}\right]_{x=-\infty}^{x=\infty} - \int_{-\infty}^{\infty}\frac{e^{-2\pi isx}}{-2\pi is}f'(x)\,dx \\ &\qquad \text{(integration by parts with } u = f(x),\ dv = e^{-2\pi isx}dx) \\ &= \frac{1}{2\pi is}\int_{-\infty}^{\infty}e^{-2\pi isx}f'(x)\,dx \quad \text{(using } f(x) \to 0 \text{ as } x \to \pm\infty) \\ &= \frac{1}{2\pi is}(\mathcal{F}f')(s)\end{aligned}$$

We then have

$$|\mathcal{F}f(s)| = \frac{1}{2\pi s}|(\mathcal{F}f')(s)| \le \frac{1}{2\pi s}\|f'\|_1\,.$$

The last inequality follows from the result: "The Fourier transform is bounded by the L^1-norm of the function". This says that $\mathcal{F}f(s)$ tends to 0 at $\pm\infty$ *like* $1/s$. (Remember that $\|f'\|_1$ is some fixed number here, independent of s.) Earlier we commented (without proof) that if f is integrable then $\mathcal{F}f$ tends to 0 at $\pm\infty$, but here with the stronger assumptions we get a stronger conclusion, that $\mathcal{F}f$ tends to zero *at a certain rate*.

Let's go one step further in this direction. Suppose $f(x)$ is *twice* differentiable, that its first and second derivatives are integrable, and that $f(x)$ and $f'(x)$ tend to 0 as $x \to \pm\infty$. The same argument gives

$$\begin{aligned}
\mathcal{F}f(s) &= \int_{-\infty}^{\infty} e^{-2\pi isx} f(x)\,dx \\
&= \frac{1}{2\pi is}\int_{-\infty}^{\infty} e^{-2\pi isx} f'(x)\,dx \quad \text{(picking up on where we were before)} \\
&= \frac{1}{2\pi is}\left(\left[f'(x)\frac{e^{-2\pi isx}}{-2\pi is}\right]_{x=-\infty}^{x=\infty} - \int_{-\infty}^{\infty}\frac{e^{-2\pi isx}}{-2\pi is} f''(x)\,dx\right) \\
&\qquad \text{(integration by parts with } u = f'(x),\ dv = e^{-2\pi isx}dx) \\
&= \frac{1}{(2\pi is)^2}\int_{-\infty}^{\infty} e^{-2\pi isx} f''(x)\,dx \quad \text{(using } f'(x) \to 0 \text{ as } x \to \pm\infty) \\
&= \frac{1}{(2\pi is)^2}(\mathcal{F}f'')(s)
\end{aligned}$$

Thus

$$|\mathcal{F}f(s)| \le \frac{1}{|2\pi s|^2}\|f''\|_1$$

and we see that $\mathcal{F}f(s)$ tends to 0 like $1/s^2$.

The headline:

- Greater smoothness of $f(x)$, plus integrability, leads to faster decay of the Fourier transform at ∞.

Remark on the derivative formula for the Fourier transform The astute reader will have noticed that in the course of our work we rederived the derivative formula

$$\mathcal{F}f'(s) = 2\pi is\mathcal{F}f(s)$$

which we've used before, but here we needed the assumption that $f(x) \to 0$, which we didn't mention before. What's up? With the technology we have available to us now, the derivation we gave, above, is the correct derivation. That is, it proceeds via integration by parts, and requires some assumption like $f(x) \to 0$ as $x \to \pm\infty$. In homework (and in the solutions to the homework) you may have given a derivation that used duality. That only works if Fourier inversion is known to hold. This was OK when the rigor police were off duty, but not now, on this day of reckoning. Later, when we develop a generalization of the Fourier transform, we'll see that the derivative formula again holds without what seem now to be extraneous conditions.

We could go on as we did above, comparing the consequences of higher differentiability, integrability, smoothness and decay, bouncing back and forth between the function and its Fourier transform. The great insight in making use of these observations is that the simplest and most useful way to coordinate all these phenomena is to allow for *arbitrarily great smoothness* and *arbitrarily fast decay*. We would like to have both phenomena in play. Here is the crucial definition.

Rapidly decreasing functions

A function $f(x)$ is said to be *rapidly decreasing* at $\pm\infty$ if

1. It is infinitely differentiable.

2. For *all* positive integers m and n,

$$\left| x^m \frac{d^n}{dx^n} f(x) \right| \to 0 \quad \text{as} \quad x \to \pm\infty$$

In words, *any* positive power of x times *any* order derivative of f tends to zero at infinity.

Note that m and n are independent in this definition. That is, we insist that, say, the 5th power of x times the 17th derivative of $f(x)$ tends to zero, and that the 100th power of x times the first derivative of $f(x)$ tends to zero; and whatever you want.

Are there any such functions? Any infinitely differentiable function that is identically zero outside some finite interval is one example, and I'll even write down a formula for one of these later. Another example is $f(x) = e^{-x^2}$. You may already be familiar with the phrase "the exponential grows faster than any power of x", and likewise with the phrase "e^{-x^2} decays faster than any power of x."[6] In fact, any derivative of e^{-x^2} decays faster than any power of x as $x \to \pm\infty$, as you can check with L'Hopital's rule, for example. We can express this exactly as in the definition:

$$\left| x^m \frac{d^n}{dx^n} e^{-x^2} \right| \to 0 \quad \text{as} \quad x \to \pm\infty$$

There are plenty of other rapidly decreasing functions. We also remark that if $f(x)$ is rapidly decreasing then it is in $L^1(\mathbf{R})$ and in $L^2(\mathbf{R})$; check that yourself.

An alternative definition An equivalent definition for a function to be rapidly decreasing is to assume that for any positive integers m and n there is a constant C_{mn} such that

$$\left| x^m \frac{d^n}{dx^n} f(x) \right| \leq C_{mn} \quad \text{as} \quad x \to \pm\infty\,.$$

In words, the mth power of x times the nth derivative of f remains bounded for all m and n, though the constant will depend on which m and n we take. This condition implies the "tends to zero" condition, above. Convince yourself of that, the key being that m and n are arbitrary and independent. We'll use this second, equivalent condition often, and it's a matter of taste which one takes as a definition.

Let us now praise famous men It was the French mathematician Laurent Schwartz who singled out this relatively simple condition to use in the service of the Fourier transform. In his honor the set of rapidly decreasing functions is usually denoted by $\mathcal{S}$ (a script S) and called the *Schwartz class* of functions.

Let's start to see why this was such a good idea.

1. The Fourier transform of a rapidly decreasing function is rapidly decreasing. Let $f(x)$ be a function in $\mathcal{S}$. We want to show that $\mathcal{F}f(s)$ is also in $\mathcal{S}$. The condition involves derivatives of $\mathcal{F}f$, so what comes in is the derivative formula for the Fourier transform and the version of that formula for higher derivatives. As we've already seen

$$2\pi i s \mathcal{F}f(s) = \left(\mathcal{F}\frac{d}{dx}f\right)(s)\,.$$

[6] I used e^{-x^2} as an example instead of e^{-x} (for which the statement is true as $x \to \infty$) because I wanted to include $x \to \pm\infty$, and I used e^{-x^2} instead of $e^{-|x|}$ because I wanted the example to be smooth. $e^{-|x|}$ has a corner at $x = 0$.

As we also noted,

$$\frac{d}{ds}\mathcal{F}f(s) = \mathcal{F}(-2\pi i x f(x))\,.$$

Because $f(x)$ is rapidly decreasing, the higher order versions of these formulas are valid; the derivations require either integration by parts or differentiating under the integral sign, both of which are justified. That is,

$$(2\pi i s)^n \mathcal{F}f(s) = \Big(\mathcal{F}\frac{d^n}{dx^n}f\Big)(s)$$
$$\frac{d^n}{ds^n}\mathcal{F}f(s) = \mathcal{F}\big((-2\pi i x)^n f(x)\big)\,.$$

(We follow the convention that the zeroth order derivative leaves the function alone.)

Combining these formulas one can show, inductively, that for all nonnegative integers m and n,

$$\mathcal{F}\Big(\frac{d^n}{dx^n}\big((-2\pi i x)^m f(x)\big)\Big) = (2\pi i s)^n \frac{d^m}{ds^m}\mathcal{F}f(s)\,.$$

Note how m and n enter in the two sides of the equation.

We use this last identity together with the estimate for the Fourier transform in terms of the L^1-norm of the function. Namely,

$$|s|^n \left|\frac{d^m}{ds^m}\mathcal{F}f(s)\right| = (2\pi)^{m-n}\left|\mathcal{F}\Big(\frac{d^n}{dx^n}(x^m f(x))\Big)\right| \le (2\pi)^{m-n}\left\|\frac{d^n}{dx^n}(x^m f(x))\right\|_1$$

The L^1-norm on the right hand side is finite because f is rapidly decreasing. Since the right hand side depends on m and n, we have shown that there is a constant C_{mn} with

$$\left|s^n \frac{d^m}{ds^m}\mathcal{F}f(s)\right| \le C_{mn}\,.$$

This implies that $\mathcal{F}f$ is rapidly decreasing. Done.

2. Fourier inversion works on $\mathcal{S}$. We first establish the inversion theorem for a *timelimited* function in $\mathcal{S}$. Suppose that $f(t)$ is smooth and for some T is *identically zero* for $|t| \ge T/2$, rather than just tending to zero at $\pm\infty$. In this case we can periodize $f(t)$ to get a smooth, periodic function of period T. Expand the periodic function as a *converging* Fourier series. Then for $-T/2 \le t \le T/2$,

$$\begin{aligned} f(t) &= \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n t/T} \\ &= \sum_{n=-\infty}^{\infty} e^{2\pi i n t/T}\left(\frac{1}{T}\int_{-T/2}^{T/2} e^{-2\pi i n x/T} f(x)\,dx\right) \\ &= \sum_{n=-\infty}^{\infty} e^{2\pi i n t/T}\left(\frac{1}{T}\int_{-\infty}^{\infty} e^{-2\pi i n x/T} f(x)\,dx\right) = \sum_{n=-\infty}^{\infty} e^{2\pi i n t/T}\mathcal{F}f\left(\frac{n}{T}\right)\frac{1}{T}\,. \end{aligned}$$

Our intention is to let T get larger and larger. What we see is a Riemann sum for the integral

$$\int_{-\infty}^{\infty} e^{2\pi i s t}\mathcal{F}f(s)\,ds = \mathcal{F}^{-1}\mathcal{F}f(t)\,,$$

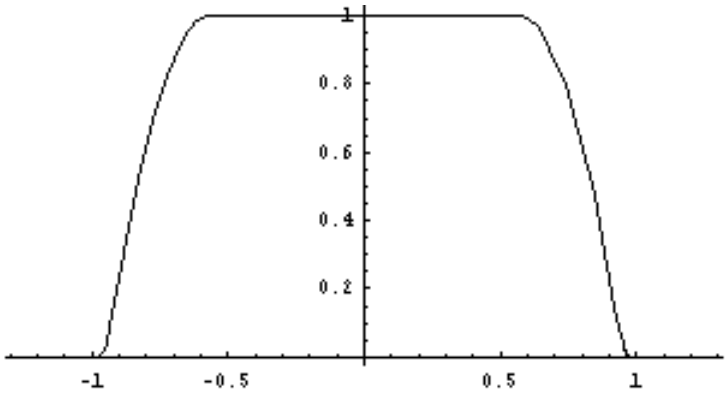
and the Riemann sum converges to the integral because of the smoothness of f. (I have not slipped anything past you here, but I don't want to quote the precise results that make all this legitimate.) Thus

$$f(t) = \mathcal{F}^{-1}\mathcal{F}f(t)\,,$$

and the Fourier inversion theorem is established for timelimited functions in $\mathcal{S}$.

When f is *not* timelimited we use "windowing". The idea is to cut $f(t)$ off *smoothly*.[7] The interesting thing in the present context — for theoretical rather than practical use — is to make the window so smooth that the "windowed" function is still in $\mathcal{S}$. Some of the details are in Section 4.20, but here's the setup.

We take a function $c(t)$ that is identically 1 for $-1/2 \le t \le 1/2$, that goes *smoothly* (infinitely differentiable) down to zero as t goes from $1/2$ to 1 and from $-1/2$ to -1, and is then identically 0 for $t \ge 1$ and $t \le -1$. This is a smoothed version of the rectangle function $\Pi(t)$; instead of cutting off sharply at $\pm 1/2$ we bring the function smoothly down to zero. You can certainly imagine drawing such a function:



In Section 4.20 I'll give an explicit formula for this.

Now scale $c(t)$ to $c_n(t) = c(t/n)$. That is, $c_n(t)$ is 1 for t between $-n/2$ and $n/2$, goes smoothly down to 0 between $\pm n/2$ and $\pm n$ and is then identically 0 for $|t| \ge n$. Next, the function $f_n(t) = c_n(t) \cdot f(t)$ is a timelimited function in $\mathcal{S}$. Hence the earlier reasoning shows that the Fourier inversion theorem holds for f_n and $\mathcal{F}f_n$. The window eventually moves past every t, that is, $f_n(t) \to f(t)$ as $n \to \infty$. Some estimates based on the properties of the cut-off function — which I won't go through — show that the Fourier inversion theorem also holds in the limit.

3. Parseval holds in $\mathcal{S}$. We'll actually derive a more general result than Parseval's identity, namely:

If $f(x)$ and $g(x)$ are complex valued functions in $\mathcal{S}$ then

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)}\,dx = \int_{-\infty}^{\infty} \mathcal{F}f(s)\overline{\mathcal{F}g(s)}\,ds\,.$$

As a special case, if we take $f = g$ then $f(x)\overline{f(x)} = |f(x)|^2$ and the identity becomes

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-\infty}^{\infty} |\mathcal{F}f(s)|^2\,ds\,.$$

[7] The design of windows, like the design of filters, is as much an art as a science.

To get the first result we'll use the fact that we can recover g from its Fourier transform via the inversion theorem. That is,

$$g(x) = \int_{-\infty}^{\infty} \mathcal{F}g(s) e^{2\pi i s x}\, ds\,.$$

The complex conjugate of the integral is the integral of the complex conjugate, hence

$$\overline{g(x)} = \int_{-\infty}^{\infty} \overline{\mathcal{F}g(s)} e^{-2\pi i s x}\, ds\,.$$

The derivation is straightforward, using one of our favorite tricks of interchanging the order of integration:

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)\overline{g(x)}\, dx &= \int_{-\infty}^{\infty} f(x) \left(\int_{-\infty}^{\infty} \overline{\mathcal{F}g(s)} e^{-2\pi i s x}\, ds \right) dx \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)\overline{\mathcal{F}g(s)} e^{-2\pi i s x}\, ds\, ds \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)\overline{\mathcal{F}g(s)} e^{-2\pi i s x}\, dx\, dx \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x) e^{-2\pi i s x}\, dx \right) \overline{\mathcal{F}g(s)}\, ds \\
&= \int_{-\infty}^{\infty} \mathcal{F}f(s)\overline{\mathcal{F}g(s)}\, ds
\end{aligned}$$

All of this works perfectly — the initial appeal to the Fourier inversion theorem, switching the order of integration — if f and g are rapidly decreasing.

4.3 A Very Little on Integrals

This section on integrals, more of a mid-chapter appendix, is not a short course on integration. It's here to provide a little, but only a little, background explanation for some of the statements made earlier. The star of this section is you. Here you go.

Integrals are first defined for positive functions In the general approach to integration (of real-valued functions) you first set out to define the integral for *nonnegative* functions. Why? Because however general a theory you're constructing, an integral is going to be some kind of limit of sums and you'll want to know when that kind of limit exists. If you work with positive (or at least nonnegative) functions then the issues for limits will be about how big the function gets, or about how big the sets are where the function is or isn't big. You feel better able to analyze accumulations than to control conspiratorial cancellations.

So you first define your integral for functions $f(x)$ with $f(x) \geq 0$. This works fine. However, you know full well that your definition won't be too useful if you can't extend it to functions which are both positive and negative. Here's how you do this. For any function $f(x)$ you let $f^+(x)$ be its *positive part*:

$$f^+(x) = \max\{f(x), 0\}$$

Likewise, you let

$$f^-(x) = \max\{-f(x), 0\}$$

be its *negative part*.[8] (Tricky: the "negative part" as you've defined it is actually a positive function; taking $-f(x)$ flips over the places where $f(x)$ is negative to be positive. You like that kind of thing.) Then

$$f = f^+ - f^-$$

[8] A different use of the notation f^- than we had before, but we'll never use this one again.

while

$$|f| = f^+ + f^- .$$

You now say that f is integrable if both f^+ and f^- are integrable — a condition which makes sense since f^+ and f^- are both nonnegative functions — and *by definition* you set

$$\int f = \int f^+ - \int f^- .$$

(For complex-valued functions you apply this to the real and imaginary parts.) You follow this approach for integrating functions on a finite interval or on the whole real line. Moreover, according to this definition $|f|$ is integrable if f is because then

$$\int |f| = \int (f^+ + f^-) = \int f^+ + \int f^-$$

and f^+ and f^- are each integrable.[9] It's also true, conversely, that if $|f|$ is integrable then so is f. You show this by observing that

$$f^+ \le |f| \quad \text{and} \quad f^- \le |f|$$

and this implies that both f^+ and f^- are integrable.

- You now know where the implication $\displaystyle\int_{-\infty}^{\infty} |f(t)|\, dt < \infty \;\Rightarrow\; \mathcal{F}f$ exists comes from.

You get an easy inequality out of this development:

$$\left|\int f\right| \le \int |f| .$$

In words, "the absolute value of the integral is at most the integral of the absolute value". And sure that's true, because $\int f$ may involve cancellations of the positive and negative values of f while $\int |f|$ won't have such cancellations. You don't shirk from a more formal argument:

$$\begin{aligned}
\left|\int f\right| &= \left|\int (f^+ - f^-)\right| = \left|\int f^+ - \int f^-\right| \\
&\le \left|\int f^+\right| + \left|\int f^-\right| = \int f^+ + \int f^- \quad \text{(since } f^+ \text{ and } f^- \text{ are both nonnegative)} \\
&= \int (f^+ + f^-) = \int |f| .
\end{aligned}$$

- You now know where the second inequality in

$$|\mathcal{F}f(s) - \mathcal{F}f(s')| = \left|\int_{-\infty}^{\infty} \left(e^{-2\pi i s t} - e^{-2\pi i s' t}\right) f(t)\, dt\right| \le \int_{-\infty}^{\infty} \left|e^{-2\pi i s t} - e^{-2\pi i s' t}\right| \, |f(t)|\, dt$$

  comes from; this came up in showing that $\mathcal{F}f$ is continuous.

[9] Some authors reserve the term "summable" for the case when $\int |f| < \infty$, i.e., for when both $\int f^+$ and $\int f^-$ *are finite*. They still define $\int f = \int f^+ - \int f^-$ but they allow the possibility that one of the integrals on the right may be ∞, in which case $\int f$ is ∞ or $-\infty$ and they don't refer to f as summable.

sinc stinks What about the sinc function and trying to make sense of the following equation?

$$\mathcal{F}\mathrm{sinc}(s) = \int_{-\infty}^{\infty} e^{-2\pi ist}\,\mathrm{sinc}\,t\,dt$$

According to the definitions you just gave, the sinc function is not integrable. In fact, the argument I gave to show that

$$\int_{-\infty}^{\infty} |\,\mathrm{sinc}\,t|\,dt = \infty$$

(the second argument) can be easily modified to show that both

$$\int_{-\infty}^{\infty} \mathrm{sinc}^{+}t\,dt = \infty \quad \text{and} \quad \int_{-\infty}^{\infty} \mathrm{sinc}^{-}t\,dt = \infty\,.$$

So if you wanted to write

$$\int_{-\infty}^{\infty} \mathrm{sinc}\,t\,dt = \int_{-\infty}^{\infty} \mathrm{sinc}^{+}t\,dt - \int_{-\infty}^{\infty} \mathrm{sinc}^{-}t\,dt$$

you'd be faced with $\infty - \infty$. Bad. The integral of sinc (and also the integral of $\mathcal{F}\,\mathrm{sinc}$) *has* to be understood as a limit,

$$\lim_{a\to-\infty,\,b\to\infty} \int_{a}^{b} e^{-2\pi ist}\,\mathrm{sinc}\,t\,dt$$

Evaluating this is a classic of contour integration and the residue theorem, which you may have seen in a class on "Functions of a Complex Variable". I won't do it. You won't do it. Ahlfors did it: See *Complex Analysis*, third edition, by Lars Ahlfors, pp. 156–159.

You can relax now. I'll take it from here.

Subtlety vs. cleverness. For the full mathematical theory of Fourier series and Fourier integrals one needs the Lebesgue integral, as I've mentioned before. Lebesgue's approach to defining the integral allows a wider class of functions to be integrated and it allows one to establish very general, very helpful results of the type "the limit of the integral is the integral of the limit", as in

$$f_n \to f \;\Rightarrow\; \lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(t)\,dt = \int_{-\infty}^{\infty} \lim_{n\to\infty} f_n(t)\,dt = \int_{-\infty}^{\infty} f(t)\,dt\,.$$

You probably do things like this routinely, and so do mathematicians, but it takes them a year or so of graduate school before they feel good about it. More on this in just a moment.

The definition of the Lebesgue integral is based on a study of the size, or *measure*, of the sets where a function is big or small, and you don't wind up writing down the same kinds of "Riemann sums" you used in calculus to define the integral. Interestingly, the constructions and definitions of *measure theory*, as Lebesgue and others developed it, were later used in reworking the foundations of probability. But now take note of the following quote of the mathematician T. Körner from his book *Fourier Analysis*:

> Mathematicians find it easier to understand and enjoy ideas which are clever rather than subtle. Measure theory is subtle rather than clever and so requires hard work to master.

More work than we're willing to do, and need to do. But here's one more thing:

The general result allowing one to pull a limit inside the integral sign is the *Lebesgue dominated convergence theorem*. It says: If f_n is a sequence of integrable functions that converges pointwise to a function f except possibly on a set of measure 0, and if there is an integrable function g with $|f_n| \leq g$ for all n (the "dominated" hypothesis) then f is integrable and

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(t)\, dt = \int_{-\infty}^{\infty} f(t)\, dt\,.$$

There's a variant of this that applies when the integrand depends on a parameter. It goes: If $f(x, t_0) = \lim_{t\to t_0} f(x, t)$ for all x, and if there is an integrable function g such that $|f(x, t)| \leq g(x)$ for all x then

$$\lim_{t\to t_0} \int_{-\infty}^{\infty} f(x, t)\, dt = \int_{-\infty}^{\infty} f(x, t_0)\, dx\,.$$

The situation described in this result comes up in many applications, and it's good to know that it holds in great generality.

Integrals are not always just like sums. Here's one way they're different, and it's important to realize this for our work on Fourier transforms. For sums we have the result that

$$\sum_n a_n \text{ converges implies } a_n \to 0\,.$$

We used this fact together with Parseval's identity for Fourier series to conclude that the Fourier coefficients tend to zero. You also all know the classic counterexample to the converse of the statement:

$$\frac{1}{n} \to 0 \quad \text{but} \quad \sum_{n=1}^{\infty} \frac{1}{n} \quad \text{diverges}\,.$$

For integrals, however, it is possible that

$$\int_{-\infty}^{\infty} f(x)\, dx$$

exists but $f(x)$ does not tend to zero at $\pm\infty$. Make $f(x)$ nonzero (make it equal to 1, if you want) on thinner and thinner intervals going out toward infinity. Then $f(x)$ doesn't decay to zero, but you can make the intervals thin enough so that the integral converges. I'll leave an exact construction up to you.

How about this example? $\sum_{n=1}^{\infty} n\Pi\left(n^3(x-n)\right)$

How shall we test for convergence of integrals? The answer depends on the context, and different choices are possible. Since the convergence of Fourier integrals is at stake, the important thing to measure is the size of a function "at infinity" — does it decay fast enough for the integrals to converge.[10] Any kind of measuring requires a "standard", and for judging the decay (or growth) of a function the easiest and most common standard is to measure using powers of x. The "ruler" based on powers of x reads:

$$\int_a^{\infty} \frac{dx}{x^p} \quad \text{is} \quad \begin{cases} \text{infinite} & \text{if } 0 < p \leq 1 \\ \text{finite} & \text{if } p > 1 \end{cases}$$

[10] For now, at least, let's assume that the only cause for concern in convergence of integrals is decay of the function at infinity, not some singularity at a finite point.

You can check this by direct integration. We take the lower limit a to be positive, but a particular value is irrelevant since the convergence or divergence of the integral depends on the decay near infinity. You can formulate the analogous statements for integrals $-\infty$ to $-a$.

To measure the decay of a function $f(x)$ at $\pm\infty$ we look at

$$\lim_{x\to\pm\infty} |x|^p |f(x)|$$

If, for some $p > 1$, this is *bounded* then $f(x)$ is integrable. If there is a $0 < p \leq 1$ for which the limit is unbounded, i.e., equals ∞, then $f(x)$ is not integrable.

Standards are good only if they're easy to use, and powers of x, together with the conditions on their integrals are easy to use. You can use these tests to show that every rapidly decreasing function is in both $L^1(\mathbf{R})$ and $L^2(\mathbf{R})$.

4.4 Distributions

Our program to extend the applicability of the Fourier transform has several steps. We took the first step last time:

> We defined $\mathcal{S}$, the collection of rapidly decreasing functions. In words, these are the infinitely differentiable functions whose derivatives decrease faster than any power of x at infinity. These functions have the properties that:
>
> 1. If $f(x)$ is in $\mathcal{S}$ then $\mathcal{F}f(s)$ is in $\mathcal{S}$.
> 2. If $f(x)$ is in $\mathcal{S}$ then $\mathcal{F}^{-1}\mathcal{F}f = f$.

We'll sometimes refer to the functions in $\mathcal{S}$ simply as Schwartz functions.

The next step is to use the functions in $\mathcal{S}$ to define a broad class of "generalized functions", or as we'll say, *tempered distributions* $\mathcal{T}$, which will include $\mathcal{S}$ as well as some nonintegrable functions, sine and cosine, δ functions, and much more, and for which the two properties, above, continue to hold.

I want to give a straightforward, no frills treatment of how to do this. There are two possible approaches.

1. Tempered distributions defined as limits of functions in $\mathcal{S}$.

This is the "classical" (vacuum tube) way of defining generalized functions, and it pretty much applies only to the delta function, and constructions based on the delta function. This is an important enough example, however, to make the approach worth our while.

The other approach, the one we'll develop more fully, is:

2. Tempered distributions defined via operating on functions in $\mathcal{S}$.

   We also use a different terminology and say that tempered distributions are *paired* with functions in $\mathcal{S}$, returning a number for the pairing of a distribution with a Schwartz function.

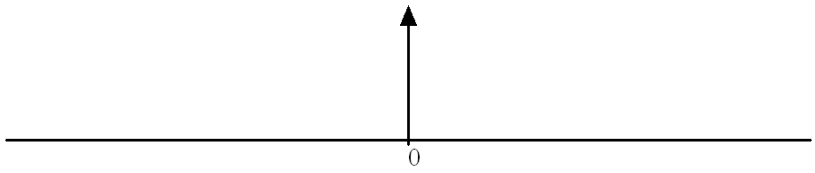
In both cases it's fair to say that "distributions are what distributions do", in that fundamentally they are defined by how they *act* on "genuine" functions, those in $\mathcal{S}$. In the case of "distributions as limits", the nature of the action will be clear but the kind of objects that result from the limiting process is sort of hazy. (That's the problem with this approach.) In the case of "distributions as operators" the nature of

the objects is clear, but just how they are supposed to act is sort of hazy. (And that's the problem with this approach, but it's less of a problem.) You may find the second approach conceptually more difficult, but removing the "take a limit" aspect from center stage really does result in a clearer and computationally easier setup. The second approach is actually present in the first, but there it's cluttered up by framing the discussion in terms of approximations and limits. Take your pick which point of view you prefer, but it's best if you're comfortable with both.

4.4.1 Distributions as limits

The first approach is to view generalized functions as some kind of limit of ordinary functions. Here we'll work with functions in $\mathcal{S}$, but other functions can be used; see Appendix 3.

Let's consider the delta function as a typical and important example. You probably met δ as a mathematical, idealized impulse. You learned: "It's concentrated at the point zero, actually infinite at the point zero, and it vanishes elsewhere." You probably learned to represent this graphically as a spike:



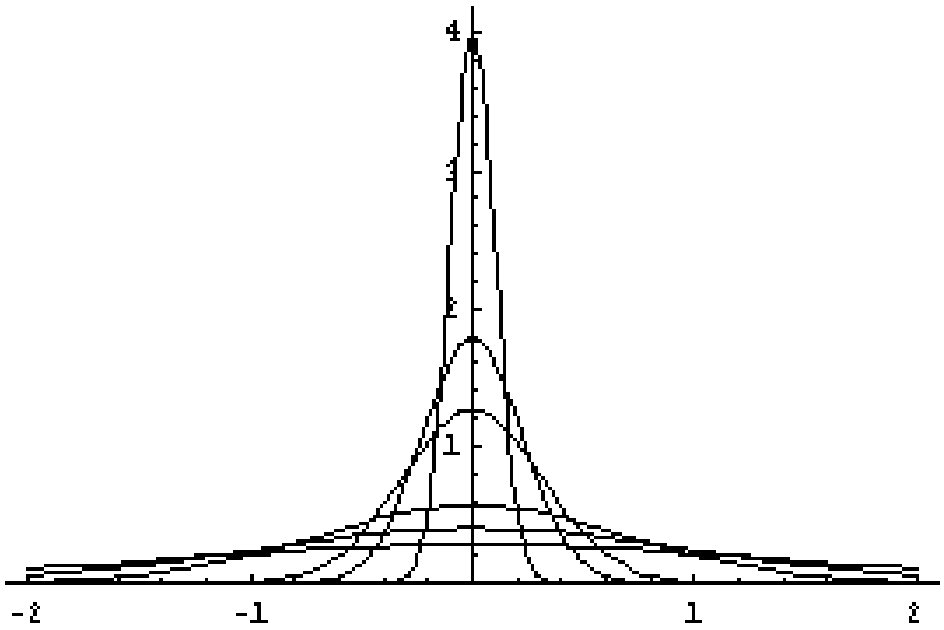
Don't worry, I *don't* want to disabuse you of these ideas, or of the picture. I just want to refine things somewhat.

As an approximation to δ through functions in $\mathcal{S}$ one might consider the family of Gaussians

$$g(x,t) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}, \quad t > 0\,.$$

We remarked earlier that the Gaussians are rapidly decreasing functions.

Here's a plot of some functions in the family for $t = 2$, 1, 0.5, 0.1, 0.05 and 0.01. The smaller the value of t, the more sharply peaked the function is at 0 (it's more and more "concentrated" there), while away from 0 the functions are hugging the axis more and more closely. These are the properties we're trying to capture, approximately.



As an idealization of a function concentrated at $x = 0$, δ should then be a limit

$$\delta(x) = \lim_{t \to 0} g(x, t)\,.$$

This limit doesn't make sense as a pointwise statement — it doesn't define a function — but it begins to make sense when one shows how the limit works *operationally* when "paired" with other functions. The pairing, by definition, is by integration, and to anticipate the second approach to distributions, we'll write this as

$$\langle g(x,t), \varphi \rangle = \int_{-\infty}^{\infty} g(x,t)\varphi(x)\,dx\,.$$

(Don't think of this as an inner product. The angle bracket notation is just a good notation for pairing.[11]) The fundamental result — what it means for the $g(x,t)$ to be "concentrated at 0" as $t \to 0$ — is

$$\lim_{t \to 0} \int_{-\infty}^{\infty} g(x,t)\varphi(x)\,dx = \varphi(0)\,.$$

Now, whereas you'll have a hard time making sense of $\lim_{t\to 0} g(x,t)$ alone, there's no trouble making sense of the limit of the integral, and, in fact, no trouble proving the statement just above. Do observe, however, that the statement: "The limit of the integral is the integral of the limit." is thus not true in this case. The limit of the integral makes sense but not the integral of the limit.[12]

We can and will define the distribution δ by this result, and write

$$\langle \delta, \varphi \rangle = \lim_{t \to 0} \int_{-\infty}^{\infty} g(x,t)\varphi(x)\,dx = \varphi(0)\,.$$

I won't go through the argument for this here, but see Section 4.6.1 for other ways of getting to δ and for a general result along these lines.

[11] Like one pairs "bra" vectors with "ket" vectors in quantum mechanics to make a $\langle A|B\rangle$ — a bracket.

[12] If you read the Appendix on integrals from the preceding lecture, where the validity of such a result is stated as a variant of the Lebesgue Dominated Convergence theorem, what goes wrong here is that $g(t,x)\varphi(x)$ will not be dominated by an integrable function since $g(0,t)$ is tending to ∞.

The Gaussians tend to ∞ at $x = 0$ as $t \to 0$, and that's why writing simply $\delta(x) = \lim_{t\to 0} g(x,t)$ doesn't make sense. One would have to say (and people *do* say, though I have a hard time with it) that the delta function has these properties:

- $\delta(x) = 0$ for $x \neq 0$
- $\delta(0) = \infty$
- $\displaystyle\int_{-\infty}^{\infty} \delta(x)\,dx = 1$

These reflect the corresponding (genuine) properties of the $g(x,t)$:

- $\lim_{t\to 0} g(x,t) = 0$ if $x \neq 0$
- $\lim_{t\to 0} g(0,t) = \infty$
- $\displaystyle\int_{-\infty}^{\infty} g(x,t)\,dx = 1$

The third property is our old friend, the second is clear from the formula, and you can begin to believe the first from the shape of the graphs. The first property is the flip side of "concentrated at a point", namely to be zero *away* from the point where the function is concentrated.

The limiting process also works with convolution:

$$\lim_{t\to 0}(g * \varphi)(a) = \lim_{t\to 0}\int_{-\infty}^{\infty} g(a-x,t)\varphi(x)\,dx = \varphi(a)\,.$$

This is written

$$(\delta * \varphi)(a) = \varphi(a)$$

as shorthand for the limiting process that got us there, and the notation is then pushed so far as to write the delta function itself under the integral, as in

$$(\delta * \varphi)(a) = \int_{-\infty}^{\infty} \delta(a-x)\varphi(x)\,dx = \varphi(a)\,.$$

Let me declare now that I am *not* going to try to talk you out of writing this.

The equation

$$(\delta * \varphi)(a) = \varphi(a)$$

completes the analogy: "δ is to 1 as convolution is to multiplication".

Why concentrate? Why would one want a function concentrated at a point in the first place? We'll certainly have *plenty* of applications of delta functions very shortly, and you've probably already seen a variety through classes on systems and signals in EE or on quantum mechanics in physics. Indeed, it would be wrong to hide the origin of the delta function. Heaviside used δ (without the notation) in his applications and reworking of Maxwell's theory of electromagnetism. In EE applications, starting with Heaviside, you find the "unit impulse" used, as an idealization, in studying how systems respond to sharp, sudden inputs. We'll come back to this latter interpretation when we talk about linear systems. The symbolism, and the three defining properties of δ listed above, were introduced later by P. Dirac in the

service of calculations in quantum mechanics. Because of Dirac's work, δ is often referred to as the "Dirac δ function".

For the present, let's take a look back at the heat equation and how the delta function comes in there. We're perfectly set up for that.

We have seen the family of Gaussians

$$g(x,t) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}, \quad t > 0$$

before. They arose in solving the heat equation for an "infinite rod". Recall that the temperature $u(x,t)$ at a point x and time t satisfies the partial differential equation

$$u_t = \tfrac{1}{2} u_{xx}\,.$$

When an infinite rod (the real line, in other words) is given an initial temperature $f(x)$ then $u(x,t)$ is given by the convolution with $g(x,t)$:

$$u(x,t) = g(x,t) * f(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} * f(x) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-(x-y)^2/2t} f(y)\, dy\,.$$

One thing I didn't say at the time, knowing that this day would come, is how one recovers the initial temperature $f(x)$ *from* this formula. The initial temperature is at $t = 0$, so this evidently requires that we take the limit:

$$\lim_{t \to 0^+} u(x,t) = \lim_{t \to 0^+} g(x,t) * f(x) = (\delta * f)(x) = f(x)\,.$$

Out pops the initial temperature. Perfect. (Well, there have to be some assumptions on $f(x)$, but that's another story.)

4.4.2 Distributions as linear functionals

Farewell to vacuum tubes The approach to distributions we've just followed, illustrated by defining δ, can be very helpful in particular cases and where there's a natural desire to have everything look as "classical" as possible. Still and all, I maintain that adopting this approach wholesale to defining and working with distributions is using technology from a bygone era. I haven't yet defined the collection of tempered distributions $\mathcal{T}$ which is supposed to be the answer to all our Fourier prayers, and I don't know how to do it from a purely "distributions as limits" point of view. It's time to transistorize.

In the preceding discussion we did wind up by considering a distribution, at least δ, in terms of how it acts when paired with a Schwartz function. We wrote

$$\langle \delta, \varphi \rangle = \varphi(0)$$

as shorthand for the result of taking the limit of the pairing

$$\langle g(x,t), \varphi(x) \rangle = \int_{-\infty}^{\infty} g(x,t)\varphi(x)\, dx\,.$$

The second approach to defining distributions takes this idea — "the outcome" of a distribution acting on a test function — as a starting point rather than as a conclusion. The question to ask is what aspects of "outcome", as present in the approach via limits, do we try to capture and incorporate in the basic definition?

Mathematical functions defined on $\mathbf{R}$, "live at points", to use the hip phrase. That is, you plug in a particular point from $\mathbf{R}$, the domain of the function, and you get a particular value in the range, as for instance in the simple case when the function is given by an algebraic expression and you plug values into the expression. Generalized functions — distributions — do not live at points. The domain of a generalized function is not a set of numbers. The value of a generalized function is not determined by plugging in a number from $\mathbf{R}$ and determining a corresponding number. Rather, a particular value of a distribution is determined by how it "operates" on a particular *test function*. The domain of a generalized function is a set of test functions. As they say in Computer Science, helpfully:

- You pass a distribution a test function and it returns a number.

That's not so outlandish. There are all sorts of operations you've run across that take a signal as an argument and return a number. The terminology of "distributions" and "test functions", from the dawn of the subject, is even supposed to be some kind of desperate appeal to physical reality to make this reworking of the earlier approaches more appealing and less "abstract". See label 4.5 for a weak attempt at this, but I can only keep up that physical pretense for so long.

Having come this far, but still looking backward a little, recall that we asked which properties of a pairing — integration, as we wrote it in a particular case in the first approach — do we want to subsume in the general definition. To get all we need, we need remarkably little. Here's the definition:

Tempered distributions A *tempered distribution* T is a complex-valued continuous linear functional on the collection $\mathcal{S}$ of Schwartz functions (called *test functions*). We denote the collection of all tempered distributions by $\mathcal{T}$.

That's the complete definition, but we can unpack it a bit:

1. If φ is in $\mathcal{S}$ then $T(\varphi)$ is a complex number. (You pass a distribution a Schwartz function, it returns a complex number.)
   - We often write this action of T on φ as $\langle T, \varphi\rangle$ and say that T is *paired* with φ. (This terminology and notation are conventions, not commandments.)
2. A tempered distribution is linear operating on test functions:
   $$T(\alpha_1\varphi_1 + \alpha_2\varphi_2) = \alpha_1 T(\varphi_1) + \alpha_2 T(\varphi_2)$$
   or, in the other notation,
   $$\langle T, \alpha_1\varphi_1 + \alpha_2\varphi_2\rangle = \alpha_1\langle T, \varphi_1\rangle + \alpha_2\langle T, \varphi_2\rangle,$$
   for test functions φ_1, φ_2 and complex numbers α_1, α_2.
3. A tempered distribution is continuous: if φ_n is a sequence of test functions in $\mathcal{S}$ with $\varphi_n \to \varphi$ in $\mathcal{S}$ then
   $$T(\varphi_n) \to T(\varphi)\,, \quad \text{also written} \quad \langle T, \varphi_n\rangle \to \langle T, \varphi\rangle\,.$$

Also note that two tempered distributions T_1 and T_2 are equal if they agree on all test functions:

$$T_1 = T_2 \quad \text{if} \quad T_1(\varphi) = T_2(\varphi) \quad (\langle T_1, \varphi\rangle = \langle T_2, \varphi\rangle) \quad \text{for all } \varphi \text{ in } \mathcal{S}\,.$$

This isn't part of the definition, it's just useful to write down.

There's a catch There is one hard part in the definition, namely, what it means for a sequence of test functions in $\mathcal{S}$ to converge in $\mathcal{S}$. To say that $\varphi_n \to \varphi$ in $\mathcal{S}$ is to control the convergence of φ_n together with all its derivatives. We won't enter into this, and it won't be an issue for us. If you look in standard mathematics books on the theory of distributions you will find long, difficult discussions of the appropriate topologies on spaces of functions that must be used to talk about convergence. And you will be discouraged from going any further. Don't go there.

It's another question to ask *why* continuity is included in the definition. Let me just say that this is important when one considers limits of distributions and approximations to distributions.

Other classes of distributions This settles the question of what a tempered distribution *is*: it's a continuous linear functional on $\mathcal{S}$. For those who know the terminology, $\mathcal{T}$ is the *dual space* of the space $\mathcal{S}$. In general, the dual space to a vector space is the set of continuous linear functionals on the vector space, the catch being to define continuity appropriately. From this point of view one can imagine defining types of distributions other than the tempered distributions. They arise by taking the dual spaces of collections of test functions other than $\mathcal{S}$. Though we'll state things for tempered distributions, most general facts (those not pertaining to the Fourier transform, yet to come) also hold for other types of distributions. We'll discuss this in the last section.

4.4.3 Two important examples of distributions

Let us now understand:

1. How $\mathcal{T}$ somehow includes the functions we'd like it to include for the purposes of extending the Fourier transform.
2. How δ fits into this new scheme.

The first item is a general construction and the second is an example of a specific distribution defined in this new way.

How functions determine tempered distributions, and why the tempered distributions include the functions we want. Suppose $f(x)$ is a function for which

$$\int_{-\infty}^{\infty} f(x)\varphi(x)\,dx$$

exists for all Schwartz functions $\varphi(x)$. This is not asking too much, considering that Schwartz functions decrease so rapidly that they're plenty likely to make a product $f(x)\varphi(x)$ integrable. We'll look at some examples, below.

In this case the function $f(x)$ *determines* ("defines" or "induces" or "corresponds to" — pick your preferred descriptive phrase) a tempered distribution T_f *by means of the formula*

$$T_f(\varphi) = \int_{-\infty}^{\infty} f(x)\varphi(x)\,dx\,.$$

In words, T_f acts on a test function φ by integration of φ against f. Alternatively, we say that the function f determines a distribution T_f through the pairing

$$\langle T_f, \varphi\rangle = \int_{-\infty}^{\infty} f(x)\varphi(x)\,dx\,, \quad \varphi \text{ a test function.}$$

This is just what we considered in the earlier approach that led to δ, pairing Gaussians with a Schwartz function. In the present terminology we would say that the Gaussian $g(x,t)$ determines a distribution T_g according to the formula

$$\langle T_g, \varphi\rangle = \int_{-\infty}^{\infty} g(x,t)\varphi(x)\,dx\,.$$

Let's check that the pairing $\langle T_f, \varphi\rangle$ meets the standard of the definition of a distribution. The pairing is linear because integration is linear:

$$\begin{aligned}\langle T_f, \alpha_1\varphi_1 + \alpha_2\varphi_2\rangle &= \int_{-\infty}^{\infty} f(x)(\alpha_1\varphi_1(x) + \alpha_2\varphi_2(x))\,dx \\ &= \int_{-\infty}^{\infty} f(x)\alpha_1\varphi_1(x)\,dx + \int_{-\infty}^{\infty} f(x)\alpha_2\varphi_2(x)\,dx \\ &= \alpha_1\langle T_f, \varphi_1\rangle + \alpha_2\langle T_f, \varphi_2\rangle\end{aligned}$$

What about continuity? We have to take a sequence of Schwartz functions φ_n converging to a Schwartz function φ and consider the limit

$$\lim_{n\to\infty}\langle T_f, \varphi_n\rangle = \lim_{n\to\infty}\int_{-\infty}^{\infty} f(x)\varphi_n(x)\,dx\,.$$

Again, we haven't said anything precisely about the meaning of $\varphi_n \to \varphi$, but the standard results on taking the limit inside the integral will apply in this case and allow us to conclude that

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} f(x)\varphi_n(x)\,dx. = \int_{-\infty}^{\infty} f(x)\varphi(x)\,dx$$

i.e., that

$$\lim_{n\to\infty}\langle T_f, \varphi_n\rangle = \langle T_f, \varphi\rangle\,.$$

This is continuity.

Using a function $f(x)$ to determine a distribution T_f this way is a very common way of constructing distributions. We will use it frequently. Now, you might ask yourself whether different functions can give rise to the same distribution. That is, if $T_{f_1} = T_{f_2}$ as distributions, then must we have $f_1(x) = f_2(x)$? Yes, fortunately, for if $T_{f_1} = T_{f_2}$ then for all test functions $\varphi(x)$ we have

$$\int_{-\infty}^{\infty} f_1(x)\varphi(x)\,dx = \int_{-\infty}^{\infty} f_2(x)\varphi(x)\,dx$$

hence

$$\int_{-\infty}^{\infty} (f_1(x) - f_2(x))\varphi(x)\,dx = 0\,.$$

Since this holds for all test functions $\varphi(x)$ we can conclude that $f_1(x) = f_2(x)$.

Because a function $f(x)$ determines a *unique* distribution, it's natural to "identify" the function $f(x)$ with the corresponding distribution T_f. Sometimes we then write just f for the corresponding distribution rather than writing T_f, and we write the pairing as

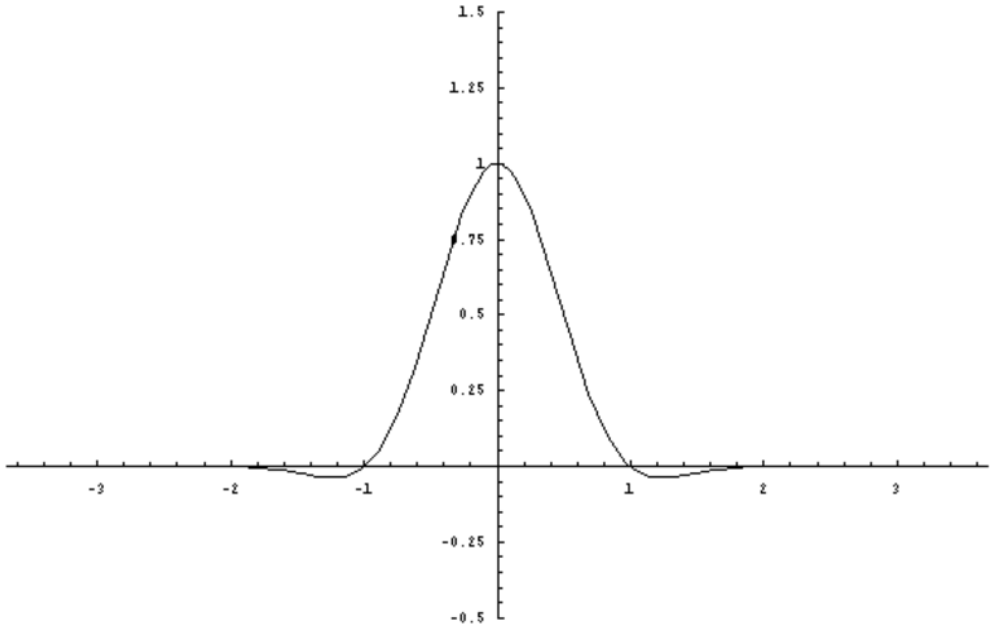
$$\langle f, \varphi\rangle = \int_{-\infty}^{\infty} f(x)\varphi(x)\,dx$$

rather than as $\langle T_f, \varphi\rangle$.

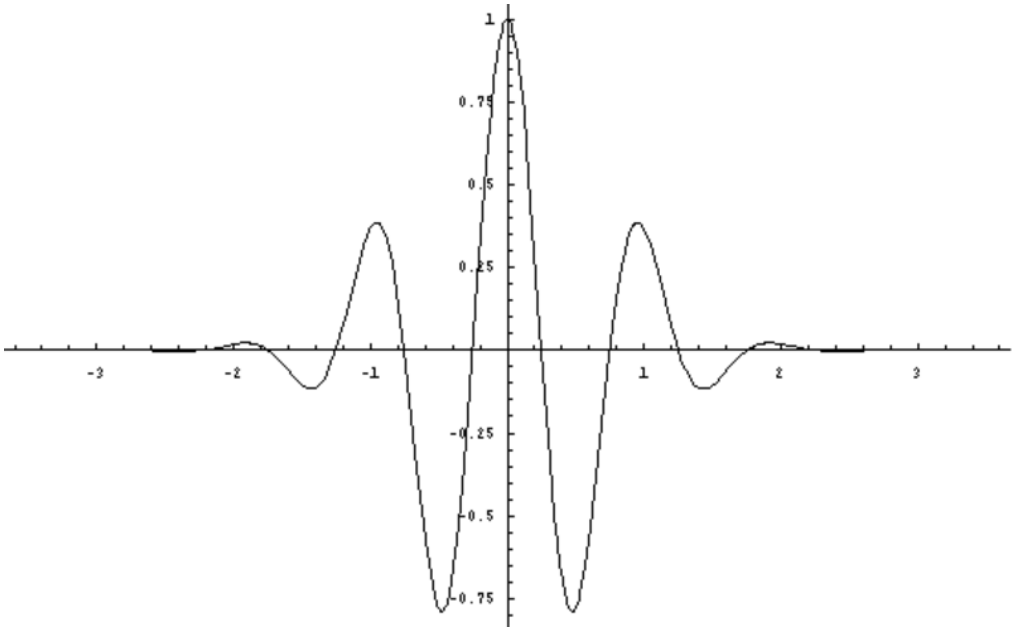
- It is in this sense — identifying a function f with the distribution T_f it determines— that a class of distributions "contains" classical functions.

Let's look at some examples.

Examples The sinc function defines a tempered distribution, because, though sinc is not integrable, $(\operatorname{sinc} x)\varphi(x)$ *is* integrable for any Schwartz function $\varphi(x)$. Remember that a Schwartz function $\varphi(x)$ dies off faster than any power of x and that's more than enough to pull sinc down rapidly enough at $\pm\infty$ to make the integral exist. I'm not going to prove this but I have no qualms asserting it. For example, here's a plot of e^{-x^2} times the sinc function on the interval $-3.5 \le x \le 3.5$:



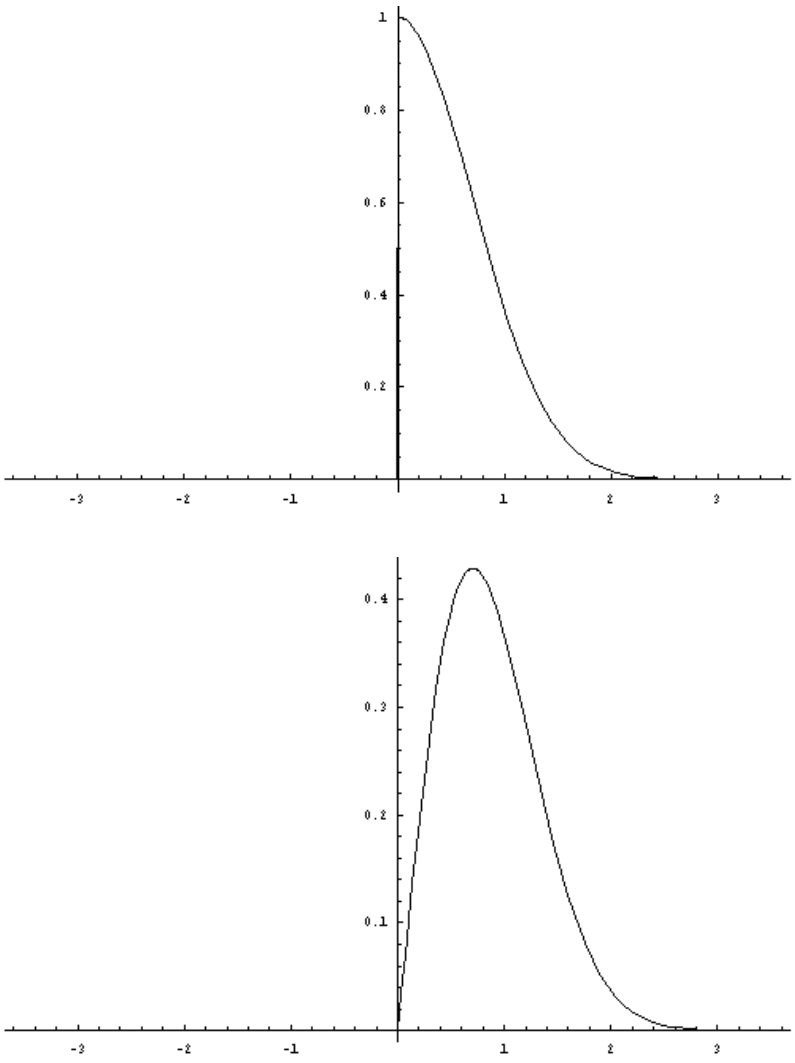
For the same reason any complex exponential, and also sine and cosine, define tempered distributions. Here's a plot of e^{-x^2} times $\cos 2\pi x$ on the range $-3.5 \le x \le 3.5$:



Take two more examples, the Heaviside unit step $H(x)$ and the unit ramp $u(x)$:

$$H(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases} \qquad u(x) = \begin{cases} 0 & x \leq 0 \\ x & x \geq 0 \end{cases}$$

Neither function is integrable; indeed, $u(x)$ even tends to ∞ as $x \to \infty$, but it does so only to the first power (exactly) of x. Multiplying by a Schwartz function brings $H(x)$ and $u(x)$ down, and they each determine tempered distributions. Here are plots of e^{-x^2} times $H(x)$ and $u(x)$, respectively:



The upshot is that the sinc, complex exponentials, the unit step, the unit ramp, and many others, can all be considered to *be* tempered distributions. This is a good thing, because we're aiming to define the Fourier transform *of a tempered distribution*, and we want to be able to apply it to the signals society needs. (We'll also get back to our good old formula $\mathcal{F}\text{sinc} = \Pi$, and all will be right with the world.)

Do all tempered distributions "come from functions" in this way? In the next section we'll define δ as a (tempered) distribution, i.e., as a linear functional. δ does not come from a function in the way we've just described (or in any way). This adds to the feeling that we really have defined something new, that "generalized functions" *include* many (classical) functions but go beyond the classical functions.

Two final points. As we've just remarked, not every distribution comes from a function and so the nature of the pairing of a given tempered distribution T with a Schwartz function φ is unspecified, so to speak. By that I mean, *don't* think that $\langle T, \varphi\rangle$ is an integral, as in

$$\langle T, \varphi\rangle = \int_{-\infty}^{\infty} T(x)\varphi(x)\,dx$$

for any old tempered distribution T.[13] The pairing *is* an integral when the distribution comes from a function, but there's more to tempered distributions than that.

Finally a note of caution. Not *every* function determines a tempered distribution. For example e^{x^2} doesn't.[14] It doesn't because e^{-x^2} is a Schwartz function and

$$\int_{-\infty}^{\infty} e^{x^2}e^{-x^2}\,dx = \int_{-\infty}^{\infty} 1\,dx = \infty\,.$$

δ as a tempered distribution The limiting approach to the delta function culminated with our writing

$$\langle \delta, \varphi\rangle = \varphi(0)$$

as the result of

$$\lim_{t\to 0}\int_{-\infty}^{\infty} g(x,t)\varphi(x)\,dx = \varphi(0)\,.$$

Now with our second approach, tempered distributions as linear functionals on $\mathcal{S}$, we can simply *define* the tempered distribution δ by how it should operate on a function φ in $\mathcal{S}$ so as to *achieve this outcome*, and obviously what we want is

$$\delta(\varphi) = \varphi(0), \quad \text{or in the bracket notation} \quad \langle \delta, \varphi\rangle = \varphi(0);$$

you pass δ a test function and it returns the value of the test function at 0.

Let's check the definition. For linearity,

$$\begin{aligned}\langle \delta, \varphi_1 + \varphi_2\rangle &= \varphi_1(0) + \varphi_2(0) = \langle \delta, \varphi_1\rangle + \langle \delta, \varphi_2\rangle\\ \langle \delta, \alpha\varphi\rangle &= \alpha\varphi(0) = \alpha\langle \delta, \varphi\rangle\,.\end{aligned}$$

For continuity, if $\varphi_n(x) \to \varphi(0)$ then in particular $\varphi_n(0) \to \varphi(0)$ and so

$$\langle \delta, \varphi_n\rangle = \varphi_n(0) \to \varphi(0) = \langle \delta, \varphi\rangle\,.$$

So the mysterious δ, clouded in controversy by statements like

$$\begin{aligned}&\delta(x) = 0 \text{ for } x \neq 0\\ &\delta(0) = \infty\\ &\int_{-\infty}^{\infty} \delta(x)\,dx = 1\end{aligned}$$

[13] For one thing it doesn't make sense, strictly speaking, even to write $T(x)$; you don't pass a distribution a number x to evaluate, you pass it a function.

[14] It does determine other kinds of distributions, ones based on other classes of test functions. See Section 4.20.

now emerges as *the simplest possible nontrivial tempered distribution* — it's just the functional described in words by "evaluate at 0"!

There was a second identity that we had from the "δ as limit" development, namely

$$(\delta_a * \varphi) = \varphi(a)\,.$$

as a result of

$$\lim_{t\to 0}\int_{-\infty}^{\infty} g(a-x,t)\varphi(x)\,dx = \varphi(a)$$

We'll get back to convolution with distributions, but there's another way we can capture this outcome without mentioning convolution. We define a tempered distribution δ_a (the δ function based at a) by the formula

$$\langle \delta_a, \varphi\rangle = \varphi(a)\,.$$

In words, you pass δ_a a test function and it returns the value of the test function at a. I won't check that δ_a satisfies the definition — it's the same argument as for δ.

δ and δ_a are two different distributions (for $a \neq 0$). Classically, if that word makes sense here, one would write δ_a as $\delta(x-a)$, just a shifted δ. We'll get to that, and use that notation too, but a bit later. As tempered distributions, δ and δ_a are *defined* to have the property we want them to have. It's air tight — no muss, no fuss. That's δ. That's δ_a.

Would we have come upon this simple, direct definition without having gone through the "distributions as limits" approach? Would we have the transistor without first having vacuum tubes? Perhaps so, perhaps not. That first approach via limits provided the basic insights that allowed people, Schwartz in particular, to reinvent the theory of distributions based on linear functionals as we have done here (as he did).

4.4.4 Other types of distributions

We have already seen that the functions in $\mathcal{S}$ work well for Fourier transforms. We'll soon see that the tempered distributions $\mathcal{T}$ based on $\mathcal{S}$ are the right objects to be operated on by a generalized Fourier transform. However, $\mathcal{S}$ isn't the only possible collection of test functions and $\mathcal{T}$ isn't the only possible collection of distributions.

Another useful set of test functions are the smooth functions that are timelimited, to use terminology from EE. That is, we let $\mathcal{C}$ be the set of infinitely differentiable functions which are identically zero beyond a point:

> $\varphi(x)$ is in $\mathcal{C}$ if $\varphi(x)$ has derivatives of all orders and if $\varphi(x) = 0$ for $|x| \geq x_0$ (where x_0 can depend on φ).

The mathematical terminology for such a function is that it has *compact support*. The *support* of a function is the complement of the largest set where the function is identically zero. (The letter $\mathcal{C}$ is supposed to connote "compact".)

The continuous linear functionals on $\mathcal{C}$ also form a collection of distributions, denoted by $\mathcal{D}$. In fact, when most people use the term "distribution" (without the adjective tempered) they are usually thinking of an element of $\mathcal{D}$. We use the same notation as before for the pairing: $\langle T, \varphi\rangle$ for T in $\mathcal{D}$ and φ in $\mathcal{C}$.

δ and δ_a belong to $\mathcal{D}$ as well as to $\mathcal{T}$, and the definition is the same:

$$\langle \delta, \varphi \rangle = \varphi(0) \quad \text{and} \quad \langle \delta_a, \varphi \rangle = \varphi(a)\,.$$

It's the same δ. It's not a new distribution, it's only operating on a different class of test functions.

$\mathcal{D}$ is a bigger collection of distributions than $\mathcal{T}$ because $\mathcal{C}$ is a smaller collection of test functions than $\mathcal{S}$. The latter point should be clear to you: To say that $\varphi(x)$ is smooth and *vanishes* identically outside some interval is a stronger condition than requiring merely that it *decays at* infinity (albeit faster than any power of x). Thus if $\varphi(x)$ is in $\mathcal{C}$ then it's also in $\mathcal{S}$. Why is $\mathcal{D}$ bigger than $\mathcal{T}$? Since $\mathcal{C}$ is contained in $\mathcal{S}$, a continuous linear functional on $\mathcal{S}$ is automatically a continuous linear functional on $\mathcal{C}$. That is, $\mathcal{T}$ is contained in $\mathcal{D}$.

Just as we did for $\mathcal{T}$, we say that a function $f(x)$ determines a distribution in $\mathcal{D}$ if

$$\int_{-\infty}^{\infty} f(x)\varphi(x)\,dx$$

exists for all test functions φ in $\mathcal{C}$. As before, we write T_f for the distribution induced by a function f, and the pairing as

$$\langle T_f, \varphi \rangle = \int_{-\infty}^{\infty} f(x)\varphi(x)\,dx\,.$$

As before, a function determines a unique distribution in this way, so we identify f with T_f and write the pairing as

$$\langle f, \varphi \rangle = \int_{-\infty}^{\infty} f(x)\varphi(x)\,dx\,.$$

It's easier to satisfy the integrability condition for $\mathcal{C}$ than for $\mathcal{S}$ because multiplying $f(x)$ by a function in $\mathcal{C}$ kills it off completely outside some interval, rather than just bringing it smoothly down to zero at infinity as would happen when multiplying by a function in $\mathcal{S}$. This is another reason why $\mathcal{D}$ is a bigger class of distributions than $\mathcal{T}$ — more functions determine distributions. For example, we observed that the function e^{x^2} doesn't determine a tempered distribution, but it *does* determine an element of $\mathcal{D}$.

4.5 A Physical Analogy for Distributions

Think of heat distributed over a region in space. A number associated with heat is temperature, and we want to measure the temperature at a point using a thermometer. But does it really make sense to ask for the temperature "at a point"? What kind of test instrument could possibly measure the temperature at a point?

What makes more sense is that a thermometer registers some overall value of the temperature near a point. That is, the temperature is whatever the thermometer says it is, and is determined by a pairing of the heat (the distribution) with the thermometer (a test function or test device). The more "concentrated" the thermometer (the more sharply peaked the test function) the more accurate the measurement, meaning the closer the reading is to being the temperature "at a point".

A pairing of a test function with the heat is somehow supposed to model how the thermometer responds to the distribution of heat. One particular way to model this is to say that if f is the heat and φ is the test function, then the reading on the thermometer is

$$\int f(x)\varphi(x)\,dx\,,$$

an integrated, average temperature. I've left limits off the integral to suggest that it is taken over some region of space where the heat is distributed.

Such measurements (temperature or other sorts of physical measurements) are supposed to obey laws of superposition (linearity) and the like, which, in this model case, translates to

$$\int f(x)(\alpha_1\varphi_1(x) + \alpha_2(x)\varphi_2(x))\,dx = \alpha_1 \int f(x)\varphi_1(x)\,dx + \alpha_2 \int f(x)\varphi_2(x)\,dx$$

for test functions φ_1 and φ_2. That's why we incorporate linearity into the definition of distributions. With enough wishful thinking you can pass from this motivation to the general definition. Sure you can.

4.6 Limits of Distributions

There's a very useful general result that allows us to define distributions by means of limits. The statement goes:

> Suppose that T_n is a sequence of tempered distributions and that $\langle T_n, \varphi\rangle$ (a sequence of numbers) converges for every Schwartz function φ. Then T_n converges to a tempered distribution T and
>
> $$\langle T, \varphi\rangle = \lim_{n\to\infty} \langle T_n, \varphi\rangle$$

Briefly, distributions can be defined by taking limits of sequences of distributions, and the result says that if the parings converge then the distributions converge. This is by no means a trivial fact, the key issue being the proper notion of convergence of distributions, and that's hard. We'll have to be content with the statement and let it go at that.

You might not spot it from the statement, but one practical consequence of this result is that if different converging sequences have the same effect on test functions then they must be converging to the same distribution. More precisely, if $\lim_{n\to\infty}\langle S_n, \varphi\rangle$ and $\lim_{n\to\infty}\langle T_n, \varphi\rangle$ both exist and are equal for every test function φ then S_n and T_n both converge to the same distribution. That's certainly possible — different sequences can have the same limit, after all.

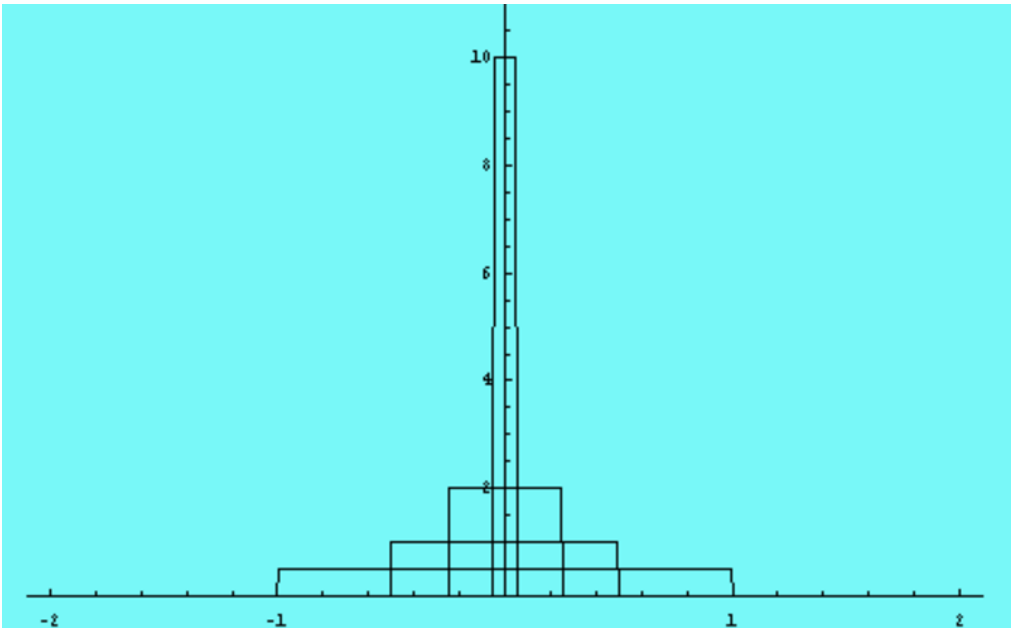
To illustrate just why this is helpful to know, let's consider different ways of approximating δ.

4.6.1 Other Approximating Sequences for δ

Go back to the idea that δ is an idealization of an impulse concentrated at a point. Earlier we used a family of Gaussians to approach δ, but there are many other ways we could try to approximate this characteristic behavior of δ in the limit. For example, take the family of scaled Π functions

$$R_\epsilon(x) = \frac{1}{\epsilon}\Pi_\epsilon(x) = \frac{1}{\epsilon}\Pi\left(\frac{x}{\epsilon}\right) = \begin{cases} \frac{1}{\epsilon} & |x| < \frac{\epsilon}{2} \\ 0 & |x| \geq \frac{\epsilon}{2} \end{cases}$$

where ϵ is a positive constant. Here's a plot of $R_\epsilon(x)$ for $\epsilon = 2,\ 1,\ 0.5,\ 0.1$, some of the same values we used for the parameter in the family of Gaussians.



What happens if we integrate $R_\epsilon(x)$ against a test function $\varphi(x)$? The function $\varphi(x)$ could be a Schwartz function, if we wanted to stay within the class of tempered distributions, or an element of $\mathcal{C}$. In fact, all that we require is that $\varphi(x)$ is smooth near the origin so that we can use a Taylor approximation (and we could get away with less than that). We write

$$\begin{aligned}\langle R_\epsilon, \varphi\rangle &= \int_{-\infty}^{\infty} R_\epsilon(x)\varphi(x)\,dx = \frac{1}{\epsilon}\int_{-\epsilon/2}^{\epsilon/2}\varphi(x)\,dx \\ &= \frac{1}{\epsilon}\int_{-\epsilon/2}^{\epsilon/2}(\varphi(0)+\varphi''(0)x+O(x^2))\,dx = \varphi(0)+\frac{1}{\epsilon}\int_{-\epsilon/2}^{\epsilon/2}O(x^2)\,dx = \varphi(0)+O(\epsilon^2)\,.\end{aligned}$$

If we let $\epsilon \to 0$ we obtain

$$\lim_{\epsilon\to 0}\langle R_\epsilon, \varphi\rangle = \varphi(0)\,.$$

In the limit, the result of pairing the R_ϵ with a test function is the same as pairing a Gaussian with a test function:

$$\lim_{\epsilon\to 0}\langle R_\epsilon, \varphi\rangle = \varphi(0) = \lim_{t\to 0}\langle g(x,t), \varphi(x)\rangle\,.$$

Thus the distributions defined by R_ϵ and by $g(x,t)$ each converge and to the same distribution, namely δ.[15]

A general way to get to δ There's a general, flexible and simple approach to getting to δ by a limit. It can be useful to know this if one model approximation might be preferred to another in a particular computation or application. Start with a function $f(x)$ having

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

and form

$$f_p(x) = pf(px)\,, \quad p > 0\,.$$

[15] Note that the convergence isn't phrased in terms of a sequential limit with $n \to \infty$, but that's not important — we could have set, for example, $\epsilon_n = 1/n$, $t_n = 1/n$ and let $n \to \infty$ to get $\epsilon \to 0$, $t \to 0$.

Then one has

$$f_p \to \delta\,.$$

How does f_p compare with f? As p increases, the scaled function $f(px)$ concentrates near $x = 0$, that is, the graph is squeezed in the horizontal direction. Multiplying by p to form $pf(px)$ then stretches the values in the vertical direction. Nevertheless

$$\int_{-\infty}^{\infty} f_p(x)\,dx = 1$$

as we see by making the change of variable $u = px$.

To show that f_p converges to δ, we pair $f_p(x)$ with a test function $\varphi(x)$ via integration and show

$$\lim_{p\to\infty} \int_{-\infty}^{\infty} f_p(x)\varphi(x)\,dx = \varphi(0) = \langle \delta, \varphi \rangle\,.$$

There is a nice argument to show this. Write

$$\begin{aligned}
\int_{-\infty}^{\infty} f_p(x)\varphi(x)\,dx &= \int_{-\infty}^{\infty} f_p(x)(\varphi(x) - \varphi(0) + \varphi(0))\,dx \\
&= \int_{-\infty}^{\infty} f_p(x)(\varphi(x) - \varphi(0))\,dx + \varphi(0)\int_{-\infty}^{\infty} f_p(x)\,dx \\
&= \int_{-\infty}^{\infty} f_p(x)(\varphi(x) - \varphi(0))\,dx + \varphi(0) \\
&= \int_{-\infty}^{\infty} f(x)(\varphi(x/p) - \varphi(0))\,dx + \varphi(0),
\end{aligned}$$

where we have used that the integral of f_p is 1 and have made a change of variable in the last integral.

The object now is to show that the integral of $f(x)(\varphi(x/p) - \varphi(0))$ goes to zero as $p \to \infty$. There are two parts to this. Since the integral of $f(x)(\varphi(x/p) - \varphi(0))$ is finite, the tails at $\pm\infty$ are arbitrarily small, meaning, more formally, that for any $\epsilon > 0$ there is an $a > 0$ such that

$$\left|\int_a^{\infty} f(x)(\varphi(x/p) - \varphi(0))\,dx\right| + \left|\int_{-\infty}^{-a} f(x)(\varphi(x/p) - \varphi(0))\,dx\right| < \epsilon\,.$$

This didn't involve letting p tend to ∞; that comes in now. Fix a as above. It remains to work with the integral

$$\int_{-a}^{a} f(x)(\varphi(x/p) - \varphi(0))\,dx$$

and show that this too can be made arbitrarily small. Now

$$\int_{-a}^{a} |f(x)|\,dx$$

is a fixed number, say M, and we can take p so large that $|\varphi(x/p) - \varphi(0)| < \epsilon/M$ for $|x/p| \le a$. With this,

$$\left|\int_{-a}^{a} f(x)(\varphi(x/p) - \varphi(0))\,dx\right| \le \int_{-a}^{a} |f(x)|\,|\varphi(x/p) - \varphi(0)|\,dx < \epsilon\,.$$

Combining the three estimates we have

$$\left|\int_{-\infty}^{\infty} f(x)(\varphi(x/p) - \varphi(0))\,dx\right| < 2\epsilon\,,$$

and we're done.

We've already seen two applications of this construction, to

$$f(x) = \Pi(x)$$

and, originally, to

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{x^2/2}\,, \quad \text{take } p = 1/\sqrt{t}\,.$$

Another possible choice, believe it or not, is

$$f(x) = \operatorname{sinc} x\,.$$

This works because the integral

$$\int_{-\infty}^{\infty} \operatorname{sinc} x\, dx$$

is the Fourier transform of sinc at 0, and you'll recall that we stated the true fact that

$$\int_{-\infty}^{\infty} e^{-2\pi i s t} \operatorname{sinc} t\, dt = \begin{cases} 1 & |t| < \frac{1}{2} \\ 0 & |t| > \frac{1}{2} \end{cases}$$

4.7 The Fourier Transform of a Tempered Distribution

It's time to show how to generalize the Fourier transform to tempered distributions.[16] It will take us one or two more steps to get to the starting line, but after that it's a downhill race passing effortlessly (almost) through all the important gates.

How to extend an operation from functions to distributions: Try a function first. To define a distribution T is to say what it does to a test function. You give me a test function φ and I have to tell you $\langle T, \varphi\rangle$ — how T operates on φ. We have done this in two cases, one particular and one general. In particular, we defined δ directly by

$$\langle \delta, \varphi\rangle = \varphi(0)\,.$$

In general, we showed how a function f determines a distribution T_f by

$$\langle T_f, \varphi\rangle = \int_{-\infty}^{\infty} f(x)\varphi(x)\, dx$$

provided that the integral exists for every test function. We also say that the distribution *comes from* a function. When no confusion can arise we identify the distribution T_f with the function f it comes from and write

$$\langle f, \varphi\rangle = \int_{-\infty}^{\infty} f(x)\varphi(x)\, dx\,.$$

When we want to extend an operation from functions to distributions — e.g., when we want to define the Fourier transform of a distribution, or the reverse of distribution, or the shift of a distribution, or the derivative of a distribution — we take our cue from the way functions determine distributions and ask how the operation works in the case when the pairing is given by integration. What we hope to see is an outcome that suggests a direct definition (as happened with δ, for example). This is a procedure to follow. It's something to try. See Appendix 1 for a discussion of why this is really the natural thing to do, but for now let's see how it works for *the* operation we're most interested in.

[16] In other words, it's time to put up, or shut up.

4.7.1 The Fourier transform defined

Suppose T is a tempered distribution. Why should such an object have a Fourier transform, and how on earth shall we define it? It can't be an integral, because T isn't a function so there's nothing to integrate. If $\mathcal{F}T$ is to be itself a tempered distribution (just as $\mathcal{F}\varphi$ is again a Schwartz function if φ is a Schwartz function) then we have to say how $\mathcal{F}T$ *pairs* with a Schwartz function, because that's what tempered distributions do. So how?

We have a toe-hold here. If ψ is a Schwartz function then $\mathcal{F}\psi$ is again a Schwartz function and we can ask: How does the Schwartz function $\mathcal{F}\psi$ pair with another Schwartz function φ? What is the outcome of $\langle \mathcal{F}\psi, \varphi\rangle$? We know how to pair a distribution that comes from a function ($\mathcal{F}\psi$ in this case) with a Schwartz function; it's

$$\langle \mathcal{F}\psi, \varphi\rangle = \int_{-\infty}^{\infty} \mathcal{F}\psi(x)\varphi(x)\,dx\,.$$

But we can work with the right hand side:

$$\begin{aligned}
\langle \mathcal{F}\psi, \varphi\rangle &= \int_{-\infty}^{\infty} \mathcal{F}\psi(x)\varphi(x)\,dx \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{-2\pi i x y}\psi(y)\,dy\right)\varphi(x)\,dx \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi i x y}\psi(y)\varphi(x)\,dy\,dx \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{-2\pi i x y}\varphi(x)\,dx\right)\psi(y)\,dy \\
&\qquad \text{(the interchange of integrals is justified because } \varphi(x)e^{-2\pi i s x} \\
&\qquad \text{and } \psi(x)e^{-2\pi i s x} \text{ are integrable)} \\
&= \int_{-\infty}^{\infty} \mathcal{F}\varphi(y)\psi(y)\,dy \\
&= \langle \psi, \mathcal{F}\varphi\rangle
\end{aligned}$$

The outcome of pairing $\mathcal{F}\psi$ with φ is:

$$\langle \mathcal{F}\psi, \varphi\rangle = \langle \psi, \mathcal{F}\varphi\rangle\,.$$

This tells us how we should make the definition in general:

- Let T be a tempered distribution. The *Fourier transform* of T, denoted by $\mathcal{F}(T)$ or $\widehat{T}$, is the tempered distribution defined by
$$\langle \mathcal{F}T, \varphi\rangle = \langle T, \mathcal{F}\varphi\rangle\,.$$
for any Schwartz function φ.

This definition makes sense because when φ is a Schwartz function so is $\mathcal{F}\varphi$; it is only then that the pairing $\langle T, \mathcal{F}\varphi\rangle$ is even defined.

We define the inverse Fourier transform by following the same recipe:

- Let T be a tempered distribution. The *inverse Fourier transform* of T, denoted by $\mathcal{F}^{-1}(T)$ or $\check{T}$, is defined by
$$\langle \mathcal{F}^{-1}T, \varphi\rangle = \langle T, \mathcal{F}^{-1}\varphi\rangle\,.$$
for any Schwartz function φ.

Now all of a sudden we have

Fourier inversion:

$$\mathcal{F}^{-1}\mathcal{F}T = T \quad \text{and} \quad \mathcal{F}\mathcal{F}^{-1}T = T$$

for any tempered distribution T.

It's a cinch. Watch. For any Schwartz function φ,

$$\begin{aligned}\langle \mathcal{F}^{-1}(\mathcal{F}T), \varphi\rangle &= \langle \mathcal{F}T, \mathcal{F}^{-1}\varphi\rangle \\ &= \langle T, \mathcal{F}(\mathcal{F}^{-1}\varphi)\rangle \\ &= \langle T, \varphi\rangle \quad \text{(because Fourier inversion works for Schwartz functions)}\end{aligned}$$

This says that $\mathcal{F}^{-1}(\mathcal{F}T)$ and T have the same value when paired with any Schwartz function. Therefore they are the same distribution: $\mathcal{F}^{-1}\mathcal{F}T = T$. The second identity is derived in the same way.

Done. The most important result in the subject, done, in a few lines.

In Section 4.10 we'll show that we've gained, and haven't lost. That is, the generalized Fourier transform "contains" the original, classical Fourier transform in the same sense that tempered distributions contain classical functions.

4.7.2 A Fourier transform hit parade

With the definition in place it's time to reap the benefits and find some Fourier transforms explicitly. We note one general property

- $\mathcal{F}$ is linear on tempered distributions.

This means that

$$\mathcal{F}(T_1 + T_2) = \mathcal{F}T_1 + \mathcal{F}T_2 \quad \text{and} \quad \mathcal{F}(\alpha T) = \alpha\mathcal{F}T\,,$$

α a number. These follow directly from the definition. To wit:

$$\begin{aligned}\langle \mathcal{F}(T_1 + T_2), \varphi\rangle &= \langle T_1 + T_2, \mathcal{F}\varphi\rangle = \langle T_1, \mathcal{F}\varphi\rangle + \langle T_2, \mathcal{F}\varphi\rangle = \langle \mathcal{F}T_1, \varphi\rangle + \langle \mathcal{F}T_2, \varphi\rangle = \langle \mathcal{F}T_1 + \mathcal{F}T_2, \varphi\rangle \\ \langle \mathcal{F}(\alpha T), \varphi\rangle &= \langle \alpha T, \mathcal{F}\varphi\rangle = \alpha\langle T, \mathcal{F}\varphi\rangle = \alpha\langle \mathcal{F}T, \varphi\rangle = \langle \alpha\mathcal{F}T, \varphi\rangle\end{aligned}$$

The Fourier transform of δ As a first illustration of computing with the generalized Fourier transform we'll find $\mathcal{F}\delta$. The result is:

- The Fourier transform of δ is

$$\mathcal{F}\delta = 1\,.$$

This must be understood as an equality between distributions, i.e., as saying that $\mathcal{F}\delta$ and 1 produce the same values when paired with any Schwartz function φ. Realize that "1" is the constant function, and this defines a tempered distribution via integration:

$$\langle 1, \varphi\rangle = \int_{-\infty}^{\infty} 1 \cdot \varphi(x)\,dx$$

That integral converges because $\varphi(x)$ is integrable (it's much more than integrable, but it's certainly integrable).

We derive the formula by appealing to the definition of the Fourier transform and the definition of δ. On the one hand,

$$\langle \mathcal{F}\delta, \varphi\rangle = \langle \delta, \mathcal{F}\varphi\rangle = \mathcal{F}\varphi(0) = \int_{-\infty}^{\infty} \varphi(x)\,dx\,.$$

On the other hand, as we've just noted,

$$\langle 1, \varphi\rangle = \int_{-\infty}^{\infty} 1\cdot\varphi(x)\,dx = \int_{-\infty}^{\infty} \varphi(x)\,dx\,.$$

The results are the same, and we conclude that $\mathcal{F}\delta = 1$ as distributions. According to the inversion theorem we can also say that $\mathcal{F}^{-1}1 = \delta$.

We can also show that

$$\mathcal{F}1 = \delta\,.$$

Here's how. By definition,

$$\langle \mathcal{F}1, \varphi\rangle = \langle 1, \mathcal{F}\varphi\rangle = \int_{-\infty}^{\infty} \mathcal{F}\varphi(s)\,ds\,.$$

But we recognize the integral as giving the *inverse* Fourier transform of $\mathcal{F}\varphi$ at 0:

$$\mathcal{F}^{-1}\mathcal{F}\varphi(t) = \int_{-\infty}^{\infty} e^{2\pi ist}\mathcal{F}\varphi(s)\,ds \quad \text{and at } t=0 \quad \mathcal{F}^{-1}\mathcal{F}\varphi(0) = \int_{-\infty}^{\infty} \mathcal{F}\varphi(s)\,ds\,.$$

And now by Fourier inversion on $\mathcal{S}$,

$$\mathcal{F}^{-1}\mathcal{F}\varphi(0) = \varphi(0)\,.$$

Thus

$$\langle \mathcal{F}1, \varphi\rangle = \varphi(0) = \langle \delta, \varphi\rangle$$

and we conclude that $\mathcal{F}1 = \delta$. (We'll also get this by duality and the evenness of δ once we introduce the reverse of a distribution.)

The equations $\mathcal{F}\delta = 1$ and $\mathcal{F}1 = \delta$ are the extreme cases of the trade-off between timelimited and bandlimited signals. δ is the idealization of the most concentrated function possible — it's the ultimate timelimited signal. The function 1, on the other hand, is uniformly spread out over its domain.

It's rather satisfying that the simplest tempered distribution, δ, has the simplest Fourier transform, 1. (Simplest other than the function that is identically zero.) Before there were tempered distributions, however, there was δ, and before there was the Fourier transform of tempered distributions there was $\mathcal{F}\delta = 1$. In the vacuum tube days this had to be established by limiting arguments, accompanied by an uneasiness (among some) over the nature of the limit and what exactly it produced. Our computation of $\mathcal{F}\delta = 1$ is simple and direct and leaves nothing in question about the meaning of all the quantities involved. Whether it is conceptually simpler than the older approach is something you will have to decide for yourself.

The Fourier transform of δ_a Recall the distribution δ_a is defined by

$$\langle \delta_a, \varphi \rangle = \varphi(a)\,.$$

What is the Fourier transform of δ_a? One way to obtain $\mathcal{F}\delta_a$ is via a generalization of the shift theorem, which we'll develop later. Even without that we can find $\mathcal{F}\delta_a$ directly from the definition, as follows.

The calculation is along the same lines as the one for δ. We have

$$\langle \mathcal{F}\delta_a, \varphi \rangle = \langle \delta_a, \mathcal{F}\varphi \rangle = \mathcal{F}\varphi(a) = \int_{-\infty}^{\infty} e^{-2\pi i a x}\varphi(x)\,dx\,.$$

This last integral, which is nothing but the definition of the Fourier transform of φ, can also be interpreted as the pairing of the function $e^{-2\pi i a x}$ with the Schwartz function $\varphi(x)$. That is,

$$\langle \mathcal{F}\delta_a, \varphi \rangle = \langle e^{-2\pi i a x}, \varphi \rangle$$

hence

$$\mathcal{F}\delta_a = e^{-2\pi i s a}\,.$$

To emphasize once again what all is going on here, $e^{-2\pi i a x}$ is *not* integrable, but it defines a tempered distribution through

$$\int_{-\infty}^{\infty} e^{-2\pi i a x}\varphi(x)\,dx$$

which exists because $\varphi(x)$ is integrable. So, again, the equality of $\mathcal{F}\delta_a$ and $e^{-2\pi i s a}$ means they have the same effect when paired with a function in $\mathcal{S}$.

To complete the picture, we can also show that

$$\mathcal{F}e^{2\pi i x a} = \delta_a\,.$$

(There's the usual notational problem here with variables, writing the variable x on the left hand side. The "variable problem" doesn't go away in this more general setting.) This argument should look familiar: if φ is in $\mathcal{S}$ then

$$\begin{aligned}\langle \mathcal{F}e^{2\pi i x a}, \varphi \rangle &= \langle e^{2\pi i x a}, \mathcal{F}\varphi \rangle \\ &= \int_{-\infty}^{\infty} e^{2\pi i x a}\mathcal{F}\varphi(x)\,dx \quad \text{(the pairing here is with respect to } x\text{)}\end{aligned}$$

But this last integral is the inverse Fourier transform of $\mathcal{F}\varphi$ at a, and so we get back $\varphi(a)$. Hence

$$\langle \mathcal{F}e^{2\pi i x a}, \varphi \rangle = \varphi(a) = \langle \delta_a, \varphi \rangle$$

whence

$$\mathcal{F}e^{2\pi i x a} = \delta_a\,.$$

Remark on notation You might be happier using the more traditional notation $\delta(x)$ for δ and $\delta(x-a)$ for δ_a (and $\delta(x+a)$ for δ_{-a}). I don't have any objection to this — it *is* a useful notation for many problems — but try to remember that the δ-function is not a function and, really, it is not to be evaluated "at points"; the notation $\delta(x)$ or $\delta(x-a)$ doesn't really make sense from the distributional point of view.

In this notation the results so far appear as:

$$\mathcal{F}\delta(x \pm a) = e^{\pm 2\pi i s a}\,, \quad \mathcal{F}e^{\pm 2\pi i x a} = \delta(s \mp a)$$

Careful how the + and − enter.

You may also be happier writing

$$\int_{-\infty}^{\infty} \delta(x)\varphi(x)\,dx = \varphi(0) \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(a-x)\varphi(x)\,dx = \varphi(a)\,.$$

I want you to be happy.

The Fourier transform of sine and cosine We can combine the results above to find the Fourier transform pairs for the sine and cosine.

$$\mathcal{F}\left(\tfrac{1}{2}(\delta_a + \delta_{-a})\right) = \tfrac{1}{2}(e^{-2\pi isa} + e^{2\pi isa}) = \cos 2\pi sa\,.$$

I'll even write the results "at points":

$$\mathcal{F}\left(\tfrac{1}{2}(\delta(x-a) + \delta(x+a))\right) = \cos 2\pi sa\,.$$

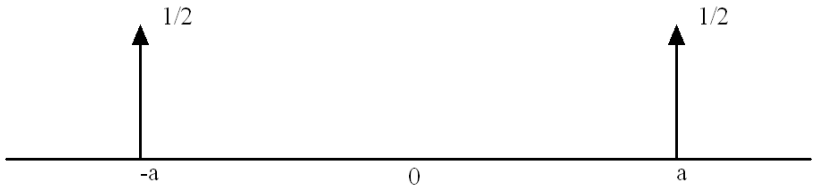
Going the other way,

$$\mathcal{F}\cos 2\pi ax = \mathcal{F}\left(\tfrac{1}{2}(e^{2\pi ixa} + e^{-2\pi ixa})\right) = \tfrac{1}{2}(\delta_a + \delta_{-a})\,.$$

Also written as

$$\mathcal{F}\cos 2\pi ax = \tfrac{1}{2}(\delta(s-a) + \delta(s+a)))\,.$$

The Fourier transform of the cosine is often represented graphically as:



I tagged the spikes with 1/2 to indicate that they have been scaled.[17]

For the sine function we have, in a similar way,

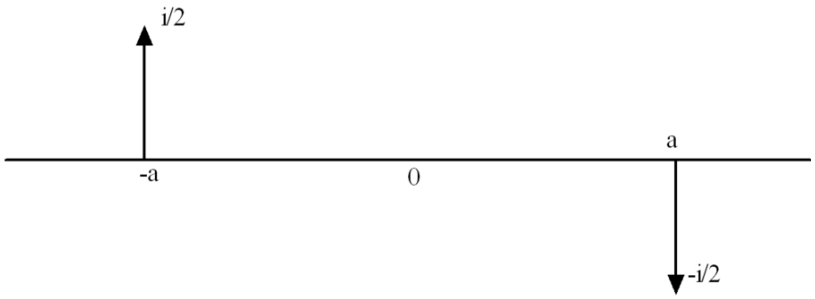
$$\mathcal{F}\left(\frac{1}{2i}(\delta(x+a) - \delta(x-a))\right) = \frac{1}{2i}(e^{2\pi isa} - e^{-2\pi isa}) = \sin 2\pi sa\,,$$

and

$$\mathcal{F}\sin 2\pi ax = \mathcal{F}\left(\frac{1}{2i}(e^{2\pi ixa} - e^{-2\pi ixa})\right) = \frac{1}{2i}(\delta(s-a) - \delta(s+a))\,.$$

The picture of $\mathcal{F}\sin 2\pi x$ is

[17] Of course, the height of a δ_a is infinite, if height means anything at all, so scaling the height doesn't mean much. Sometimes people speak of $\alpha\delta$, for example, as a δ-function "of strength α", meaning just $\langle \alpha\delta, \varphi\rangle = \alpha\varphi(0)$.



Remember that $1/i = -i$. I've tagged the spike δ_a with $-i/2$ and the spike δ_{-a} with $i/2$.

We'll discuss symmetries of the generalized Fourier transform later, but you can think of $\mathcal{F}\cos 2\pi ax$ as real and even and $\mathcal{F}\sin 2\pi ax$ as purely imaginary and odd.

We should reflect a little on what we've done here and not be too quick to move on. The sine and cosine do not have Fourier transforms in the original, classical sense. It is impossible to do anything with the integrals

$$\int_{-\infty}^{\infty} e^{-2\pi isx}\cos 2\pi x\,dx \quad \text{or} \quad \int_{-\infty}^{\infty} e^{-2\pi isx}\sin 2\pi x\,dx\,.$$

To find the Fourier transform of such basic, important functions we *must* abandon the familiar, classical terrain and plant some spikes in new territory. It's worth the effort.

4.8 Fluxions Finis: The End of Differential Calculus

I will continue the development of the generalized Fourier transform and its properties later. For now let's show how introducing distributions "completes" differential calculus; how we can define the derivative of a distribution, and consequently how we can differentiate functions you probably thought had no business being differentiated. We'll make use of this for Fourier transforms, too.

The motivation for how to bring about this remarkable state of affairs goes back to integration by parts, a technique we've used often in our calculations with the Fourier transform. If φ is a test function and f is a function for which $f(x)\varphi(x) \to 0$ as $x \to \pm\infty$ (not too much to ask), and *if* f is differentiable *then* we can use integration by parts to write

$$\begin{aligned}\int_{-\infty}^{\infty} f'(x)\varphi(x)\,dx &= \Big[f(x)\varphi(x)\Big]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)\varphi'(x)\,dx \quad (u=\varphi\,,\ dv = f'(x)\,dx)\\ &= -\int_{-\infty}^{\infty} f(x)\varphi'(x)\,dx.\end{aligned}$$

The derivative has shifted from f to φ.

We can find similar formulas for higher derivatives. For example, supposing that the boundary terms in

the integration by parts tend to 0 as $x \to \pm\infty$, we find that

$$\begin{aligned}\int_{-\infty}^{\infty} f''(x)\varphi(x)\,dx &= \Big[f'(x)\varphi(x)\Big]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x)\varphi'(x)\,dx \quad (u=\varphi(x)\,,\ dv = f''(x)\,dx)\\ &= -\int_{-\infty}^{\infty} f'(x)\varphi'(x)\,dx\\ &= -\left(\Big[f(x)\varphi'(x)\Big]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)\varphi''(x)\,dx\right) \quad (u=\varphi'(x)\,,\ dv = f'(x)\,dx)\\ &= \int_{-\infty}^{\infty} f(x)\varphi''(x)\,dx\,.\end{aligned}$$

Watch out — there's no minus sign out front when we've shifted the *second* derivative from f to φ.

We'll concentrate just on the formula for the first derivative. Let's write it again:

$$\int_{-\infty}^{\infty} f'(x)\varphi(x)\,dx = -\int_{-\infty}^{\infty} f(x)\varphi'(x)\,dx\,.$$

The right hand side may make sense even if the left hand side does not, that is, we can view the right hand side as a way of saying how the derivative of f *would* act if it had a derivative. Put in terms of our "try a function first" procedure, if a distribution comes from a function $f(x)$ then this formula tells us how the "derivative" $f'(x)$ *as a distribution*, should be paired with a test function $\varphi(x)$. It should be paired according to the equation above:

$$\langle f', \varphi\rangle = -\langle f, \varphi'\rangle\,.$$

Turning this outcome into a definition, as our general procedure tells us we should do when passing from functions to distributions, we *define* the derivative of a distribution *as another* distribution according to:

- If T is a distribution, then its derivative T' is the distribution defined by

$$\langle T', \varphi\rangle = -\langle T, \varphi'\rangle$$

Naturally, $(T_1 + T_2)' = T_1' + T_2'$ and $(\alpha T)' = \alpha T'$. However, there is *no* product rule in general because there's no way to multiply two distributions. I'll discuss this later in connection with convolution.

You can go on to define derivatives of higher orders in a similar way, and I'll let you write down what the general formula for the pairing should be. The striking thing is that you don't have to stop: *distributions are infinitely differentiable*!

Let's see how differentiating a distribution works in practice.

Derivative of the unit step function The *unit step function*, also called the Heaviside function[18] is defined by[19]

$$H(x) = \begin{cases} 0 & x \le 0 \\ 1 & x > 0 \end{cases}$$

[18] After Oliver Heaviside (1850–1925), whose work we have mentioned several times before.

[19] There's a school of thought that says $H(0)$ should be $1/2$.

$H(x)$ determines a tempered distribution because for any Schwartz function φ the paring

$$\langle H, \varphi \rangle = \int_{-\infty}^{\infty} H(x)\varphi(x)\, dx = \int_{0}^{\infty} \varphi(x)\, dx$$

makes sense (φ is integrable).

From the definition of the derivative of a distribution, if $\varphi(x)$ is any test function then

$$\langle H', \varphi \rangle = -\langle H, \varphi' \rangle = -\int_{-\infty}^{\infty} H(x)\varphi'(x)\, dx = -\int_{0}^{\infty} 1 \cdot \varphi'(x)\, dx = -(\varphi(\infty) - \varphi(0)) = \varphi(0)\,.$$

We see that pairing H' with a test function produces the same result as if we had paired δ with a test function:

$$\langle H', \varphi \rangle = \varphi(0) = \langle \delta, \varphi \rangle\,.$$

We conclude that

$$H' = \delta\,.$$

Derivative of the unit ramp The *unit ramp* function is defined by

$$u(x) = \begin{cases} 0 & x \le 0 \\ x & x > 0 \end{cases}$$

If this were an introductory calculus class and you were asked "What is the derivative of $u(x)$?" you might have said, "It's 0 if $x \le 0$ and 1 if $x > 0$, so it looks like the unit step $H(x)$ to me." You'd be right, but your jerk of a teacher would probably say you were wrong because, according to the rigor police, $u(x)$ is not differentiable at $x = 0$. But now that you know about distributions, here's why you were right. For a test function $\varphi(x)$,

$$\begin{aligned}
\langle u'(x), \varphi(x) \rangle &= -\langle u(x), \varphi'(x) \rangle = -\int_{-\infty}^{\infty} u(x)\varphi'(x)\, dx = -\int_{0}^{\infty} x\varphi'(x)\, dx \\
&= -\left(\Big[x\varphi(x) \Big]_0^{\infty} - \int_{0}^{\infty} \varphi(x)\, dx \right) = \int_{0}^{\infty} \varphi(x)\, dx \\
&\qquad (x\varphi(x) \to 0 \text{ as } x \to \infty \text{ because } \varphi(x) \text{ decays faster than any power of } x) \\
&= \langle H, \varphi \rangle
\end{aligned}$$

Since $\langle u'(x), \varphi(x) \rangle = \langle H, \varphi \rangle$ we conclude that $u' = H$ as distributions. Then of course, $u'' = \delta$.

Derivative of the signum (or sign) function The *signum (or sign) function* is defined by

$$\operatorname{sgn}(x) = \begin{cases} +1 & x > 0 \\ -1 & x < 0 \end{cases}$$

Note that sgn is not defined at $x = 0$, but that's not an issue in the derivation to follow.

Let $\varphi(x)$ be any test function. Then

$$\begin{aligned}
\langle \operatorname{sgn}', \varphi \rangle &= -\langle \operatorname{sgn}, \varphi' \rangle = -\int_{-\infty}^{\infty} \operatorname{sgn}(x)\varphi'(x)\, dx \\
&= -\left(\int_{-\infty}^{0} (-1)\varphi'(x)\, dx + \int_{0}^{\infty} (+1)\varphi'(x)\, dx \right) \\
&= (\varphi(0) - \varphi(-\infty)) - (\varphi(\infty) - \varphi(0)) = 2\varphi(0)
\end{aligned}$$

The result of pairing sgn' with φ is the same as if we had paired φ with 2δ;

$$\langle \text{sgn}', \varphi \rangle = 2\varphi(0) = \langle 2\delta, \varphi \rangle$$

Hence

$$\text{sgn}' = 2\delta\,.$$

Observe that $H(x)$ has a unit jump up at 0 and its derivative is δ, whereas sgn jumps up by 2 at 0 and its derivative is 2δ.

Derivative of δ To find the derivative of the δ-function we have, for any test function φ,

$$\langle \delta', \varphi \rangle = -\langle \delta, \varphi' \rangle = -\varphi'(0)\,.$$

That's really as much of a formula as we can write. δ itself acts by pulling out the value of a test function at 0, and δ' acts by pulling out *minus* the value of the derivative of the test function at 0. I'll let you determine the higher derivatives of δ.

Derivative of $\ln|x|$ Remember that famous formula from calculus:

$$\frac{d}{dx}\ln|x| = \frac{1}{x}\,.$$

Any chance of something like that being true for distributions? Yes, *with the proper interpretation.* This is an important example because it leads to the *Hilbert transform*, a tool that communications engineers use everyday. For your information, the Hilbert transform is given by convolution of a signal with $1/\pi x$. Once we learn how to take the Fourier transform of $1/x$, which is coming up, we'll then see that the Hilbert transform is a filter with the interesting property that magnitudes of the spectral components are unchanged but their phases are shifted by $\pm\pi/2$.

Because of their usefulness in applications it's worth going through the analysis of the distributions $\ln|x|$ and $1/x$. This takes more work than the previous examples, however, so I've put the details in Section 4.21.

4.9 Approximations of Distributions and Justifying the "Try a Function First" Principle

We started off by enunciating the principle that to see how to extend an operation from functions to distributions one should start by considering the case when the distribution comes from a function (and hence that the pairing is by integration). Let me offer a justification of why this works.

It's true that not every distribution comes from a function (δ doesn't), but it's also true that any distribution can be approximated by ones that comes from functions. The statement is:

> If T is any tempered distribution then there are Schwartz functions f_n such that T_{f_n} converge to T.

This says that for any Schwartz function φ

$$\langle T_{f_n}, \varphi \rangle = \int_{-\infty}^{\infty} f_n(x)\varphi(x)\,dx \to \langle T, \varphi \rangle\,,$$

that is, the pairing of any tempered distribution with a Schwartz function can be expressed as a limit of the natural pairing with approximating functions via integration. We're not saying that $T_{f_n} \to T_f$ for

some function f, because it's not the Schwartz functions f_n that are converging to a function, it's the *associated distributions* that are converging to a distribution. You *don't* necessarily have $T = T_f$ for some function f. (Also, this result doesn't say how you're supposed to find the approximating functions, just that they exist.)

Consider how we might apply this to justify our approach to defining the Fourier transform of a tempered distribution. According to the approximation result, any tempered distribution T is a limit of distributions that come from Schwartz functions, and we would have, say,

$$\langle T, \varphi\rangle = \lim_{n\to\infty} \langle \psi_n, \varphi\rangle \,.$$

Then if $\mathcal{F}T$ is to make sense we might understand it to be given by

$$\langle \mathcal{F}T, \varphi\rangle = \lim_{n\to\infty} \langle \mathcal{F}\psi_n, \varphi\rangle = \lim_{n\to\infty} \langle \psi_n, \mathcal{F}\varphi\rangle = \langle T, \mathcal{F}\varphi\rangle \,.$$

There's our definition.

4.10 The Generalized Fourier Transform Includes the Classical Fourier Transform

Remember that we identify a function f with the distribution T_f it defines and it is in this way we say that the tempered distributions contain many of the classical functions. Now suppose a function $f(x)$ defines a distribution and that $f(x)$ has a (classical) Fourier transform $\mathcal{F}f(s)$ which also defines a distribution, i.e.,

$$\int_{-\infty}^{\infty} \mathcal{F}f(s)\varphi(s)\,ds$$

exists for every Schwartz function φ (which isn't asking too much). Writing $T_{\mathcal{F}f}$ for the tempered distribution determined by $\mathcal{F}f$,

$$\begin{aligned}
\langle T_{\mathcal{F}f}, \varphi\rangle &= \int_{-\infty}^{\infty} \mathcal{F}f(s)\varphi(s)\,ds \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{-2\pi i s x} f(x)\,dx \right) \varphi(s)\,ds = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi i s x} f(x)\varphi(s)\,ds\,dx \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{-2\pi i s x} \varphi(s)\,ds \right) f(x)\,dx = \int_{-\infty}^{\infty} \mathcal{F}\varphi(x) f(x)\,dx = \langle T_f, \mathcal{F}\varphi\rangle
\end{aligned}$$

But now, by our definition of the generalized Fourier transform

$$\langle T_f, \mathcal{F}\varphi\rangle = \langle \mathcal{F}T_f, \varphi\rangle \,.$$

Putting this together with the start of the calculation we obtain

$$\langle T_{\mathcal{F}f}, \varphi\rangle = \langle \mathcal{F}T_f, \varphi\rangle \,,$$

whence

$$T_{\mathcal{F}f} = \mathcal{F}T_f \,.$$

In words, if the classical Fourier transform of a function defines a distribution ($T_{\mathcal{F}f}$), then that distribution is the Fourier transform of the distribution that the function defines ($\mathcal{F}T_f$). This is a precise way of saying that the generalized Fourier transform "includes" the classical Fourier transform.

4.11 Operations on Distributions and Fourier Transforms

We want to relive our past glories — duality between $\mathcal{F}$ and $\mathcal{F}^{-1}$, evenness and oddness, shifts and stretches, convolution — in the more general setting we've developed. The new versions of the old results will ultimately look the same as they did before; it's a question of setting things up properly to apply the new definitions. There will be some new results, however. Among them will be formulas for the Fourier transform of $\operatorname{sgn} x$, $1/x$, and the unit step $H(x)$, to take a representative sample. None of these would have been possible before. We'll also point out special properties of δ along the way. Pay particular attention to these because we'll be using them a lot in applications.

Before you dive in, let me offer a reader's guide. There's a lot of material in here — *way* more than you need to know for your day-to-day working life. Furthermore, almost all the results are accompanied by some necessary extra notation; the truth is that it's somewhat more cumbersome to define operations on distributions than on functions, and there's no way of getting around it. We *have* to have this material *in some fashion* but you should probably treat the sections to follow mostly as a reference. Feel free to use the formulas you need when you need them, and remember that our aim is to recover the formulas we know from earlier work in pretty much the same shape as you first learned them.

4.12 Duality, Changing Signs, Evenness and Oddness

One of the first things we observed about the Fourier transform and its inverse is that they're pretty much the same thing except for a change in sign; see Chapter 2. The relationships are

$$\mathcal{F}f(-s) = \mathcal{F}^{-1}f(s)$$
$$\mathcal{F}^{-1}f(-t) = \mathcal{F}f(t)$$

We had similar results when we changed the sign of the variable first and then took the Fourier transform. The relationships are

$$\mathcal{F}(f(-t)) = \mathcal{F}^{-1}f(s)$$
$$\mathcal{F}^{-1}(f(-s)) = \mathcal{F}f(s)$$

We referred to these collectively as the "duality" between Fourier transform pairs, and we'd like to have similar duality formulas when we take the Fourier transforms of distributions.

The problem is that for distributions we don't really have "variables" to change the sign of. We don't really write $\mathcal{F}T(s)$, or $\mathcal{F}T(-s)$, or $T(-s)$, because distributions don't operate on points s — they operate on test functions. What we can do easily is to define a "reversed distribution", and once this is done the rest is plain sailing.

Reversed distributions Recall that we introduced the reversed signal of a signal $f(x)$ by means of

$$f^-(x) = f(-x)$$

and this helped us to write clean, "variable free" versions of the duality results. Using this notation the above results become

$$(\mathcal{F}f)^- = \mathcal{F}^{-1}f, \quad (\mathcal{F}^{-1}f)^- = \mathcal{F}f\,, \quad \mathcal{F}f^- = \mathcal{F}^{-1}f, \quad \mathcal{F}^{-1}f^- = \mathcal{F}f\,.$$

A variant version is to apply $\mathcal{F}$ or $\mathcal{F}^{-1}$ twice, resulting in

$$\mathcal{F}\mathcal{F}f = f^-, \quad \mathcal{F}^{-1}\mathcal{F}^{-1}f = f^- .$$

My personal favorites among formulas of this type are:

$$\mathcal{F}f^- = (\mathcal{F}f)^-, \quad \mathcal{F}^{-1}f^- = (\mathcal{F}^{-1}f)^- .$$

What can "sign change", or "reversal" mean for a distribution T? Our standard approach is first to take the case when the distribution comes from a function $f(x)$. The pairing of T_f with a test function φ is

$$\langle T_f, \varphi\rangle = \int_{-\infty}^{\infty} f(x)\varphi(x)\,dx .$$

We might well believe that reversing T_f (i.e., a possible definition of $(T_f)^-$) should derive from reversing f, that is, integrating f^- against a test function. The paring of T_{f^-} with φ is

$$\begin{aligned}\langle T_{f^-}, \varphi\rangle &= \int_{-\infty}^{\infty} f(-x)\varphi(x)\,dx \\ &= \int_{\infty}^{-\infty} f(u)\varphi(-u)\,(-du) \quad \text{(making the change of variable } u = -x) \\ &= \int_{-\infty}^{\infty} f(u)\varphi(-u)\,du.\end{aligned}$$

This says that f^- is paired with $\varphi(x)$ in the same way as f is paired with φ^-, more precisely:

$$\langle T_{f^-}, \varphi\rangle = \langle T_f, \varphi^-\rangle .$$

Wouldn't it then make sense to say we have found a meaning for $(T_f)^-$ (i.e., have defined $(T_f)^-$) via the formula

$$\langle (T_f)^-, \varphi\rangle = \langle T_f, \varphi^-\rangle \quad \text{(the right-hand-side is defined because } \varphi^- \text{ is defined)} .$$

The "outcome" — how this result should be turned into a general definition — is before our eyes:

- If T is a distribution we define the reversed distribution T^- according to

$$(T^-, \varphi) = (T, \varphi^-) .$$

Note that with this definition we have, quite agreeably,

$$(T_f)^- = T_{f^-} .$$

If you understand what's just been done you'll understand this last equation. Understand it.

Duality It's now easy to state the duality relations between the Fourier transform and its inverse. Adopting the notation, above, we want to look at $(\mathcal{F}T)^-$ and how it compares to $\mathcal{F}^{-1}T$. For a test function φ,

$$\begin{aligned}((\mathcal{F}T)^-, \varphi) &= (\mathcal{F}T, \varphi^-) \\ &= (T, \mathcal{F}(\varphi^-)) \quad \text{(that's how the Fourier transform is defined)} \\ &= (T, \mathcal{F}^{-1}\varphi) \quad \text{(because of duality for ordinary Fourier transforms)} \\ &= (\mathcal{F}^{-1}T, \varphi) \quad \text{(that's how the inverse Fourier transform is defined)}\end{aligned}$$

Pretty slick, really. We can now write simply

$$(\mathcal{F}T)^- = \mathcal{F}^{-1}T\,.$$

We also then have

$$\mathcal{F}T = (\mathcal{F}^{-1}T)^-\,.$$

Same formulas as in the classical setting.

To take one more example,

$$\langle \mathcal{F}(T^-), \varphi\rangle = \langle T^-, \mathcal{F}\varphi\rangle = \langle T, (\mathcal{F}\varphi)^-\rangle = \langle T, \mathcal{F}^{-1}\varphi\rangle = \langle \mathcal{F}^{-1}T, \varphi\rangle\,,$$

and there's the identity

$$\mathcal{F}(T^-) = \mathcal{F}^{-1}T$$

popping out. Finally, we have

$$\mathcal{F}^{-1}(T^-) = \mathcal{F}T\,.$$

Combining these,

$$\mathcal{F}T^- = (\mathcal{F}T)^-, \quad \mathcal{F}^{-1}T^- = (\mathcal{F}^{-1}T)^-\,.$$

Applying $\mathcal{F}$ or $\mathcal{F}^{-1}$ twice leads to

$$\mathcal{F}\mathcal{F}T = T^-, \quad \mathcal{F}^{-1}\mathcal{F}^{-1}T = T^-\,.$$

That's all of them.

Even and odd distributions: δ is even Now that we know how to reverse a distribution we can define what it means for a distribution to be even or odd.

- A distribution T is *even* if $T^- = T$. A distribution is *odd* if $T^- = -T$.

Observe that if $f(x)$ determines a distribution T_f and if $f(x)$ is even or odd then T_f has the same property. For, as we noted earlier,

$$(T_f)^- = T_{f^-} = T_{\pm f} = \pm T_f\,.$$

Let's next establish the useful fact:

- δ is even.

This is quick:

$$\langle \delta^-, \varphi\rangle = \langle \delta, \varphi^-\rangle = \varphi^-(0) = \varphi(-0) = \varphi(0) = \langle \delta, \varphi\rangle$$

Let's now use this result plus duality to rederive $\mathcal{F}1 = \delta$. This is quick, too:

$$\mathcal{F}1 = (\mathcal{F}^{-1}1)^- = \delta^- = \delta\,.$$

$\delta_a + \delta_{-a}$ is even. $\delta_a - \delta_{-a}$ is odd. Any distribution is the sum of an even and an odd distribution.

You can now show that *all* of our old results on evenness and oddness of a signal and its Fourier transform extend in like form to the Fourier transform of distributions. For example, if T is even then so is $\mathcal{F}T$, for

$$(\mathcal{F}T)^- = \mathcal{F}T^- = \mathcal{F}T\,,$$

and if T is odd then

$$(\mathcal{F}T)^- = \mathcal{F}T^- = \mathcal{F}(-T) = -\mathcal{F}T\,,$$

thus $\mathcal{F}T$ is odd.

Notice how this works for the cosine (even) and the sine (odd) and their respective Fourier transforms:

$$\mathcal{F}\cos 2\pi ax = \tfrac{1}{2}(\delta_a + \delta_{-a})$$

$$\mathcal{F}\sin 2\pi ax = \frac{1}{2i}(\delta_a - \delta_{-a})$$

I'll let you define what it means for a distribution to be real, or purely imaginary.

Fourier transform of sinc

$$\begin{aligned}\mathcal{F}\,\mathrm{sinc} &= \mathcal{F}(\mathcal{F}\Pi)\\ &= \Pi^- \quad \text{(one of the duality equaltions)}\\ &= \Pi \quad (\Pi \text{ is even})\end{aligned}$$

At last. To be really careful here: $\mathcal{F}\,\mathrm{sinc}$ makes sense only as a tempered distribution. So the equality $\mathcal{F}\,\mathrm{sinc} = \Pi$ has to be understood as an equation between distributions, meaning that $\mathcal{F}\,\mathrm{sinc}$ and Π give the same result when paired with any Schwartz function. But you should lose no sleep over this. From now on, write $\mathcal{F}\,\mathrm{sinc} = \Pi$, think in terms of functions, and start your company.

4.13 A Function Times a Distribution Makes Sense

There's no way to define the product of two distributions that works consistently with all the rest of the definitions and properties — try as you might, it just won't work. However, it is possible (and easy) to define the product of a function and a distribution.

Say T is a distribution and g is a function. What is gT as a distribution? I have to tell you what $\langle gT, \varphi\rangle$ is for a test function φ. We take our usual approach to looking for the outcome when T comes from a function, $T = T_f$. The pairing of gT_f and φ is given by

$$\langle gT_f, \varphi\rangle = \int_{-\infty}^{\infty} g(x)f(x)\varphi(x)\,dx = \int_{-\infty}^{\infty} f(x)(g(x)\varphi(x))\,dx$$

As long as $g\varphi$ is still a test function (so, certainly, g has to be infinitely differentiable) this last integral is the pairing $\langle T_f, g\varphi\rangle$. The outcome is $\langle gT_f, \varphi\rangle = \langle T_f, g\varphi\rangle$. We thus make the following definition:

- Let T be a distribution. If g is a smooth function such that $g\varphi$ is a test function whenever φ is a test function, then gT is the distribution defined by

$$\langle gT, \varphi\rangle = \langle T, g\varphi\rangle\,.$$

This looks as simple as can be, and it is. You may wonder why I even singled out this operation for comment. In fact, some funny things can happen, as we'll now see.

4.13.1 A function times δ

Watch what happens if we multiply δ by $g(x)$:

$$\langle g\delta, \varphi\rangle = \langle \delta, g\varphi\rangle = g(0)\varphi(0)$$

This is the same result as if we had paired $g(0)\delta$ with φ. Thus

$$g(x)\delta = g(0)\delta$$

In particular if $g(0) = 0$ then the result is 0! For example

$$x\delta = 0$$

or for that matter

$$x^n\delta = 0$$

for any positive power of x.

Along with $g\delta = g(0)\delta$ we have

$$g(x)\delta_a = g(a)\delta_a\,.$$

To show this:

$$\langle g\delta_a, \varphi\rangle = \langle \delta_a, g\varphi\rangle = g(a)\varphi(a) = g(a)\langle \delta_a, \varphi\rangle = \langle g(a)\delta_a, \varphi\rangle\,.$$

If you want to write this identity more classically, it is

$$g(x)\delta(x-a) = g(a)\delta(x-a)\,.$$

We'll use this property in many applications, for example when we talk about sampling.

More on a function times δ There's a converse to one of the above properties that's interesting in itself and that we'll use in the next section when we find some particular Fourier transforms.

- If T is a distribution and $xT = 0$ then $T = c\delta$ for some constant c.

I'll show you the proof of this, but you can skip it if you want. The argument is more involved than the simple statement might suggest, but it's a nice example, and a fairly typical example, of the kind of tricks that are used to prove things in this area. Each to their own tastes.

Knowing where this is going, let me start with an innocent observation.[20] If ψ is a smooth function then

$$\begin{aligned}\psi(x) &= \psi(0) + \int_0^x \psi'(t)\,dt \\ &= \psi(0) + \int_0^1 x\psi'(xu)\,du \quad \text{(using the substitution } u = t/x\text{)} \\ &= \psi(0) + x\int_0^1 \psi'(xu)\,du\,.\end{aligned}$$

Let

$$\Psi(x) = \int_0^1 \psi'(xu)\,du$$

[20] This innocent observation is actually the beginning of deriving Taylor series "with remainder".

so that

$$\psi(x) = \psi(0) + x\Psi(x)\,.$$

We'll now use this innocent observation in the case when $\psi(0) = 0$, for then

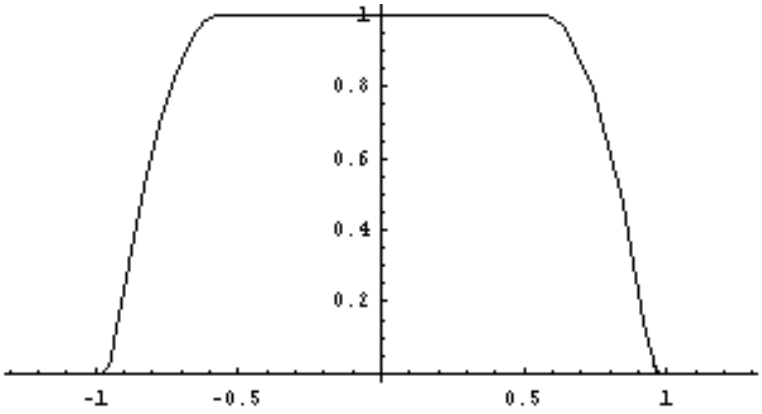
$$\psi(x) = x\Psi(x)\,.$$

It's clear from the definition of Ψ that Ψ is as smooth as ψ is and that if, for example, ψ is rapidly decreasing then so is Ψ. Put informally, we've shown that if $\psi(0) = 0$ we can "factor out an x" and still have a function that's as good as ψ.

Now suppose $xT = 0$, meaning that

$$\langle xT, \varphi\rangle = 0$$

for *every* test function φ. Fix a smooth windowing function φ_0 that is identically 1 on an interval about $x = 0$, goes down to zero smoothly and is identically zero far enough away from $x = 0$; we mentioned smooth windows earlier — see Section 4.20, below.



Since φ_0 is fixed in this argument, T operating on φ_0 gives some fixed number, say

$$\langle T, \varphi_0\rangle = c\,.$$

Now write

$$\varphi(x) = \varphi(0)\varphi_0(x) + (\varphi(x) - \varphi(0)\varphi_0(x)) = \varphi(0)\varphi_0(x) + \psi(x)$$

where, by this clever way of writing φ, the function $\psi(x) = \varphi(x) - \varphi(0)\varphi_0(x)$ has the property that

$$\psi(0) = \varphi(0) - \varphi(0)\varphi_0(0) = \varphi(0) - \varphi(0) = 0$$

because $\varphi_0(0) = 1$. This means that we can factor out an x and write

$$\psi(x) = x\Psi(x)$$

where Ψ is again a test function, and then

$$\varphi(x) = \varphi(0)\varphi_0(x) + x\Psi(x)\,.$$

But now

$$\begin{aligned}
\langle T, \varphi(x)\rangle &= \langle T, \varphi(0)\varphi_0 + x\Psi\rangle \\
&= \langle T, \varphi(0)\varphi_0\rangle + \langle T, x\Psi\rangle \\
&= \varphi(0)\langle T, \varphi_0\rangle + \langle T, x\Psi\rangle \quad \text{(linearity)} \\
&= \varphi(0)\langle T, \varphi_0\rangle + \langle xT, \Psi\rangle \quad \text{(that's how mutiplying } T \text{ by the smooth function } x \text{ works)} \\
&= \varphi(0)\langle T, \varphi_0\rangle + 0 \quad \text{(because } \langle xT, \Psi\rangle = 0\text{!)} \\
&= c\varphi(0) \\
&= \langle c\delta, \varphi\rangle
\end{aligned}$$

We conclude that

$$T = c\delta\,.$$

4.14 The Derivative Theorem

Another basic property of the Fourier transform is how it behaves in relation to differentiation — "differentiation becomes multiplication" is the shorthand way of describing the situation. We know how to differentiate a distribution, and it's an easy step to bring the Fourier transform into the picture. We'll then use this to *find* the Fourier transform for some common functions that heretofore we have not been able to treat.

Let's recall the formulas for functions, best written:

$$f'(t) \rightleftharpoons 2\pi i s F(s) \quad \text{and} \quad -2\pi i t f(t) \rightleftharpoons F'(s)$$

where $f(t) \rightleftharpoons F(s)$.

We first want to find $\mathcal{F}T'$ for a distribution T. For any test function φ,

$$\begin{aligned}
\langle \mathcal{F}T', \varphi\rangle &= \langle T', \mathcal{F}\varphi\rangle = -\langle T, (\mathcal{F}\varphi)'\rangle \\
&= -\langle T, \mathcal{F}(-2\pi i s\varphi)\rangle \quad \text{(from the second formula above)} \\
&= -\langle \mathcal{F}T, -2\pi i s\varphi\rangle \quad \text{(moving } \mathcal{F} \text{ back over to } T\text{)} \\
&= \langle 2\pi i s\mathcal{F}T, \varphi\rangle \\
&\text{(cancelling minus signs and moving the smooth function } 2\pi i s \text{ back onto } \mathcal{F}T\text{)}
\end{aligned}$$

So the second formula for functions has helped us derive the version of the first formula for distributions:

$$\mathcal{F}T' = 2\pi i s\mathcal{F}T\,.$$

On the right hand side, that's the smooth function $2\pi i s$ times the distribution $\mathcal{F}T$.

Now let's work with $(\mathcal{F}T)'$:

$$\begin{aligned}
\langle (\mathcal{F}T)', \varphi\rangle &= -\langle \mathcal{F}T, \varphi'\rangle = -\langle T, \mathcal{F}(\varphi')\rangle \\
&= -\langle T, 2\pi i s\mathcal{F}\varphi\rangle \quad \text{(from the first formula for functions)} \\
&= \langle -2\pi i s T, \mathcal{F}\varphi\rangle \\
&= \langle \mathcal{F}(-2\pi i s T), \varphi\rangle
\end{aligned}$$

Therefore

$$(\mathcal{F}T)' = \mathcal{F}(-2\pi i s T)\,.$$

4.14.1 Fourier transforms of sgn, $1/x$, and the unit step

We can put the derivative formula to use to find the Fourier transform of the sgn function, and from that the Fourier transform of the unit step.

On the one hand, $\mathrm{sgn}' = 2\delta$, from an earlier calculation, so $\mathcal{F}\mathrm{sgn}' = 2\mathcal{F}\delta = 2$. On the other hand, using the derivative theorem,

$$\mathcal{F}\mathrm{sgn}' = 2\pi i s\,\mathcal{F}\mathrm{sgn}\,.$$

Hence

$$2\pi i s\,\mathcal{F}\mathrm{sgn} = 2\,.$$

We'd like to say that

$$\mathcal{F}\text{sgn} = \frac{1}{\pi i s}$$

where $1/s$ is the Cauchy principal value distribution. In fact this *is* the case, but it requires a little more of an argument. From $2\pi i s\,\mathcal{F}\text{sgn} = 2$ we can say that

$$\mathcal{F}\text{sgn} = \frac{1}{\pi i s} + c\delta$$

where c is a constant. Why the extra δ term? We need it for generality. If T is such that $sT = 0$ then $2\pi i s\,\mathcal{F}\text{sgn}$ and $2 + sT$, will have the same effect when paired with a test function. But earlier we showed that such a T must be $c\delta$ for some constant c. Thus we write

$$\mathcal{F}\text{sgn} = \frac{1}{\pi i s} + c\delta\,.$$

Now, sgn is odd and so is its Fourier transform, and so is $1/2\pi i s$. But δ is even, and the only way $1/\pi i s + c\delta$ can be odd is to have $c = 0$.

To repeat, we have now found

$$\mathcal{F}\text{sgn} = \frac{1}{\pi i s}\,.$$

Gray and Goodman p. 217 (and also Bracewell) give a derivation of this result using limiting arguments.

By duality we also now know the Fourier transform of $1/x$. The distributions are odd, hence

$$\mathcal{F}\left(\frac{1}{x}\right) = -\pi i \,\text{sgn}\, s\,.$$

Having found $\mathcal{F}\text{sgn}$ it's easy to find the Fourier transform of the unit step H. Indeed,

$$H(t) = \tfrac{1}{2}(1 + \text{sgn}\, t)$$

and from this

$$\mathcal{F}H = \tfrac{1}{2}\left(\delta + \frac{1}{\pi i s}\right)\,.$$

4.15 Shifts and the Shift Theorem

Let's start with shifts. What should we make of $T(x \pm b)$ for a distribution T when, once again, it doesn't make sense to evaluate T at a point $x \pm b$? We use the same strategy as before, starting by assuming that T comes from a function f and asking how we should pair, say, $f(x - b)$ with a test function $\varphi(x)$. For that, we want

$$\int_{-\infty}^{\infty} f(x-b)\varphi(x)\,dx = \int_{-\infty}^{\infty} f(u)\varphi(u+b)\,du \quad \text{(making the substitution } u = x - b.\text{)}$$

As we did when we analyzed "changing signs" our work on shifts is made easier (really) if we introduce a notation.

The shift or delay operator It's pretty common to let τ_b stand for "translate by b", or "delay by b". That is, for any function φ the delayed signal, $\tau_b\varphi$, is the new function defined by

$$(\tau_b\varphi)(x) = \varphi(x-b)\,.$$

Admittedly there's some awkwardness in the notation here; one has to remember that τ_b corresponds to $x-b$.

In terms of τ_b the integrals above can be written (using x as a variable of integration in both cases):

$$\langle \tau_b f, \varphi\rangle = \int_{-\infty}^{\infty} (\tau_b f)(x)\varphi(x)\,dx = \int_{-\infty}^{\infty} f(x)(\tau_{-b}\varphi)(x)\,dx = \langle f, \tau_{-b}\varphi\rangle\,.$$

Note that on the left hand side f is shifted by b while on the right hand side φ is shifted by $-b$. This result guides us in making the general definition:

- If T is a distribution we define $\tau_b T$ (T delayed by b) by

$$\langle \tau_b T, \varphi\rangle = \langle T, \tau_{-b}\varphi\rangle\,.$$

You can check that for a distribution T_f coming from a function f we have

$$\tau_b T_f = T_{\tau_b f}\,.$$

δ_a is a shifted δ To close the loop on some things we said earlier, watch what happens when we delay δ by a:

$$\begin{aligned}\langle \tau_a\delta, \varphi\rangle &= \langle \delta, \tau_{-a}\varphi\rangle \\ &= (\tau_{-a}\varphi)(0) \\ &= \varphi(a) \quad \text{(remember, } \tau_{-a}\varphi(x) = \varphi(x+a)\text{)} \\ &= \langle \delta_a, \varphi\rangle\end{aligned}$$

We have shown that

$$\tau_a\delta = \delta_a\,.$$

This is the variable-free way of writing $\delta(x-a)$.

The shift theorem: We're now ready for the general form of the shift theorem:

If T is a distribution then

$$\mathcal{F}(\tau_b T) = e^{-2\pi i b x}\mathcal{F}T\,.$$

To verify this, first

$$\langle \mathcal{F}(\tau_b T), \varphi\rangle = \langle \tau_b T, \mathcal{F}\varphi\rangle = \langle T, \tau_{-b}\mathcal{F}\varphi\rangle\,.$$

We can evaluate the test function in the last term:

$$\begin{aligned}\tau_{-b}(\mathcal{F}\varphi)(s) &= \mathcal{F}\varphi(s+b)\\ &= \int_{-\infty}^{\infty} e^{-2\pi i(s+b)x}\varphi(x)\,dx\\ &= \int_{-\infty}^{\infty} e^{-2\pi isx}e^{-2\pi ibx}\varphi(x)\,dx = \mathcal{F}(e^{-2\pi ibx}\varphi)(s)\end{aligned}$$

Now plug this into what we had before:

$$\begin{aligned}\langle \mathcal{F}(\tau_b T), \varphi\rangle &= \langle T, \tau_{-b}\mathcal{F}\varphi\rangle\\ &= \langle T, \mathcal{F}(e^{-2\pi ibx}\varphi)\rangle\\ &= \langle \mathcal{F}T, e^{-2\pi ibx}\varphi\rangle = \langle e^{-2\pi ibx}\mathcal{F}T, \varphi\rangle\end{aligned}$$

Thus, keeping track of what we're trying to show,

$$\langle \mathcal{F}(\tau_b T), \varphi\rangle = \langle e^{-2\pi ibx}\mathcal{F}T, \varphi\rangle$$

for all test functions φ, and hence

$$\mathcal{F}(\tau_b T) = e^{-2\pi ibx}\mathcal{F}T\,.$$

As one quick application of this let's see what happens to the shifted δ. By the shift theorem

$$\mathcal{F}\tau_a\delta = e^{-2\pi ias}\mathcal{F}\delta = e^{-2\pi isa}$$

in accord with what we found earlier for $\mathcal{F}\delta_a$ directly from the definitions of δ_a and $\mathcal{F}$.

4.16 Scaling and the Stretch Theorem

To find the appropriate form of the Stretch Theorem, or Similarity Theorem, we first have to consider how to define $T(ax)$. Following our now usual procedure, we check what happens when T comes from a function f. We need to look at the pairing of $f(ax)$ with a test function $\varphi(x)$, and we find for $a > 0$ that

$$\int_{-\infty}^{\infty} f(ax)\varphi(x)\,dx = \int_{-\infty}^{\infty} f(u)\varphi(u/a)\frac{1}{a}\,du\,,$$

making the substitution $u = ax$, and for $a < 0$ that

$$\int_{-\infty}^{\infty} f(ax)\varphi(x)\,dx = \int_{\infty}^{-\infty} f(u)\varphi(u/a)\frac{1}{a}\,du = -\int_{-\infty}^{\infty} f(u)\varphi(u/a)\frac{1}{a}\,du\,.$$

We combine the cases and write

$$\int_{-\infty}^{\infty} f(ax)\varphi(x)\,dx = \int_{-\infty}^{\infty} f(u)\frac{1}{|a|}\varphi(u/a)\,du\,.$$

The scaling operator As we did to write shifts in a variable-free way, we do the same for similarities. We let σ_a stand for the operator "scale by a". That is,

$$(\sigma_a\varphi)(x) = \varphi(ax)\,.$$

The integrals above can then be written as

$$\langle \sigma_a f\, \varphi\rangle = \int_{-\infty}^{\infty} (\sigma_a f)(x)\varphi(x)\,dx = \int_{-\infty}^{\infty} f(x)\frac{1}{|a|}(\sigma_{1/a}\varphi)(x)\,dx = \langle f, \frac{1}{|a|}(\sigma_{1/a}\varphi)\rangle\,.$$

Thus for a general distribution:

- If T is a distribution we define $\sigma_a T$ via

$$\langle \sigma_a T, \varphi\rangle = \langle T, \frac{1}{|a|}\sigma_{1/a}\varphi\rangle\,.$$

Note also that then

$$\langle \frac{1}{|a|}\sigma_{1/a}T, \varphi\rangle = \langle T, \sigma_a\varphi\rangle\,.$$

For a distribution T_f coming from a function f the relation is

$$\sigma_a T_f = T_{\sigma_a f}\,.$$

Scaling δ Since δ is concentrated at a point, however you want to interpret that, you might not think that scaling $\delta(x)$ to $\delta(ax)$ should have any effect. But it does:

$$\begin{aligned}\langle \sigma_a\delta, \varphi\rangle &= \langle \delta, \frac{1}{|a|}\sigma_{1/a}\varphi\rangle = \frac{1}{|a|}(\sigma_{1/a}\varphi)(0)\\ &= \frac{1}{|a|}\varphi(0/a) = \frac{1}{|a|}\varphi(0) = \langle \frac{1}{|a|}\delta, \varphi\rangle\end{aligned}$$

Hence

$$\sigma_a\delta = \frac{1}{|a|}\delta\,.$$

This is most often written "at points", as in

$$\delta(ax) = \frac{1}{|a|}\delta(x)$$

The effect of "scaling the variable" is to "scale the strength" of δ by the reciprocal amount.

The stretch theorem With the groundwork we've done it's now not difficult to state and derive the general stretch theorem:

If T is a distribution then

$$\mathcal{F}(\sigma_a T) = \frac{1}{|a|}\sigma_{1/a}(\mathcal{F}T)\,.$$

To check this,

$$\langle \mathcal{F}(\sigma_a T), \varphi\rangle = \langle \sigma_a T, \mathcal{F}\varphi\rangle = \langle T, \frac{1}{|a|}\sigma_{1/a}\mathcal{F}\varphi\rangle\,.$$

But now by the stretch theorem for functions

$$\frac{1}{|a|}(\sigma_{1/a}\mathcal{F}\varphi)(s) = \frac{1}{|a|}\mathcal{F}\varphi\left(\frac{s}{a}\right) = \mathcal{F}(\sigma_a\varphi)(s)\,.$$

Plug this back into what we had:

$$\langle \mathcal{F}(\sigma_a T), \varphi\rangle = \langle T, \frac{1}{|a|}\sigma_{1/a}\mathcal{F}\varphi\rangle = \langle T, \mathcal{F}(\sigma_a\varphi)\rangle = \langle \mathcal{F}T, \sigma_a\varphi\rangle = \langle \frac{1}{|a|}\sigma_{1/a}(\mathcal{F}T), \varphi\rangle\,.$$

This proves that

$$\mathcal{F}(\sigma_a T) = \frac{1}{|a|}\sigma_{1/a}(\mathcal{F}T)\,.$$

4.17 Convolutions and the Convolution Theorem

Convolution of distributions presents some special problems and we're not going to go into this too deeply. It's not so hard figuring out formally how to define $S*T$ for distributions S and T, it's setting up conditions under which the convolution exists that's somewhat tricky. This is related to the fact of nature that it's impossible to define (in general) the product of two distributions, for we also want to have a convolution theorem that says $\mathcal{F}(S*T) = (\mathcal{F}S)(\mathcal{F}T)$ and both sides of the formula should make sense.

What works easily is the convolution of a distribution with a test function. This goes through as you might expect (with a little twist) but in case you want to skip the following discussion I am pleased to report right away that the convolution theorem on Fourier transforms continues to hold: If ψ is a test function and T is a distribution then

$$\mathcal{F}(\psi * T) = (\mathcal{F}\psi)(\mathcal{F}T)\,.$$

The right hand side is the product of a test function and a distribution, which *is* defined.

Here's the discussion that supports the development of convolution in this setting. First we consider how to define convolution of ψ and T. As in every other case of extending operations from functions to distributions, we suppose first that a distribution T comes from a function f. If ψ is a test function we want to look at the pairing of $\psi * f$ with a test function φ. This is

$$\begin{aligned}
\langle \psi * f, \varphi\rangle &= \int_{-\infty}^{\infty} (\psi * f)(x)\varphi(x)\,dx \\
&= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\psi(x-y)f(y)\,dy\right)\varphi(x)\,dx \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\psi(x-y)\varphi(x)f(y)\,dy\,dx \\
&= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\psi(x-y)\varphi(x)\,dx\right)f(y)\,dy
\end{aligned}$$

(The interchange of integration in the last line is justified because every function in sight is as nice as can be.) We *almost* see a convolution $\psi*\varphi$ in the inner integral — but the sign is wrong. However, bringing back our notation $\psi^-(x) = \psi(-x)$, we *can* write the inner integral as the convolution $\psi^-*\varphi$ (or as $\psi*\varphi^-$ by a change of variable). That is

$$\langle \psi * f, \varphi\rangle = \int_{-\infty}^{\infty}(\psi * f)(x)\varphi(x)\,dx = \int_{-\infty}^{\infty}(\psi^- * \varphi)(x)f(x)\,dx = \langle f, \psi^- * \varphi\rangle\,.$$

This tells us what to do in general:

- If T is a distribution and ψ is a test function then $\psi * T$ is defined by

$$\langle \psi * T, \varphi \rangle = \langle T, \psi^- * \varphi \rangle \, .$$

Convolution property of δ Let's see how this works to establish the basic convolution property of the δ-function:

$$\psi * \delta = \psi$$

where on the right hand side we regard ψ as a distribution. To check this:

$$\begin{aligned} \langle \psi * \delta, \varphi \rangle &= \langle \delta, \psi^- * \varphi \rangle = (\psi^- * \varphi)(0) \\ &= \int_{-\infty}^{\infty} \psi^-(-y)\varphi(y)\, dy = \int_{-\infty}^{\infty} \psi(y)\varphi(y)\, dy = \langle \psi, \varphi \rangle \, . \end{aligned}$$

Look at this carefully, or rather, simply. It says that $\psi * \delta$ has the same outcome as ψ does when paired with ϕ. That is, $\psi * \delta = \psi$. Works like a charm. Air tight.

As pointed out earlier, it's common practice to write this property of δ as an integral,

$$\psi(x) = \int_{-\infty}^{\infty} \delta(x - y)\psi(y)\, dy \, .$$

This is sometimes called the *sifting property* of δ. Generations of distinguished engineers and scientists have written this identity in this way, and no harm seems to have befallen them.

We can even think of Fourier inversion as a kind of convolution identity, in fact as exactly the sifting property of δ. The inversion theorem is sometimes presented in this way (proved, according to some people, though it's circular reasoning). We need to write (formally)

$$\int_{-\infty}^{\infty} e^{2\pi i s x}\, ds = \delta(x)$$

viewing the left hand side as the inverse Fourier transform of 1, and then, shifting,

$$\int_{-\infty}^{\infty} e^{2\pi i s x} e^{-2\pi i s t}\, ds = \delta(x - t) \, .$$

And now, shamelessly,

$$\begin{aligned} \mathcal{F}^{-1}\mathcal{F}\varphi(x) &= \int_{-\infty}^{\infty} e^{2\pi i s x} \left(\int_{-\infty}^{\infty} e^{-2\pi i s t} \varphi(t)\, dt \right) ds \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{2\pi i s x} e^{-2\pi i s t} \varphi(t)\, dt\, dt \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{2\pi i s x} e^{-2\pi i s t}\, ds \right) \varphi(t)\, dt = \int_{-\infty}^{\infty} \delta(x - t)\varphi(t)\, dt = \varphi(x) \, . \end{aligned}$$

At least these manipulations didn't lead to a contradiction! I don't mind if you think of the inversion theorem in this way, as long as you know what's behind it, and as long as you don't tell anyone where you saw it.

The convolution theorem Having come this far, we can now derive the convolution theorem for the Fourier transform:

$$\begin{aligned}\langle \mathcal{F}(\psi * T), \varphi\rangle &= \langle \psi * T, \mathcal{F}\varphi\rangle = \langle T, \psi^- * \mathcal{F}\varphi\rangle \\ &= \langle T, \mathcal{F}\mathcal{F}\psi * \mathcal{F}\varphi\rangle \quad \text{(using the identity } \mathcal{F}\mathcal{F}\psi = \psi^-\text{)} \\ &= \langle T, \mathcal{F}(\mathcal{F}\psi \cdot \varphi)\rangle \\ &\qquad \text{(for functions the convolution of the Fourier transfoms is} \\ &\qquad \text{the Fourier transform of the product)} \\ &= \langle \mathcal{F}T, \mathcal{F}\psi \cdot \varphi\rangle \quad \text{(bringing } \mathcal{F} \text{ back to } T\text{)} \\ &= \langle (\mathcal{F}\psi)(\mathcal{F}T), \varphi\rangle \quad \text{(how multiplication by a function is defined)}\end{aligned}$$

Comparing where we started and where we ended up:

$$\langle \mathcal{F}(\psi * T), \varphi\rangle = \langle (\mathcal{F}\psi)(\mathcal{F}T), \varphi\rangle\,.$$

that is,

$$\mathcal{F}(\psi * T) = (\mathcal{F}\psi)(\mathcal{F}T)\,.$$

Done.

One can also show the dual identity:

$$\mathcal{F}(\psi T) = \mathcal{F}\psi * \mathcal{F}T$$

Pay attention to how everything makes sense here and has been previously defined. The product of the Schwartz function ψ and the distribution T is defined, and as a tempered distribution it has a Fourier transform. Since ψ is a Schwartz function so is its Fourier transform $\mathcal{F}\psi$, and hence $\mathcal{F}\psi * \mathcal{F}T$ is defined.

I'll leave it to you to check that the algebraic properties of the convolution continue to hold for distributions, whenever all the quantities are defined.

Note that the convolution identities are consistent with $\psi * \delta = \psi$, and with $\psi\delta = \psi(0)\delta$. The first of these convolution identities says that

$$\mathcal{F}(\psi * \delta) = \mathcal{F}\psi\mathcal{F}\delta = \mathcal{F}\psi\,,$$

since $\mathcal{F}\delta = 1$, and that jibes with $\psi * \delta = \psi$. The other identity is a little more interesting. We have

$$\mathcal{F}(\psi\delta) = \mathcal{F}\psi * \mathcal{F}\delta = \mathcal{F}\psi * 1 = \int_{-\infty}^{\infty} 1 \cdot \mathcal{F}\psi(x)\,dx = \mathcal{F}^{-1}\mathcal{F}\psi(0) = \psi(0)\,.$$

This is consistent with $\mathcal{F}(\psi\delta) = \mathcal{F}(\psi(0)\delta) = \psi(0)\mathcal{F}\delta = \psi(0)$.

Convolution in general I said earlier that convolution can't be defined for every pair of distributions. I want to say a little more about this, but only a little, and give a few examples of cases when it works out OK.

At the beginning of this section we considered, as we always do, what convolution looks like for distributions in the case when the distribution comes from a function. With f playing the role of the distribution and

ψ a Schwartz function we wrote

$$\begin{aligned}\langle \psi * f, \varphi\rangle &= \int_{-\infty}^{\infty} (\psi * f)(x)\varphi(x)\,dx \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \psi(x-y)f(y)\,dy\right)\varphi(x)\,dx \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi(x-y)\varphi(x)f(y)\,dy\,dy \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \psi(x-y)\varphi(x)\,dx\right) f(y)\,dy\,.\end{aligned}$$

At this point we stopped and wrote this as the pairing

$$\langle \psi * f, \varphi\rangle = \langle f, \psi^- * \varphi\rangle$$

so that we could see how to define $\psi * T$ when T is a distribution.

This time, and for a different reason, I want to take the inner integral one step further and write

$$\int_{-\infty}^{\infty} \psi(x-y)\varphi(x)\,dx = \int_{-\infty}^{\infty} \psi(u)\varphi(u+y)\,du \quad \text{(using the substituion } u = x-y).$$

This latter integral is the pairing $\langle \psi(x), \varphi(x+y)\rangle$, where I wrote the variable of the paring (the integration variable) as x and I included it in the notation for pairing to indicate that what results from the pairing is a function y. In fact, what we see from this is that $\langle \psi * f, \varphi\rangle$ can be written as a "nested" pairing, namely

$$\langle \psi * f, \varphi\rangle = \langle f(y), \langle \psi(x), \varphi(x+y)\rangle\rangle$$

where I included the variable y in the outside pairing to keep things straight and to help recall that in the end everything gets integrated away and the result of the nested pairing is a number.

Now, this nested pairing tells us how we might define the convolution $S * T$ of two distributions S and T. It is, with a strong proviso:

> **Convolution of two distributions** If S and T are two distributions then their convolution is the distribution $S * T$ defined by
>
> $$\langle S * T, \varphi\rangle = \langle S(y), \langle T(x), \varphi(x+y)\rangle\rangle$$
>
> provided the right-hand-side exists.
>
> We've written $S(y)$ and $T(x)$ "at points" to keep straight what gets paired with what; $\varphi(x+y)$ makes sense, is a function of x and y, and it's necessary to indicate which variable x or y is getting hooked up with T in the inner pairing and then with S in the outer pairing.

Why the proviso? Because the inner paring $\langle T(x), \varphi(x+y)\rangle$ produces a function of y which *might not be a test function*. Sad, but true. One can state some general conditions under which $S * T$ exists, but this requires a few more definitions and a little more discussion.[21] Enough is enough. It can be dicey, but we'll play a little fast and loose with existence of convolution and applications of the convolution theorem. Tell the rigor police to take the day off.

[21] It inevitably brings in questions about associativity of convolution, which might not hold in general, as it turns out, and, a more detailed treatment of the convolution theorem.

Convolving δ with itself. For various applications you may find yourself wanting to use the identity

$$\delta * \delta = \delta\,.$$

By all means, use it. In this case the convolution makes sense and the formula follows:

$$\begin{aligned}\langle \delta * \delta, \varphi\rangle &= \langle \delta(y), \langle \delta(x), \varphi(x+y)\rangle\rangle \\ &= \langle \delta(y), \varphi(y)\rangle = \varphi(0) = \langle \delta, \varphi\rangle\,.\end{aligned}$$

A little more generally, we have

$$\delta_a * \delta_b = \delta_{a+b}\,,$$

a nice formula! We can derive this easily from the definition:

$$\begin{aligned}\langle \delta_a * \delta_b, \varphi\rangle &= \langle \delta_a(y), \langle \delta_b(x), \varphi(x+y)\rangle\rangle \\ &= \langle \delta_a(y), \varphi(b+y)\rangle = \varphi(b+a) = \langle \delta_{a+b}, \varphi\rangle\,.\end{aligned}$$

It would be more common to write this identity as

$$\delta(x-a) * \delta(x-b) = \delta(x-a-b)\,.$$

In this notation, here's the down and dirty version of what we just did (so you know how it looks):

$$\begin{aligned}\delta(x-a) * \delta(x-b) &= \int_{-\infty}^{\infty} \delta(y-a)\delta(x-b-y)\,dy \\ &= \int_{-\infty}^{\infty} \delta(u-b-a)\delta(x-u)\,du \quad (\text{using } u = b+y) \\ &= \delta(x-b-a) \quad (\text{by the sifting property of } \delta).\end{aligned}$$

Convolution really is a "smoothing operation" (most of the time) I want to say a little more about general properties of convolution (first for functions) and why convolution is a smoothing operation. In fact, it's often taken as a maxim when working with convolutions that:

- The function $f * g$ has the good properties of f and g.

This maxim is put to use through a result called the *derivative theorem for convolutions*:

$$(f * g)'(x) = (f * g')(x) = (f' * g)(x)\,.$$

On the left hand side is the derivative of the convolution, while on the right hand side we put the derivative on whichever factor *has* a derivative.

We allow ourselves to differentiate under the integral sign — sometimes a delicate business, but set that aside — and the derivation is easy. If g is differentiable, then

$$\begin{aligned}(f * g)'(x) &= \frac{d}{dx}\int_{-\infty}^{\infty} f(u)g(x-u)\,du \\ &= \int_{-\infty}^{\infty} f(u)\frac{d}{dx}g(x-u)\,du = \int_{-\infty}^{\infty} f(u)g'(x-u)\,du = (f * g')(x)\end{aligned}$$

The second formula follows similarly if f is differentiable.

The importance of this is that the convolution of two functions may have more smoothness than the individual factors. We've seen one example of this already, where it's not smoothness but continuity that's

improved. Remember $\Pi * \Pi = \Lambda$; the convolution of the rectangle function with itself is the triangle function. The rectangle function is not continuous — it has jump discontinuities at $x = \pm 1/2$ — but the convolved function *is* continuous.[22] We also saw that repeated convolution of a function with itself will lead to a Gaussian.

The derivative theorem is saying: If f is rough, but g is smooth then $f * g$ will be smoother than f because we can differentiate the convolution *by putting the derivative on* g. We can also compute higher order derivatives in the same way. If g is n-times differentiable then

$$(f * g)^{(n)}(x) = (f * g^{(n)})(x)\,.$$

Thus convolving a rough function f with an n-times differentiable function g produces an n-times differentiable function $f * g$. It is in this sense that convolution is a "smoothing" operation.

The technique of smoothing by convolution can also be applied to distributions. There one works with $\psi * T$ where ψ is, for example, a Schwartz function. Using the family of Gaussians $g_t(x) = (1/\sqrt{2\pi t})e^{-x^2/2t}$ to form $g_t * T$ produces the so-called *regularization* of T. This is the basis of the theorem on approximating a general distribution by a sequence of distributions that come from Schwartz functions.

The distribution δ is the breakeven point for smoothing by convolution — it doesn't do any smoothing, it leaves the function alone, as in

$$\delta * f = f\,.$$

Going further, convolving a differentiable function with derivatives of δ produces derivatives of the function, for example,

$$\delta' * f = f'\,.$$

You can derive this from scratch using the definition of the derivative of a distribution and the definition of convolution, or you can also think of

$$\delta' * f = \delta * f' = f'\,.$$

(Careful here: This is δ' *convolved* with f, not δ' *paired* with f.) A similar result holds for higher derivatives:

$$\delta^{(n)} * f = f^{(n)}\,.$$

Sometimes one thinks of taking a derivative as making a function less smooth, so counterbalancing the maxim that convolution is a smoothing operation, one should add that convolving with derivatives of δ may roughen a function up.

4.18 δ Hard at Work

We've put a lot of effort into general theory and now it's time to see a few applications. They range from finishing some work on filters, to optics and diffraction, to X-ray crystallography. The latter will even lead us toward the sampling theorem. The one thing all these examples have in common is their use of δ's.

The main properties of δ we'll need, along with its Fourier transform, are what happens with convolution with a function φ and with multiplication by a function φ:

$$\delta * \varphi = \varphi \quad \text{and} \quad \varphi\delta = \varphi(0)\delta\,.$$

[22] In fact, it's a general result that if f and g are merely integrable then $f * g$ is already continuous.

We'll tend to "write the variables" in this section, so these identities appear as

$$\int_{-\infty}^{\infty} \delta(x-y)\varphi(y)\,dy = \varphi(x) \quad \text{and} \quad \varphi(x)\delta(x) = \varphi(0)\delta(x)\,.$$

(I can live with it.) There are useful variations of these formulas for a shifted δ:

$$\delta(x-b) * \varphi(x) = \varphi(x-b)$$
$$\delta(x-b)\varphi(x) = \varphi(b)\delta(x-b)$$

We also need to recall the Fourier transform for a scaled rect:

$$\mathcal{F}\Pi_a(x) = \mathcal{F}\Pi(x/a) = a\,\mathrm{sinc}\,a\,.$$

4.18.1 Filters, redux

One of our first applications of convolution was to set up and study some simple filters. Let's recall the terminology and some work left undone; see Section3.4. The input $v(t)$ and the output $w(t)$ are related via convolution with the *impulse response* $h(t)$:

$$w(t) = (h * v)(t)\,.$$

(We're not quite ready to explain why h is called the impulse response.) The action of the filter is easier to understand in the frequency domain, for there, by the convolution theorem, it acts by multiplication

$$W(s) = H(s)V(s)$$

where

$$W = \mathcal{F}w, \quad H = \mathcal{F}h, \quad \text{and} \quad V = \mathcal{F}v\,.$$

$H(s)$ is called the *transfer function.*

The simplest example, out of which the others can be built, is the low-pass filter with transfer function

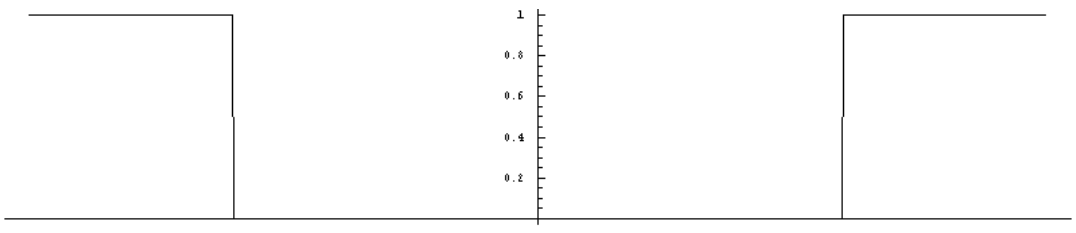
$$\mathrm{Low}(s) = \Pi_{2\nu_c}(s) = \Pi\Big(\frac{s}{2\nu_c}\Big) = \begin{cases} 1 & |s| < \nu_c \\ 0 & |s| \geq \nu_c \end{cases}$$

The impulse response is

$$\mathrm{low}(t) = 2\nu_c\,\mathrm{sinc}(2\nu_c t)$$

a scaled sinc function.[23]

High-pass filter Earlier we saw the graph of the transfer function for an ideal high pass filter:



[23] What do you think of this convention of using "Low" for the transfer function (uppercase) and "low" for the impulse response (lower case)? Send me your votes.

and a formula for the transfer function

$$\text{High}(s) = 1 - \text{Low}(s) = 1 - \Pi_{2\nu_c}(s)$$

where ν_c is the cut-off frequency. At the time we couldn't finish the analysis because we didn't have δ. Now we do. The impulse response is

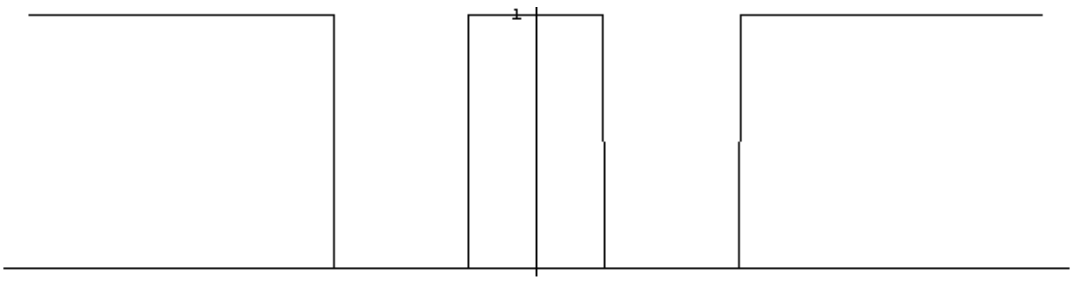
$$\text{high}(t) = \delta(t) - 2\nu_c \,\text{sinc}(2\nu_c t)\,.$$

For an input $v(t)$ the output is then

$$\begin{aligned} w(t) &= (\text{high} * v)(t) \\ &= \big(\delta(t) - 2\nu_c \,\text{sinc}(2\nu_c t)\big) * v(t) \\ &= v(t) - 2\nu_c \int_{-\infty}^{\infty} \text{sinc}(2\nu_c(t-s))v(s)\,ds\,. \end{aligned}$$

The role of the convolution property of δ in this formula shows us that the high pass filter literally subtracts part of the signal away.

Notch filter The transfer function for the notch filter is just $1 -$ (transfer function for band pass filter) and it looks like this:



Frequencies in the "notch" are filtered out and all others are passed through unchanged. Suppose that the notches are centered at $\pm\nu_0$ and that they are ν_c wide. The formula for the transfer function, in terms of transfer function for the low-pass filter with cutoff frequency ν_c, is

$$\text{Notch}(s) = 1 - \big(\text{Low}(s-\nu_0) + \text{Low}(s+\nu_0)\big)\,.$$

For the impulse response we obtain

$$\begin{aligned} \text{notch}(t) &= \delta(t) - (e^{-2\pi i\nu_0 t}\text{low}(t) + e^{2\pi i\nu_0 t}\text{low}(t)) \\ &= \delta(t) - 4\nu_c \cos(2\pi\nu_0 t)\,\text{sinc}(2\nu_c t)\,. \end{aligned}$$

Thus

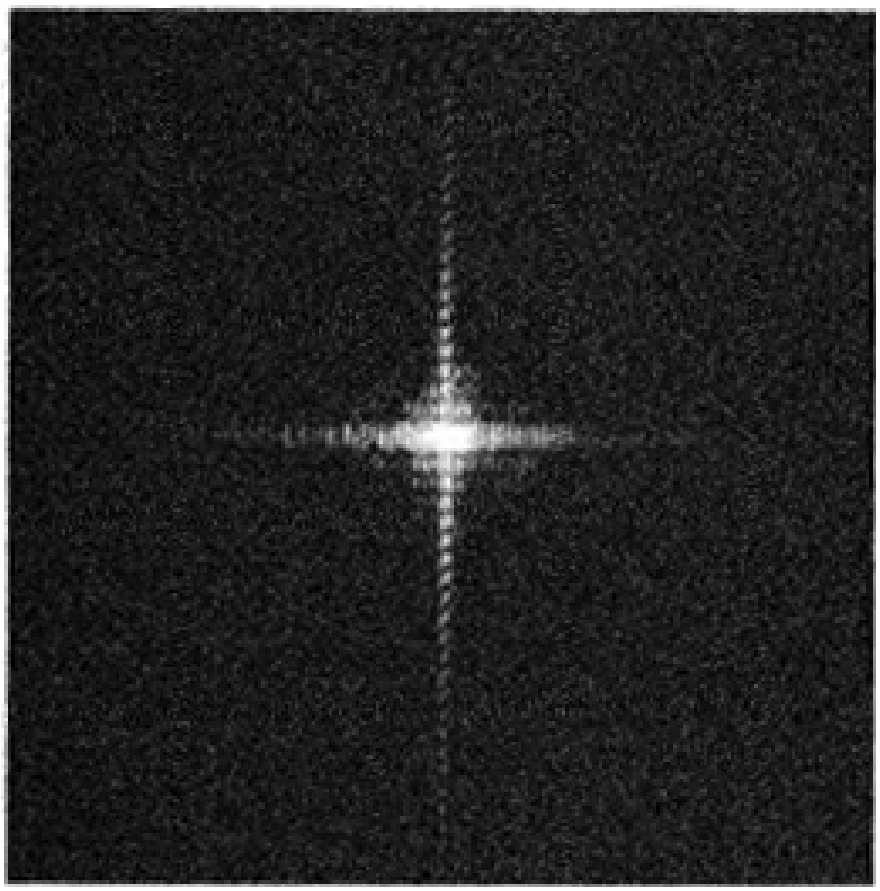
$$\begin{aligned} w(t) &= (\delta(t) - 4\nu_c \cos(2\pi\nu_0 t)\,\text{sinc}(2\nu_c t)) * v(t) \\ &= v(t) - 4\nu_c \int_{-\infty}^{\infty} \cos(2\pi\nu_0(t-s))\ \text{sinc}(2\nu_c(t-s))\,v(s)\,ds\,, \end{aligned}$$

and again we see the notch filter subtracting away part of the signal.

4.18.2 Diffraction: The sinc function, live and in pure color

Some of the most interesting applications of the Fourier transform are in the field of optics, understood broadly to include most of the electromagnetic spectrum in its purview. An excellent book on the subject is *Fourier Optics*, by Stanford's own J. W. Goodman — highly recommended.

The fundamental phenomenon associated with the wave theory of light is *diffraction* or *interference*. Sommerfeld says that diffraction is "any deviation of light rays from rectilinear paths which cannot be interpreted as reflection or refraction." Very helpful. Is there a difference between diffraction and interference? In his *Lectures on Physics*, Feynman says "No one has ever been able to define the difference between interference and diffraction satisfactorily. It is just a question of usage, and there is no specific, important physical difference between them." He does go on to say that "interference" is usually associated with patterns caused by a few radiating sources, like two, while "diffraction" is due to many sources. Whatever the definition, or nondefinition, you probably know what the picture is:



Such pictures, most notably the "Two Slits" experiments of Thomas Young (1773–1829), which we'll analyze, below, were crucial in tipping the balance away from Newton's corpuscular theory to the wave theory propounded by Christiaan Huygens (1629–1695). The shock of the diffraction patterns when first seen was that light + light could be dark. Yet the experiments were easy to perform. Spoke Young in 1803 to the Royal Society: "The experiments I am about to relate ... may be repeated with great ease, whenever the sun shines, and without any other apparatus than is at hand to every one."[24]

[24] Young also did important work in studying Egyptian hieroglyphics, completely translating a section of the Rosetta Stone.

We are thus taking sides in the grand battle between the armies of "light is a wave" and those of "light is a particle". It may be that light is truly like nothing you've ever seen before, but for this discussion it's a wave. Moreover, jumping ahead to Maxwell, we assume that light is an electromagnetic wave, and for our discussion we assume further that the light in our problems is:

- Monochromatic
  - Meaning that the periodicity in time is a single frequency, so described by a simple sinusoid.
- Linearly polarized
  - Meaning that the electric field vector stays in a plane as the wave moves. (Hence so too does the magnetic field vector.)

With this, the diffraction problem can be stated as follows:

> Light — an electromagnetic wave — is incident on an (opaque) screen with one or more apertures (transparent openings) of various shapes. What is the intensity of the light on a screen some distance from the diffracting screen?

We're going to consider only a case where the analysis is fairly straightforward, the *Fraunhofer approximation*, or *Fraunhofer diffraction*. This involves a number of simplifying assumptions, but the results are used widely. Before we embark on the analysis let me point out that reasoning very similar to what we'll do here is used to understand the radiation patterns of antennas. For this take on the subject see Bracewell, Chapter 15.

Light waves We can describe the properties of light that satisfy the above assumptions by a *scalar*-valued function of time and position. We're going to discuss "scalar" diffraction theory, while more sophisticated treatments handle the "vector" theory. The function is the *magnitude* of the electric field vector, say a function of the form

$$u(x, y, z, t) = a(x, y, z)\cos(2\pi\nu t - \phi(x, y, z))$$

Here, $a(x, y, z)$ is the amplitude as a function only of position in space, ν is the (single) frequency, and $\phi(x, y, z)$ is the phase at $t = 0$, also as a function only of position.[25]

The equation

$$\phi(x, y, z) = \text{constant}$$

describes a surface in space. At a fixed time, all the points on such a surface have the same phase, by definition, or we might say equivalently that the traveling wave reaches all points of such a surface $\phi(x, y, z) =$ constant at the same time. Thus any one of the surfaces $\phi(x, y, z) =$ constant is called a *wavefront*. In general, the wave propagates through space in a direction normal to the wavefronts.

The function $u(x, y, z, t)$ satisfies the 3-dimensional wave equation

$$\Delta u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

where

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

[25] It's also common to refer to the whole argument of the cosine, $2\pi\nu t - \phi$, simply as "the phase".

is the Laplacian and c is the speed of light in vacuum. For many problems it's helpful to separate the spatial behavior of the wave from its temporal behavior and to introduce the *complex amplitude*, defined to be

$$u(x, y, z) = a(x, y, z)e^{i\phi(x,y,z)}.$$

Then we get the time-dependent function $u(x, y, z, t)$ as

$$u(x, y, z, t) = \mathrm{Re}\left(\overline{u(x, y, z)}e^{2\pi i\nu t}\right).$$

If we know $u(x, y, z)$ we can get $u(x, y, z, t)$. It turns out that $u(x, y, z)$ satisfies the differential equation

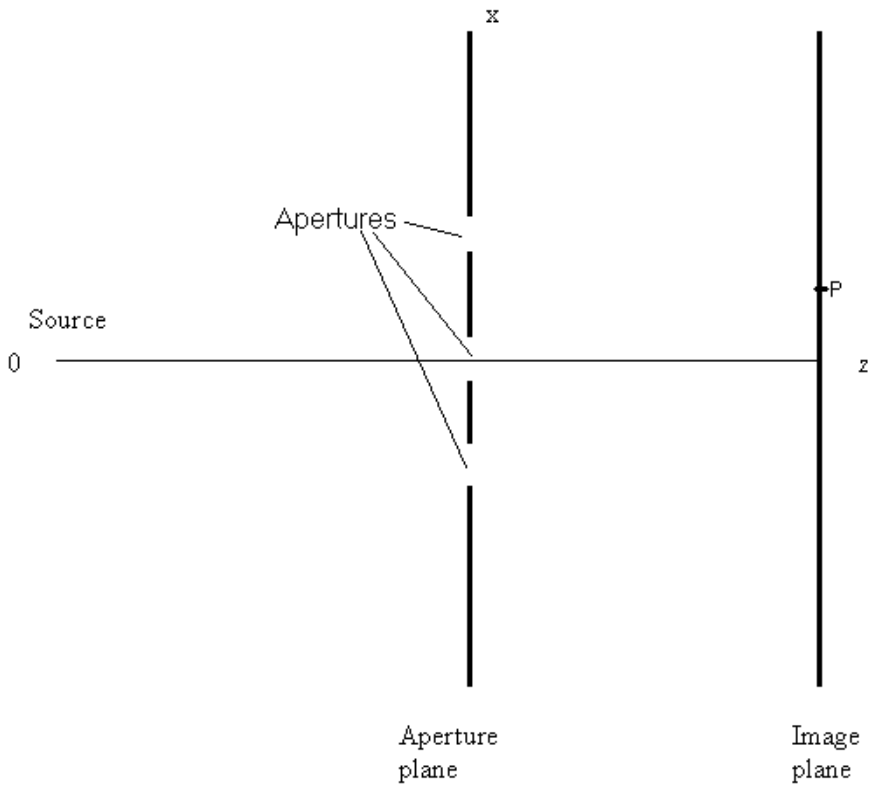
$$\Delta u(x, y, z) + k^2 u(x, y, z) = 0$$

where $k = 2\pi\nu/c$. This is called the Helmholtz equation, and the fact that it is time independent makes it simpler than the wave equation.

Fraunhofer diffraction We take a sideways view of the situation. Light is coming from a source at a point O and hits a plane S. We assume that the source is so far away from S that the magnitude of the electric field associated with the light is constant on S and has constant phase, i.e., S is a wavefront and we have what is called a *plane wave field*. Let's say the frequency is ν and the wavelength is λ. Recall that $c = \lambda\nu$, where c is the speed of light. (We're also supposing that the medium the light is passing through is isotropic, meaning that the light is traveling at velocity c in any direction, so there are no special effects from going through different flavors of jello or something like that.)

Set up coordinates so that the z-axis is perpendicular to S and the x-axis lies in S, perpendicular to the z-axis. (In most diagrams it is traditional to have the z-axis be horizontal and the x-axis be vertical.)

In S we have one or more rectangular apertures. We allow the length of the side of the aperture along the x-axis to vary, but we assume that the other side (perpendicular to the plane of the diagram) has length 1. A large distance from S is another parallel plane. Call this the image plane.



The diffraction problem is:

- What is the electric field at a point P in the image plane?

The derivation I'm going to give to answer this question is not as detailed as is possible (for details see Goodman's book), but we'll get the correct form of the answer and the point is to see how the Fourier transform enters.

The basis for analyzing diffraction is *Huygens' principle* which states, roughly, that the apertures on S (which is a wavefront of the original source) may be regarded as (secondary) sources, and the field at P is the sum (integral) of the fields coming from these sources on S. Putting in a little more symbolism, if E_0 is the strength of the electric field on S then an aperture of area dS is a source of strength $dE = E_0\,dS$. At a distance r from this aperture the field strength is $dE'' = E_0\,dS/r$, and we get the electric field at this distance by integrating over the apertures the elements dE'', "each with its proper phase". Let's look more carefully at the phase.

The wave leaves a point on an aperture in S, a new source, and arrives at P sometime later. Waves from different points on S will arrive at P at different times, and hence there will be a phase difference between the arriving waves. They also drop off in amplitude like one over the distance to P, and so by different amounts, but if, as we'll later assume, the size of the apertures on S are small compared to the distance between S and the image plane then this is not as significant as the phase differences. Light is moving so fast that even a small differences between locations of secondary point sources on S may lead to significant differences in the phases when the waves reach P.

The phase on S is constant and we might as well assume that it's zero. Then we write the electric field on

S in complex form as

$$E = E_0 e^{2\pi i \nu t}$$

where E_0 is constant and ν is the frequency of the light. Suppose P is at a distance r from a point x on S. Then the phase change from x to P depends on how big r is compared to the wavelength λ — how many wavelengths (or fractions of a wavelength) the wave goes through in going a distance r from x to P. This is $2\pi(r/\lambda)$. To see this, the wave travels a distance r in a time r/c seconds, and in that time it goes through $\nu(r/c)$ cycles. Using $c = \lambda\nu$ that's $\nu r/c = r/\lambda$. This is $2\pi r/\lambda$ radians, and that's the phase shift.

Take a thin slice of width dx at a height x above the origin of an aperture on S. Then the field at P due to this source is, on account of the phase change,

$$dE = E_0 e^{2\pi i \nu t} e^{2\pi i r/\lambda}\, dx\,.$$

The total field at P is

$$E = \int_{\text{apertures}} E_0 e^{2\pi i \nu t} e^{2\pi i r/\lambda}\, dx = E_0 e^{2\pi i \nu t} \int_{\text{apertures}} e^{2\pi i r/\lambda}\, dx$$

There's a Fourier transform coming, but we're not there yet.

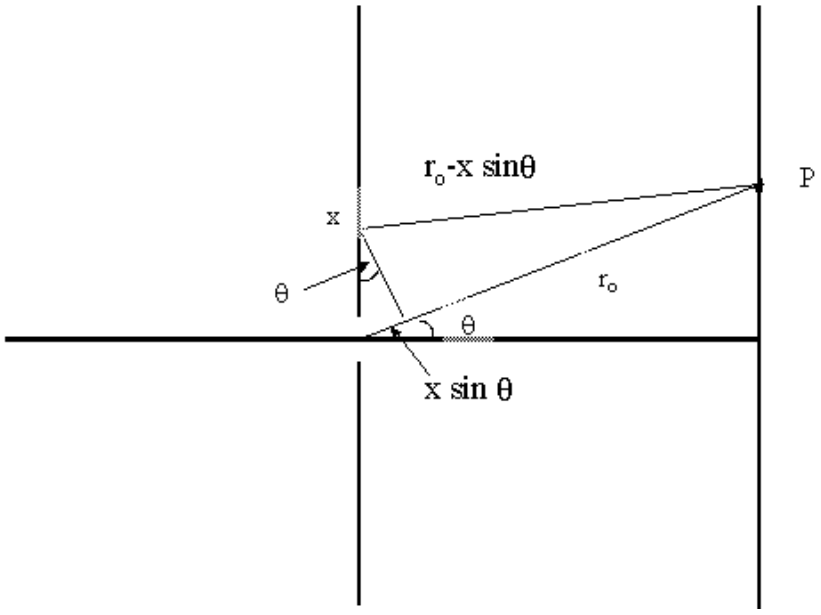
The key assumption that is now made in this argument is to suppose that

$$r \gg x\,,$$

that is, the distance between the plane S and the image plane is much greater than any x in any aperture, in particular r is large compared to any aperture size. This assumption is what makes this *Fraunhofer diffraction*; it's also referred to as *far field* diffraction. With this assumption we have, approximately,

$$r = r_0 - x\sin\theta\,,$$

where r_0 is the distance between the origin of S to P and θ is the angle between the z-axis and P.



Plug this into the formula for E:

$$E = E_0 e^{2\pi i \nu t} e^{2\pi i r_0/\lambda} \int_{\text{apertures}} e^{-2\pi i x \sin\theta/\lambda}\, dx$$

Drop that constant out front — as you'll see, it won't be important for the rest of our considerations.

We describe the apertures on S by a function $A(x)$, which is zero most of the time (the opaque parts of S) and 1 some of the time (apertures). Thus we can write

$$E \propto \int_{-\infty}^{\infty} A(x) e^{-2\pi i x \sin\theta/\lambda}\, dx$$

It's common to introduce the variable

$$p = \frac{\sin\theta}{\lambda}$$

and hence to write

$$E \propto \int_{-\infty}^{\infty} A(x) e^{-2\pi i p x}\, dx\,.$$

There you have it. With these approximations (the *Fraunhofer approximations*) the electric field (up to a multiplicative constant) is the Fourier transform of the aperture! Note that the variables in the formula are x, a spatial variable, and $p = \sin\theta/\lambda$, in terms of an angle θ. It's the θ that's important, and one always speaks of diffraction "through an angle."

Diffraction by a single slit Take the case of a single rectangular slit of width a, thus described by $A(x) = \Pi_a(x)$. Then the field at P is

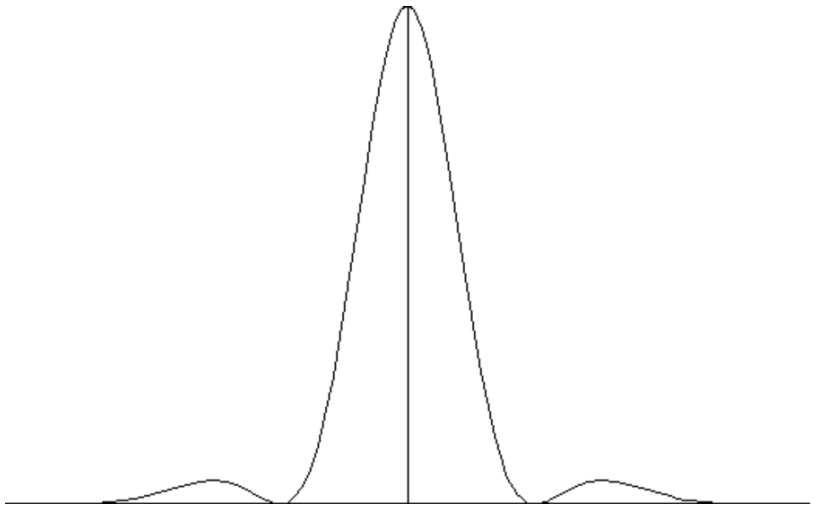
$$E \propto a\,\text{sinc}\, ap = a\,\text{sinc}\left(\frac{a\sin\theta}{\lambda}\right)\,.$$

Now, the *intensity* of the light, which is what we see and what photodetectors register, is proportional to the energy of E, i.e., to $|E|^2$. (This is why we dropped the factors $E_0 e^{2\pi i \nu t} e^{2\pi i r_0/\lambda}$ multiplying the integral. They have magnitude 1.) So the diffraction pattern you see from a single slit, those alternating bright and dark bands, is

$$\text{intensity} = a^2 \text{sinc}^2\left(\frac{a\sin\theta}{\lambda}\right)\,.$$

Pretty good. The sinc function, or at least its square, live and in color. Just as promised.

We've seen a plot of sinc^2 before, and you may very well have seen it, without knowing it, as a plot of the intensity from a single slit diffraction experiment. Here's a plot for $a = 2$, $\lambda = 1$ and $-\pi/2 \le \theta \le \pi/2$:



Young's experiment As mentioned earlier, Thomas Young observed diffraction caused by light passing through two slits. To analyze his experiment using what we've derived we need an expression for the apertures that's convenient for taking the Fourier transform.

Suppose we have two slits, each of width a, centers separated by a distance b. We can model the aperture function by the sum of two shifted rect functions,

$$A(x) = \Pi_a(x - b/2) + \Pi_a(x + b/2)\,.$$

(Like the transfer function of a bandpass filter.) That's fine, but we can also shift the Π_a's by convolving with shifted δ's, as in

$$\begin{aligned} A(x) &= \delta(x - b/2) * \Pi_a(x) + \delta(x + b/2) * \Pi_a(x) \\ &= (\delta(x - b/2) + \delta(x + b/2)) * \Pi_a(x)\,, \end{aligned}$$

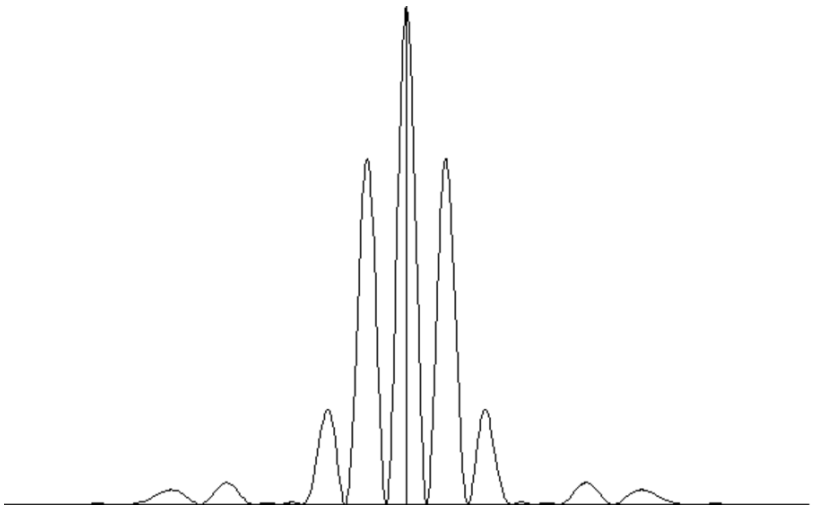
and the advantage of writing $A(x)$ in this way is that the convolution theorem applies to help in computing the Fourier transform. Namely,

$$\begin{aligned} E(p) &\propto (2\cos \pi b p)(a\,\mathrm{sinc}\, ap) \\ &= 2a\cos\left(\frac{\pi b \sin\theta}{\lambda}\right)\mathrm{sinc}\left(\frac{a\sin\theta}{\lambda}\right) \end{aligned}$$

Young saw the intensity, and so would we, which is then

$$\text{intensity} = 4a^2\cos^2\left(\frac{\pi b \sin\theta}{\lambda}\right)\mathrm{sinc}^2\left(\frac{a\sin\theta}{\lambda}\right)$$

Here's a plot for $a = 2$, $b = 6$, $\lambda = 1$ for $-\pi/2 \le \theta \le \pi/2$:



This is quite different from the diffraction pattern for one slit.

Diffraction by two point-sources Say we have two point-sources — the apertures — and that they are at a distance b apart. In this case we can model the apertures by a pair of δ-functions:

$$A(x) = \delta(x - b/2) + \delta(x + b/2)\,.$$

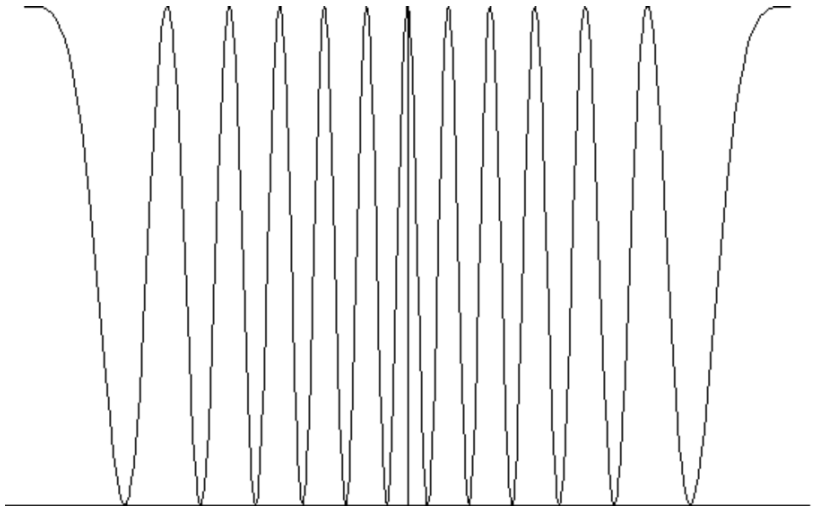
Taking the Fourier transform then gives

$$E(p) \propto 2\cos \pi bp = 2\cos\left(\frac{\pi b \sin\theta}{\lambda}\right) .$$

and the intensity as the square magnitude:

$$\text{intensity} = 4\cos^2\left(\frac{\pi b \sin\theta}{\lambda}\right) .$$

Here's a plot of this for $b = 6$, $\lambda = 1$ for $-\pi/2 \le \theta \le \pi/2$:



Incidentally, two radiating point sources covers the case of two antennas "transmitting in phase from a single oscillator".

An optical interpretation of $\mathcal{F}\delta = 1$ What if we had light radiating from a single point source? What would the pattern be on the image plane in this circumstance? For a single point source there is no diffraction (a *point source*, not a circular aperture of some definite radius) and the image plane is illuminated uniformly. Thus the strength of the field is constant on the image plane. On the other hand, if we regard the aperture as δ and plug into the formula we have the Fourier transform of δ,

$$E \propto \int_{-\infty}^{\infty} \delta(x) e^{-2\pi i p x}\, dx$$

This gives a physical reason why the Fourier transform of δ should be constant (if not 1).

Also note what happens to the intensity as $b \to 0$ of the diffraction due to two point sources at a distance b. Physically, we have a single point source (of strength 2) and the formula gives

$$\text{intensity} = 4\cos^2\left(\frac{\pi b \sin\theta}{\lambda}\right) \to 4 .$$

4.19 Appendix: The Riemann-Lebesgue lemma

The result of this section, a version of what is generally referred to as the Riemann-Lebesgue lemma, is:

- If $\int_{-\infty}^{\infty} |f(t)|\,dt < \infty$ then $|\mathcal{F}f(s)| \to 0$ as $s \to \pm\infty$.

We showed that $\mathcal{F}f$ is continuous given that f is integrable; that was pretty easy. It's a much stronger statement to say that $\mathcal{F}f$ tends to zero at infinity.

We'll derive the result from another important fact, which we won't prove and which you may find interesting. It says that a function in $L^1(\mathbf{R})$ can be approximated in the $L^1(\mathbf{R})$ norm by functions in $\mathcal{S}$, the rapidly decreasing functions. Now, functions in $L^1(\mathbf{R})$ can be quite wild and functions in $\mathcal{S}$ are about as nice as you can imagine so this is quite a useful statement, not to say astonishing. We'll use it in the following way. Let f be in $L^1(\mathbf{R})$ and choose a sequence of functions f_n in $\mathcal{S}$ so that

$$\|f - f_n\|_1 = \int_{-\infty}^{\infty} |f(t) - f_n(t)|\,dt < \tfrac{1}{n}\,.$$

We then use an earlier result that the Fourier transform of a function is bounded by the $L^1(\mathbf{R})$-norm of the function, so that

$$|\mathcal{F}f(s) - \mathcal{F}f_n(s)| \le \|f - f_n\|_1 < \tfrac{1}{n}\,.$$

Therefore

$$|\mathcal{F}f(s)| \le |\mathcal{F}f_n(s)| + \tfrac{1}{n}\,.$$

But since f_n is rapidly decreasing, so is $\mathcal{F}f_n$, and hence $\mathcal{F}f_n(s)$ tends to zero as $s \to \pm\infty$. Thus

$$\lim_{s\to\infty} |\mathcal{F}f(s)| < \tfrac{1}{n}$$

for all $n \ge 1$. Now let $n \to \infty$.

4.20 Appendix: Smooth Windows

One way of cutting off a function is simply to multiply by a rectangle function. For example, we can cut a function $f(x)$ off outside the interval $[-n/2, +n/2]$ via

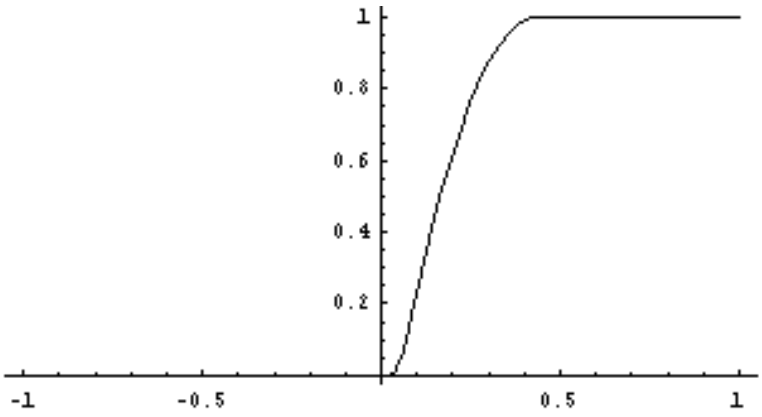
$$\Pi(x/n)f(x) = \begin{cases} f(x) & |x| < n/2 \\ 0 & |x| \ge n/2 \end{cases}$$

We can imagine letting $n \to \infty$ and in this way approximate $f(x)$ by functions which are nonzero only in a finite interval. The problem with this particular way of cutting off is that we may introduce discontinuities in the cut-off.

There are smooth ways of bringing a function down to zero. Here's a model for doing this, sort of a smoothed version of the rectangle function. It's amazing that you can write it down, and if any of you are ever looking for smooth windows here's one way to get them. The function

$$g(x) = \begin{cases} 0 & x \le 0 \\ \exp\left(-\left(\dfrac{1}{2x}\right)\exp\left(\dfrac{1}{2x-1}\right)\right) & 0 < x < \frac{1}{2} \\ 1 & x \ge \frac{1}{2} \end{cases}$$

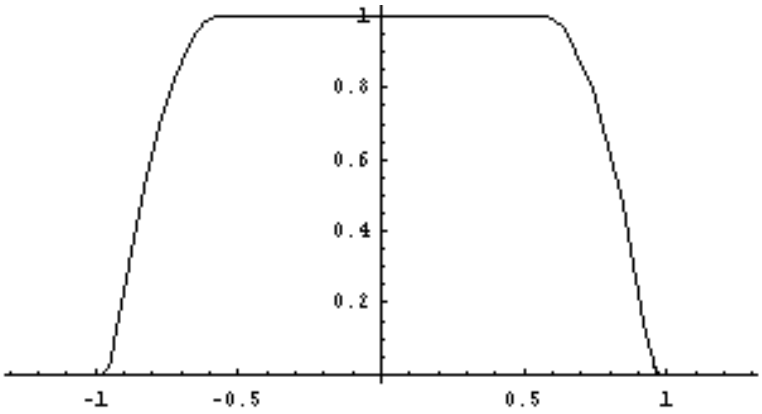
is a smooth function, i.e., infinitely differentiable! It goes from the constant value 0 to the constant value 1 smoothly on the interval from 0 to 1/2.



Then the function $g(1+x)$ goes up smoothly from 0 to 1 over the interval from -1 to $-1/2$ and the function $g(1-x)$ goes down smoothly from 1 to 0 over the interval from $1/2$ to 1. Their product

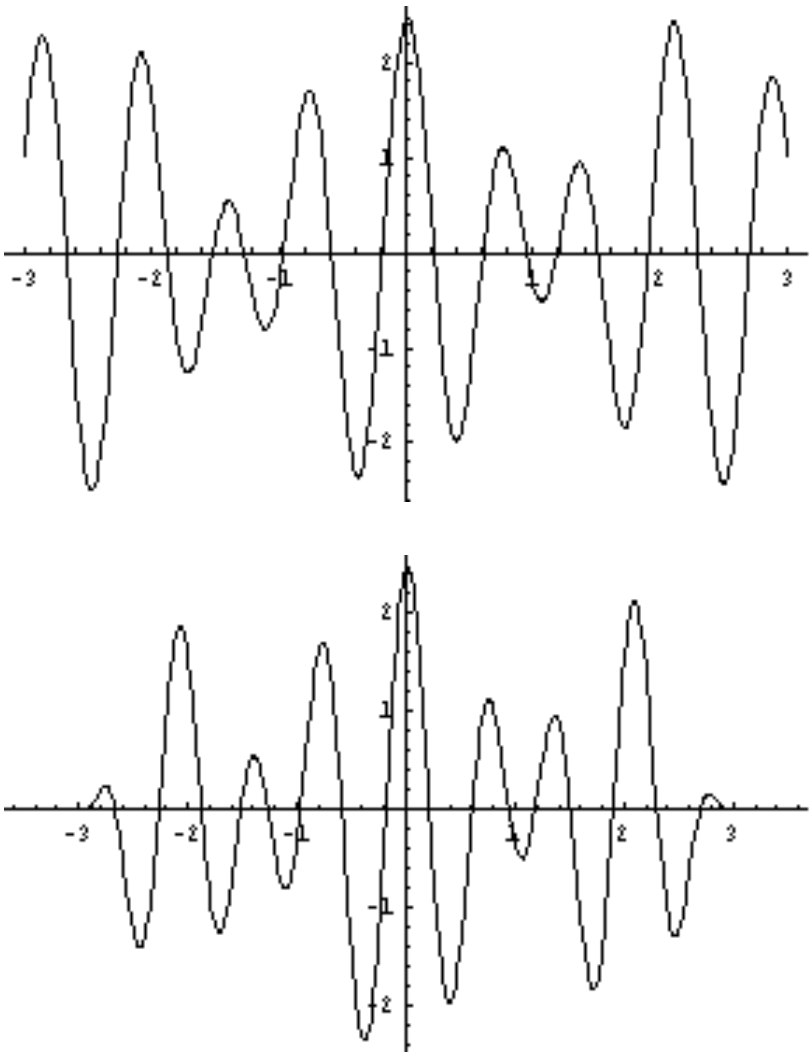
$$c(x) = g(1+x)g(1-x)$$

is 1 on the interval from $-1/2$ to $1/2$, goes down smoothly to 0 between $\pm 1/2$ and ± 1, and is zero for $x \leq -1$ and for $x \geq 1$. Here's the graph of $c(x)$, the one we had earlier in the notes.

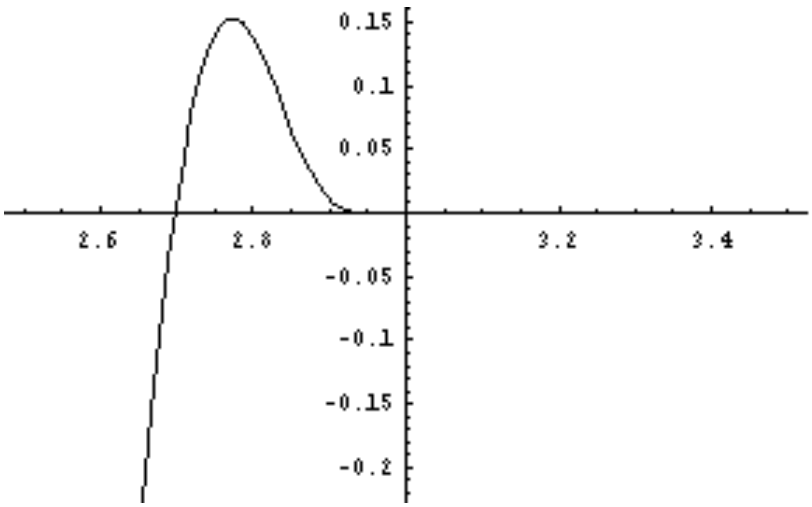


The function $c(x)$ is a smoothed rectangle function. By scaling, say to $c_n(x) = c(x/n)$, we can smoothly cut off a function to be zero outside a given interval $[-n/2, n/2]$ via $c_n(x)f(x)$. As we let the interval become larger and larger we see we are approximating a general (smooth) infinite function by a sequence of smooth functions that are zero beyond a certain point.

For example, here's a function and its smooth window (to be identically 0 after ± 3):



Here's a blow-up near the endpoint 3 so you can see that it really is coming into zero smoothly.



4.21 Appendix: $1/x$ as a Principal Value Distribution

We want to look at the formula

$$\frac{d}{dx}\ln|x| = \frac{1}{x}\,.$$

from a distributional point of view. First, does $\ln|x|$ — much less its derivative — even make sense as a distribution? It has an infinite discontinuity at the origin, so there's a question about the existence of the integral

$$\langle \ln|x|, \varphi\rangle = \int_{-\infty}^{\infty} \ln|x|\,\varphi(x)\,dx$$

when φ is a Schwartz function. Put another way, $\ln|x|$ can be defined as a distribution if we can define a pairing with test functions (that satisfies the linearity and continuity requirements). Is the pairing by simple integration, as above? Yes, but it takes some work.

The problem is at the origin not at $\pm\infty$, since $\varphi(x)\ln|x|$ will go down to zero fast enough to make the tails of the integral converge. To analyze the integral near zero, let me remind you of some general facts: When a function $f(x)$ has a discontinuity (infinite or not) at a finite point, say at 0, then

$$\int_a^b f(x)\,dx, \quad a<0,\ b>0$$

is an *improper integral* and has to be defined via a limit

$$\lim_{\epsilon_1\to 0}\int_a^{\epsilon_1} f(x)\,dx + \lim_{\epsilon_2\to 0}\int_{\epsilon_2}^{b} f(x)\,dx$$

with ϵ_1 and ϵ_2 tending to zero separately. If both limits exist then so does the integral — this is the definition, i.e., you first have to take the separate limits, then add the results. If neither or only one of the limits exists then the integral does not exist.

What's the situation for $\int_{-\infty}^{\infty}\ln|x|\,\varphi(x)\,dx$? We'll need to know two facts:

1. An antiderivative of $\ln x$ is $x\ln x - x$.
2. $\lim_{|x|\to 0}|x|^k\ln|x| = 0$ for any $k>0$.

   This is so because while $\ln|x|$ is tending to $-\infty$ as $x\to 0$, it's doing so slowly enough that multiplying it by any positive power of x will force the product to go to zero. (You can check this with L'Hospital's rule, for instance.)

Now write

$$\int_{-\infty}^{-\epsilon_1}\ln(-x)\varphi(x)\,dx + \int_{\epsilon_2}^{\infty}\ln x\,\varphi(x)\,dx =$$
$$\int_{-\infty}^{-1}\ln(-x)\varphi(x)\,dx + \int_{-1}^{-\epsilon_1}\ln(-x)\varphi(x)\,dx + \int_{\epsilon_2}^{1}\ln|x|\varphi(x)\,dx + \int_{1}^{\infty}\ln|x|\varphi(x)\,dx\,.$$

To repeat what I said earlier, the integrals going off to $\pm\infty$ aren't a problem and only the second and third integrals need work. For these, use a Taylor approximation to $\varphi(x)$, writing $\varphi(x) = \varphi(0) + O(x)$, where

$O(x)$ is a term of order x for $|x|$ small. Then

$$\begin{aligned}
&\int_{-1}^{-\epsilon_1} \ln(-x)(\varphi(0)+O(x))\,dx + \int_{\epsilon_2}^{1} \ln x(\varphi(0)+O(x))\,dx \\
&\qquad = \varphi(0)\left(\int_{-1}^{-\epsilon_1} \ln(-x)\,dx + \int_{\epsilon_2}^{1} \ln x\,dx\right) + \int_{-1}^{-\epsilon_1} O(x)\ln(-x)\,dx + \int_{\epsilon_2}^{1} O(x)\ln x\,dx \\
&\qquad = \varphi(0)\left(\int_{\epsilon_1}^{1} \ln x\,dx + \int_{\epsilon_2}^{1} \ln x\,dx\right) + \int_{-1}^{-\epsilon_1} O(x)\ln(-x)\,dx + \int_{\epsilon_2}^{1} O(x)\ln x\,dx
\end{aligned}$$

We want to let $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$. You can now use Point 1, above, to check that the limits of the first pair of integrals exist, and by Point 2 the second pair of integrals aren't even improper. We've shown that

$$\int_{-\infty}^{\infty} \ln|x|\,\varphi(x)\,dx$$

exists, hence $\ln|x|$ is a distribution. (The pairing, by integration, is obviously linear. We haven't checked continuity, but we never check continuity.)

Now the derivative of $\ln|x|$ is $1/x$, but how does the latter define a distribution? This is trickier. We would have to understand the pairing as a limit

$$\langle \frac{1}{x}, \varphi\rangle = \lim_{\epsilon_1\to 0}\int_{-\infty}^{\epsilon_1} \frac{\varphi(x)}{x}\,dx + \lim_{\epsilon_2\to 0}\int_{\epsilon_2}^{\infty} \frac{\varphi(x)}{x}\,dx$$

and this limit *need not exist*. What *is* true is that the symmetric sum

$$\int_{-\infty}^{-\epsilon} \frac{\varphi(x)}{x}\,dx + \int_{\epsilon}^{\infty} \frac{\varphi(x)}{x}\,dx$$

has a limit as $\epsilon \to 0$. This limit is called the *Cauchy principal value* of the improper integral, and one writes

$$\langle \frac{1}{x}, \varphi(x)\rangle = \text{pr.v.}\int_{-\infty}^{\infty} \frac{\varphi(x)}{x}\,dx$$

(There's not a universal agreement on the notation for a principal value integral.)

Why does the principal value exist? The analysis is much the same as we did for $\ln|x|$. As before, write

$$\begin{aligned}
\text{pr.v.}\int_{-\infty}^{\infty} \frac{\varphi(x)}{x}\,dx &= \lim_{\epsilon\to 0}\left(\int_{-\infty}^{-\epsilon} \frac{\varphi(x)}{x}\,dx + \int_{\epsilon}^{\infty} \frac{\varphi(x)}{x}\,dx\right) \\
&= \lim_{\epsilon\to 0}\left(\int_{-\infty}^{-1} \frac{\varphi(x)}{x}\,dx + \int_{-1}^{-\epsilon} \frac{\varphi(x)}{x}\,dx + \int_{\epsilon}^{1} \frac{\varphi(x)}{x}\,dx + \int_{1}^{\infty} \frac{\varphi(x)}{x}\,dx\right) \\
&= \int_{-\infty}^{-1} \frac{\varphi(x)}{x}\,dx + \int_{1}^{\infty} \frac{\varphi(x)}{x}\,dx + \lim_{\epsilon\to 0}\left(\int_{-1}^{-\epsilon} \frac{\varphi(x)}{x}\,dx + \int_{\epsilon}^{1} \frac{\varphi(x)}{x}\,dx\right)
\end{aligned}$$

To take the limit we do the same thing we did before and use $\varphi(x) = \varphi(0) + O(x)$. The terms that matter are

$$\int_{-1}^{-\epsilon} \frac{\varphi(0)}{x}\,dx + \int_{\epsilon}^{1} \frac{\varphi(0)}{x}\,dx$$

and this sum is zero.

To summarize, $1/x$ does define a distribution, but the pairing of $1/x$ with a test function is via the Cauchy Principal Value, not just direct, uncommented upon integration. The distribution $1/x$ is thus often referred to as the "Principal Value Distribution".

Chapter 5

III, Sampling, and Interpolation

5.1 X-Ray Diffraction: Through a Glass Darkly[1]

Diffraction is not only an interesting phenomenon to look at, it is an important experimental tool, the tool being *diffraction gratings*. A diffraction grating is an aperture plane with a large number of parallel slits, closely spaced. See, for example

`http://hyperphysics.phy-astr.gsu.edu/hbase/phyopt/grating.html`.

Diffraction gratings are used to separate light of different wavelengths, and to *measure* wavelengths. I want to look briefly at this latter application.

X-rays were discovered by William Roentgen in 1895. It was not known whether they were particles or waves, but the wave hypothesis put their wavelength at about 10^{-8} cm. Using diffraction gratings was out of the question for experiments on X-rays because diffraction effects are only seen if the width of the slits is comparable to the wavelength. It was possible to build such gratings for experiments on visible light, where the wavelengths are between 400 and 700 nanometers (10^{-7} cm), but that extra order of magnitude to get to X-rays couldn't be done.

A related set of mysteries had to do with the structure of crystals. It was thought that the macroscopic structure of crystals could be explained by a periodic arrangement of atoms, but there was no way to test this. In 1912 Max von Laue proposed that the purported periodic structure of crystals could be used to diffract X-rays, just as gratings diffracted visible light. He thus had three hypotheses:

1. X-rays are waves.
2. Crystals are periodic.
3. The spacing between atoms is of the order 10^{-8} cm.

Friedrich and Kniping carried out experiments that confirmed von Laue's hypotheses and the subject of X-ray crystallography was born.

But you need to know some math.

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[1] 1 Corinthians 13: When I was a child, I spake as a child, I understood as a child, I thought as a child: but when I became a man, I put away childish things. For now we see through a glass, darkly, but then face to face: now I know in part; but then shall I know even as also I am known.

Electron density distribution An important quantity to consider in crystallography is how the electrons are distributed among the atoms in the crystal. This is usually referred to as the *electron density distribution* of the crystal. We want to see how we might represent this as a function, and consider what happens to the function in the course of an X-ray diffraction experiment.

Let's take the one-dimensional case as an illustration; we'll look at the (more realistic) higher dimensional case later in the course. We view a one-dimensional crystal as an evenly spaced collection of atoms along a line. In fact, for purposes of approximation, we suppose that an infinite number of them are strung out along a line. If we describe the electron density distribution of a *single* atom by a function $\rho(x)$ then the electron density distribution of the crystal with spacing p is the *periodic* function

$$\rho_p(x) = \sum_{k=-\infty}^{\infty} \rho(x - kp)\,.$$

As our discussion of diffraction might indicate, the Fourier transform of $\rho_p(x)$ is proportional to the "scattered amplitude" of X-rays diffracted by the crystal. Thus we want to write $\rho_p(x)$ in a form that's amenable to taking the Fourier transform. (Incidentally, it's not unreasonable to suppose that ρ is rapidly decreasing — the electron density of a single atom dies off as we move away from the atom.)

As we'll see, it's convenient to write the periodized density as a convolution with a sum of shifted δ's:

$$\rho_p(x) = \sum_{k=-\infty}^{\infty} \rho(x - pk) = \sum_{k=-\infty}^{\infty} \delta(x - kp) * \rho(x) = \left(\sum_{k=-\infty}^{\infty} \delta(x - kp) \right) * \rho(x)\,.$$

Now introduce

$$\text{Ш}_p(x) = \sum_{k=-\infty}^{\infty} \delta(x - kp)\,,$$

so that, simply,

$$\rho_p = \text{Ш}_p * \rho\,.$$

Ш_p is the star of the show. Bracewell calls it the "shah function", after the Cyrillic letter, and this has caught on. It's also referred to as the *Dirac comb* (with spacing p).

Using the convolution theorem, we have

$$\mathcal{F}\rho_p = \mathcal{F}\rho \cdot \mathcal{F}\text{Ш}_p\,.$$

What is $\mathcal{F}\text{Ш}_p$? That's a really interesting question.

5.2 The Ш Distribution

We want to develop the properties of Ш_p, particularly its Fourier transform. In fact, we met this distribution earlier, in Chapter 1. Rather, we met its Fourier transform — it's the continuous buzz signal, as we'll discuss further, below.

As a "standard" we take the spacing p to be 1, so we sum over the integer points and define

$$\text{Ш}(x) = \sum_{k=-\infty}^{\infty} \delta(x - k) \quad \text{or} \quad \text{Ш} = \sum_{k=-\infty}^{\infty} \delta_k\,.$$

As above, for a series of δ's spaced p apart we write

$$\text{Ш}_p(x) = \sum_{k=-\infty}^{\infty} \delta(x - kp) \quad \text{or} \quad \text{Ш}_p = \sum_{k=-\infty}^{\infty} \delta_{kp}\,.$$

I'll mostly write δ's "at points" in this section. It seems the more natural thing to do.

To see that the series for Ш_p makes sense as a distribution, let φ be a test function; then

$$\langle \text{Ш}_p, \varphi \rangle = \left\langle \sum_{k=-\infty}^{\infty} \delta_{kp}, \varphi \right\rangle = \sum_{k=-\infty}^{\infty} \langle \delta_{kp}, \varphi \rangle = \sum_{k=-\infty}^{\infty} \varphi(kp)\,.$$

This sum converges because of the rapid decrease of φ at $\pm\infty$.

There are two facets to the Ш's versatility: *periodizing* and *sampling*. We'll consider each in turn.

5.2.1 Periodizing with Ш

Our first application of Ш was as above, to write the periodization of the electron density function ρ of a single atom in a crystal as a convolution. The purpose was to periodize ρ to reflect the physical structure of the crystal. This is a general procedure. The Ш function furnishes a handy way of generating and working with periodic functions and distributions. Take that as an aphorism.

If f is a function or distribution for which convolution with Ш makes sense, then

$$(f * \text{Ш}_p)(t) = \sum_{k=-\infty}^{\infty} f(t - pk)$$

is periodic with period p. Note that

$$f(at + b) * \text{Ш}_p(t) = \sum_{k=-\infty}^{\infty} f(at + b - apk)$$

also has period p, and this can just as well be written in terms of a shifted Ш:

$$\sum_{k=-\infty}^{\infty} f(at + b - apk) = f(at) * \text{Ш}_p(t + \frac{b}{a})\,.$$

Convolving with Ш_p now emerges as the basic, familiar way to produce a periodic function. However, some care must be taken; convolving with Ш to periodize doesn't shift the graph and link them up, it shifts the graph and *adds* them up.

In many cases the series

$$\sum_{k=-\infty}^{\infty} f(t - pk)$$

will converge in some reasonable sense, often at least to define a periodic distribution (see Section 5.4). A common application is to form $f * \text{Ш}_p$ when f is zero for $|t| \geq p/2$. In this case the convolution exists and we naturally say that $f * \text{Ш}_p$ is the p-periodic extension of f.

I want to look at this operation in a little more detail, one reason being that it will make the discussion of sampling and aliasing, soon to come, much cleaner and easier. Recall the scaled rect function

$$\Pi_p(x) = \begin{cases} 1 & |x| < p/2 \\ 0 & |x| \geq p/2 \end{cases}$$

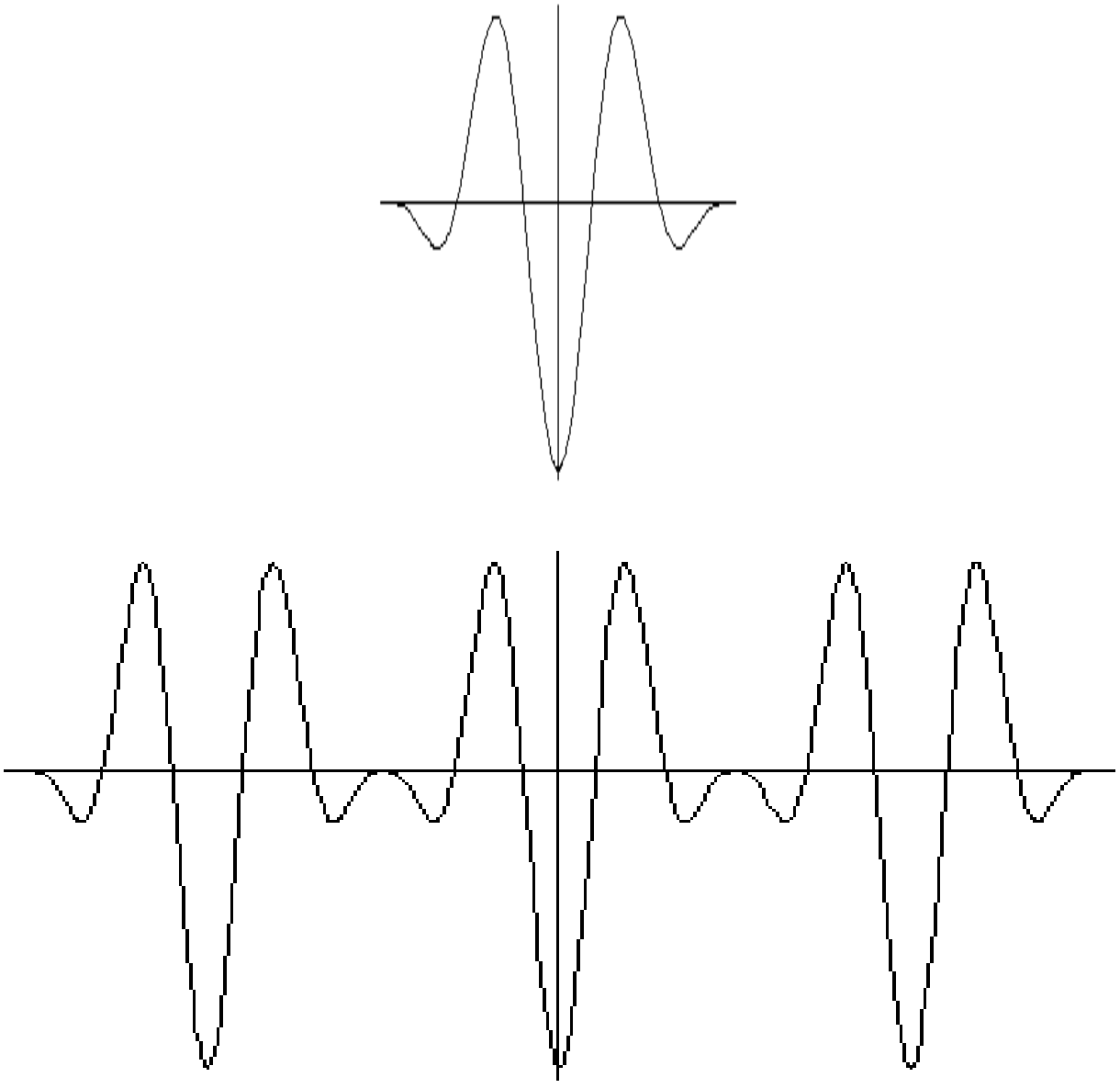
If f is zero when $|t| \geq p/2$ (note the $\geq$ not $>$) then

$$\Pi_p f = f$$

and

$$f = \Pi_p(f * \text{Ш}_p)\,.$$

In fact these two conditions are equivalent. That should be clear if you have the geometric picture in mind. For example, shown below are the graphs of a function $f(x)$ that is zero outside of $|t| < p/2$ and of three cycles of its periodization; that's $f(x+p) + f(x) + f(x-p) = f(x) * \sum_{k=-1}^{1} \delta(x - kp)$.



Here are the algebraic details that go from the picture to the formulas. If $\Pi_p f = f$ then

$$\begin{aligned}\Pi_p(t)(f * \text{Ш}_p)(t) &= \Pi_p(t)((\Pi_p f) * \text{Ш}_p)(t) \\ &= \Pi_p(t) \sum_{k=-\infty}^{\infty} \Pi_p(t-kp) f(t-kp) \\ &= \sum_{k=-\infty}^{\infty} \Pi_p(t)\Pi_p(t-kp) f(t-kp) = \Pi_p(t) f(t) = f(t)\end{aligned}$$

since

$$\Pi_p(t)\Pi_p(t-kp) = \begin{cases} \Pi_p(t) & k = 0 \\ 0 & k \neq 0 \end{cases}$$

On the other hand, if $f = \Pi_p(f * \text{Ш}_p)$ then

$$\Pi_p f = \Pi_p(\Pi_p(f * \text{Ш}_p)) = \Pi_p^2(f * \text{Ш}_p) = \Pi_p(f * \text{Ш}_p) = f\,.$$

If we had defined Π_p differently at $\pm p/2$ (in other cultures either $\Pi_p(\pm p/2) = 1$ or $\Pi_p(\pm p/2) = 1/2$ are typical) then the calculations and results above would hold *except* for the translates of $\pm p/2$, a discrete set of points. Such an exceptional set generally poses no problems in applications.

This all seems pretty innocent, but cutting off a *distribution* by Π_p (a discontinuous function) is not part of the theory. We only defined the product of a distribution and a smooth function. In general we'll proceed as though all is well, though careful justifications can take some work (which we won't do). Be not afraid.

5.2.2 Sampling with Ш

The flip side of periodizing with Ш is sampling with Ш. Here's what this means. Suppose we multiply Ш by a function f. Then as a distribution

$$f(x)\text{Ш}(x) = \sum_{k=-\infty}^{\infty} f(x)\delta(x-k) = \sum_{k=-\infty}^{\infty} f(k)\delta(x-k)\,.$$

Multiplying Ш by f "samples" f at the integer points, in the sense that it "records" the values of f at those points in the sum.

There's nothing sacred about sampling at the integers of course. Sampling using Ш_p means

$$f(x)\text{Ш}_p(x) = \sum_{k=-\infty}^{\infty} f(kp)\delta(x-kp)\,,$$

so f is sampled at the points kp. Scaled or not, the thing to keep in mind about the shah function is that it takes *evenly spaced samples* of a function f.

To summarize:

- Convolving a function with Ш (with Ш_p) produces a periodic function with period 1 (with period p).
- Multiplying a function by Ш (by Ш_p) samples the function at the integer points (at the points pk).

5.2.3 Scaling identity for $Ш_p$

There's a simple scaling identity for $Ш_p$ that comes up often enough in formulas and derivations to make it worth pointing out. We've defined

$$Ш_p(x) = \sum_{k=-\infty}^{\infty} \delta(x - kp)\,,$$

scaling the spacing of the impulses by p, but it's also natural to consider

$$Ш(px) = \sum_{k=-\infty}^{\infty} \delta(px - k)\,.$$

Now recall the scaling property of δ; for $p > 0$,

$$\delta(px) = \frac{1}{p}\delta(x)\,.$$

Plugging this into the formula for $Ш(px)$ gives

$$\begin{aligned} Ш(px) &= \sum_{k=-\infty}^{\infty} \delta(px - k) \\ &= \sum_{k=-\infty}^{\infty} \delta\left(p(x - \frac{k}{p})\right) \\ &= \sum_{k=-\infty}^{\infty} \frac{1}{p}\delta\left(x - \frac{k}{p}\right) = \frac{1}{p}Ш_{1/p}(x) \end{aligned}$$

To give it its own display:

$$Ш(px) = \frac{1}{p}Ш_{1/p}(x)$$

(It would be a good exercise to derive this in a variable-free environment, using the delay operator τ_p and the scaling operator σ_p.) By the same token,

$$Ш_p(x) = \frac{1}{p}Ш\left(\frac{1}{p}x\right)\,.$$

5.3 The Fourier Transform of Ш, or, The deepest fact about the integers is well known to every electrical engineer and spectroscopist

The most interesting thing about Ш is what happens when we take its Fourier transform. If we start with the definition

$$Ш(x) = \sum_{k=-\infty}^{\infty} \delta(x - k)\,.$$

and apply what we know about the Fourier transform of δ (it's 1) plus the shift theorem, we obtain

$$\mathcal{F}Ш(s) = \sum_{k=-\infty}^{\infty} e^{-2\pi i k s}\,.$$

Since we're summing over all positive and negative k we can write this as

$$\mathcal{F}\text{Ш}(s) = \sum_{k=-\infty}^{\infty} e^{2\pi iks}\,.$$

which looks more like a Fourier series. We did see this when we introduced the buzz signal. It sounds like a signal with every harmonic present in equal amounts. It sounds terrible.

The expression

$$\sum_{k=-\infty}^{\infty} e^{2\pi iks}$$

actually does make sense as a distribution, as we'll see, but it's not yet a helpful expression. Instead, to find the Fourier transform of Ш we go back to the definition in terms of tempered distributions. If φ is a Schwartz function then

$$\langle \mathcal{F}\text{Ш}, \varphi\rangle = \langle \text{Ш}, \mathcal{F}\varphi\rangle\,.$$

On the right hand side,

$$\langle \text{Ш}, \mathcal{F}\varphi\rangle = \left\langle \sum_{k=-\infty}^{\infty} \delta_k, \mathcal{F}\varphi \right\rangle = \sum_{k=-\infty}^{\infty} \langle \delta_k, \mathcal{F}\varphi\rangle = \sum_{k=-\infty}^{\infty} \mathcal{F}\varphi(k)$$

And now we have something absolutely remarkable.

The Poisson summation formula: Let φ be a Schwartz function. Then

$$\sum_{k=-\infty}^{\infty} \mathcal{F}\varphi(k) = \sum_{k=-\infty}^{\infty} \varphi(k)$$

This result actually holds for other classes of functions (the Schwartz class was certainly not known to Poisson!) but that's not important for us.

The Poisson summation formula is the deepest fact known about the integers. It's known to every electrical engineer and every spectroscopist because of what it says about the Fourier transform of $\mathcal{F}$Ш. We'll settle that now and come back to the derivation of the formula afterward.

We pick up our calculation of $\mathcal{F}$Ш where we left off:

$$\begin{aligned}
\langle \mathcal{F}\text{Ш}, \varphi\rangle &= \sum_{k=-\infty}^{\infty} \mathcal{F}\varphi(k) \\
&= \sum_{k=-\infty}^{\infty} \varphi(k) \quad \text{(because of the Poisson summation formula)} \\
&= \sum_{k=-\infty}^{\infty} \langle \delta_k, \varphi\rangle \quad \text{(definition of } \delta_k\text{)} \\
&= \left\langle \sum_{k=-\infty}^{\infty} \delta_k, \varphi \right\rangle \\
&= \langle \text{Ш}, \varphi\rangle\,.
\end{aligned}$$

Comparing where we started to where we ended up, we conclude that

$$\mathcal{F}\text{Ш} = \text{Ш}\,.$$

Outstanding. The Ш distribution is its own Fourier transform. (See also Section 5.10.)

Proof of the Poisson Summation Formula The proof of the Poisson summation formula is an excellent example of the power of having two different representations of the same thing, an idea certainly at the heart of Fourier analysis. Remember the maxim: If you can evaluate an expression in two different ways it's likely you've done something significant.

Given a test function $\varphi(t)$ we periodize to $\Phi(t)$ of period 1:

$$\Phi(t) = (\varphi * \text{Ш})(t) = \sum_{k=-\infty}^{\infty} \varphi(t-k)\,.$$

As a periodic function, Φ has a Fourier series:

$$\Phi(t) = \sum_{m=-\infty}^{\infty} \hat{\Phi}(m) e^{2\pi i m t}\,.$$

Let's find the Fourier coefficients of $\Phi(t)$.

$$\begin{aligned}
\hat{\Phi}(m) &= \int_0^1 e^{-2\pi i m t}\Phi(t)\,dt \\
&= \int_0^1 \sum_{k=-\infty}^{\infty} e^{-2\pi i m t}\varphi(t-k)\,dt = \sum_{k=-\infty}^{\infty}\int_0^1 e^{-2\pi i m t}\varphi(t-k)\,dt \\
&= \sum_{k=-\infty}^{\infty}\int_{-k}^{-k+1} e^{-2\pi i m(t+k)}\varphi(t)\,dt \\
&= \sum_{k=-\infty}^{\infty}\int_{-k}^{-k+1} e^{-2\pi i m t}e^{-2\pi i m k}\varphi(t)\,dt \quad (\text{using } e^{-2\pi i m k} = 1) \\
&= \int_{-\infty}^{\infty} e^{-2\pi i m t}\varphi(t)\,dt \\
&= \mathcal{F}\varphi(m)\,.
\end{aligned}$$

Therefore

$$\Phi(t) = \sum_{m=-\infty}^{\infty} \mathcal{F}\varphi(m) e^{2\pi i m t}\,.$$

(We've actually seen this calculation before, in a disguised form; look back to Section 3.5 on the relationship between the solutions of the heat equation on the line and on the circle.)

Since Φ is a smooth function, the Fourier series converges. Now compute $\Phi(0)$ two ways, one way from plugging into its definition and the other from plugging into its Fourier series:

$$\Phi(0) = \sum_{k=-\infty}^{\infty} \varphi(-k) = \sum_{k=-\infty}^{\infty} \varphi(k)$$

$$\Phi(0) = \sum_{k=-\infty}^{\infty} \mathcal{F}\varphi(k) e^{2\pi i n 0} = \sum_{k=-\infty}^{\infty} \mathcal{F}\varphi(k)$$

Done.

The Fourier transform of Ш_p From $\mathcal{F}\text{Ш} = \text{Ш}$ we can easily deduce the formula for $\mathcal{F}\text{Ш}_p$. Using the identities

$$\text{Ш}_p(x) = \frac{1}{p}\text{Ш}\left(\frac{1}{p}x\right) \quad \text{and} \quad \text{Ш}(px) = \frac{1}{p}\text{Ш}_{1/p}(x)\,.$$

we have

$$\begin{aligned}
\mathcal{F}\text{Ш}_p(s) &= \frac{1}{p}\mathcal{F}\left(\text{Ш}\left(\frac{x}{p}\right)\right) \\
&= \frac{1}{p}p\mathcal{F}\text{Ш}(ps) \quad \text{(stretch theorem)} \\
&= \text{Ш}(ps) \\
&= \frac{1}{p}\text{Ш}_{1/p}(s)
\end{aligned}$$

5.3.1 Crystal gazing

Let's return now to the setup for X-ray diffraction for a one-dimensional crystal. We described the electron density distribution of a *single* atom by a function $\rho(x)$ and the electron density distribution of the crystal with spacing p as

$$\rho_p(x) = \sum_{k=-\infty}^{\infty} \rho(x - kp) = (\rho * \text{Ш}_p)(x)\,.$$

Then

$$\begin{aligned}
\mathcal{F}\rho_p(s) &= \mathcal{F}(\rho * \text{Ш}_p)(s) \\
&= (\mathcal{F}\rho \cdot \mathcal{F}\text{Ш}_p)(s) \\
&= \mathcal{F}\rho(s)\frac{1}{p}\text{Ш}_{1/p}(s) \\
&= \sum_{k=-\infty}^{\infty} \frac{1}{p}\mathcal{F}\rho\left(\frac{k}{p}\right)\delta\left(s - \frac{k}{p}\right)
\end{aligned}$$

Here's the significance of this. In an X-ray diffraction experiment what you see on the X-ray film is a bunch of spots, corresponding to $\mathcal{F}\rho_p$. The intensity of each spot is proportional to the magnitude of the Fourier transform of the electron density ρ and the spots are spaced a distance $1/p$ apart, *not* p apart. If you were an X-ray crystallographer and didn't know your Fourier transforms, you might assume that there is a relation of direct proportion between the spacing of the dots on the film and the atoms in the crystal, but it's a *reciprocal* relation — kiss your Nobel prize goodbye. Every spectroscopist knows this.

We'll see a similar relation when we consider higher dimensional Fourier transforms and higher dimensional Ш-functions. A Ш-function will be associated with a *lattice* and the Fourier transform will be a Ш-function associated with the *reciprocal* or *dual* lattice. This phenomenon has turned out to be important in image processing; see, for example, *Digital Video Processing* by A. M. Tekalp.

5.4 Periodic Distributions and Fourier series

I want to collect a few facts about periodic distributions and Fourier series, and show how we can use Ш as a convenient tool for "classical" Fourier series.

Periodicity The notion of periodicity for distributions is invariance under the delay operator τ_p, i.e., a distribution (or a function, for that matter) is periodic with period p if

$$\tau_p S = S\,.$$

This is the "variable free" definition, since we're not supposed to write

$$S(x-p) = S(x) \quad \text{or} \quad S(x+p) = S(x)$$

which is the usual way of expressing periodicity. It's a pleasure to report that Ш_p is periodic with period p. You can see that most easily by doing what we're not supposed to do:

$$\text{Ш}_p(x+p) = \sum_{k=-\infty}^{\infty} \delta(x+p-kp) = \sum_{k=-\infty}^{\infty} \delta(x-(k-1)p) = \sum_{k=-\infty}^{\infty} \delta(x-kp) = \text{Ш}_p(x).$$

It's also easy to give a variable-free demonstration, which amounts to the same thing:

$$\tau_p \text{Ш}_p = \sum_{k=-\infty}^{\infty} \tau_p \delta_{kp} = \sum_{k=-\infty}^{\infty} \delta_{kp+p} = \sum_{k=-\infty}^{\infty} \delta_{p(k+1)} = \sum_{k=-\infty}^{\infty} \delta_{kp} = \text{Ш}_p\,.$$

When we periodize a test function φ by forming the convolution,

$$\Phi(x) = (\varphi * \text{Ш}_p)(x)\,,$$

it's natural to view the periodicity of Φ as a *consequence* of the periodicity of Ш_p. By this I mean we can appeal to:

- If S or T is periodic of period p then $S * T$ (when it is defined) is periodic of period p.

Let me show this for functions (something we could have done way back) and I'll let you establish the general result. Suppose f is periodic of period p. Consider $(f*g)(x+p)$. We have

$$(f*g)(x+p) = \int_{-\infty}^{\infty} f(x+p-y)g(y)\,dy = \int_{-\infty}^{\infty} f(x-y)g(y)\,dy = (f*g)(x).$$

The same argument works if instead g is periodic.

So, on the one hand, convolving with Ш_p produces a periodic function. On the other hand, suppose Φ is periodic of period p and we cut out one period of it by forming $\Pi_p\Phi$. We get Φ back, *in toto*, by forming the convolution with Ш_p; that is,

$$\Phi = \varphi * \text{Ш}_p = (\Pi_p\Phi) * \text{Ш}_p$$

(Well, this is almost right. The cut-off $\Pi_p\Phi$ is zero at $\pm p/2$ while $\Phi(\pm p/2)$ certainly may not be zero. These "exceptions" at the end-points won't affect the discussion here in any substantive way.[2])

The upshot of this is that something is periodic if and only if it is a convolution with Ш_p. This is a nice point of view. I'll take this up further in Section 5.10.

[2] We can either: (a) ignore this problem; (b) jigger the definition of Π_p to make it really true, which has other problems; or (c) say that the statement is true as an equality between distributions, and tell ourselves that modifying the functions at a discrete set of points will not affect that equality.

Fourier series for Ш Taking the Fourier series of Ш term by term we arrived at

$$\mathcal{F}Ш = \sum_{k=-\infty}^{\infty} e^{2\pi ikt}\,,$$

and if we next use $\mathcal{F}Ш = Ш$ we would then have

$$Ш = \sum_{k=-\infty}^{\infty} e^{2\pi ikt}\,.$$

The series

$$\sum_{k=-\infty}^{\infty} e^{2\pi ikt}$$

does define a distribution, for

$$\left\langle \sum_{k=-\infty}^{\infty} e^{2\pi ikt}, \varphi \right\rangle = \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} e^{2\pi ikt}\varphi(t)\,dt$$

exists for any test function φ because φ is rapidly decreasing. There's a pretty straightforward development of Fourier series for tempered distributions, and while we won't enter into it, suffice it to say we do indeed have

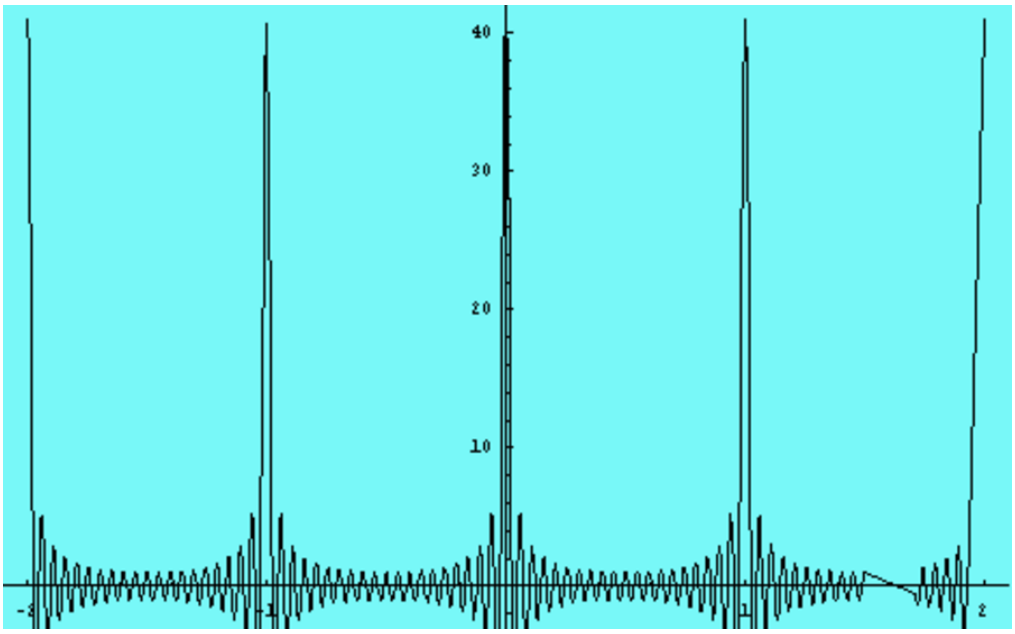
$$Ш = \sum_{k=-\infty}^{\infty} e^{2\pi ikt}\,.$$

The right hand side *really is* the Fourier series for Ш. But, by the way, you can't prove this *without* proving the Poisson summation formula and that $\mathcal{F}Ш = Ш$, so Fourier series isn't a shortcut to the latter in this case.

Remember that we saw the finite version of the Fourier series for Ш back in Fourier series section:

$$D_N(t) = \sum_{n=-N}^{N} e^{2\pi int} = \frac{\sin(\pi(2N+1)t)}{\sin \pi t}\,.$$

Here's the graph for $N = 20$:



It's now really true that

$$D_N \to \text{Ш}$$

as $N \to \infty$, where the convergence is in the sense of distributions.

Fourier transform of a Fourier series When we first started to work with tempered distributions, I said that we would be able to take the Fourier transform of functions that didn't have one, i.e., functions for which the integral defining the (classical) Fourier transform does not exist. We've made good on that promise, including complex exponentials, for which

$$\mathcal{F}e^{2\pi ikt/p} = \delta\left(s - \frac{k}{p}\right).$$

With this we can now find the Fourier transform of a Fourier series. If

$$\varphi(t) = \sum_{k=-\infty}^{\infty} c_k e^{2\pi ikt/p}$$

then

$$\mathcal{F}\varphi(s) = \sum_{k=-\infty}^{\infty} c_k \mathcal{F}e^{2\pi ikt/p} = \sum_{k=-\infty}^{\infty} c_k \delta\left(s - \frac{k}{p}\right)$$

It may well be that the series $\sum_{k=-\infty}^{\infty} c_k e^{2\pi ikt/p}$ converges to define a tempered distribution — that's not asking too much[3] — even if it doesn't converge pointwise to $\varphi(t)$. Then it still makes sense to consider its Fourier transform and the formula, above, is OK.

Rederiving Fourier series for a periodic function We can turn this around and rederive the formula for Fourier series as a consequence of our work on Fourier transforms. Suppose Φ is periodic of period p and write, as we know we can,

$$\Phi = \varphi * \text{Ш}_p$$

where φ is one period of Φ, say $\varphi = \Pi_p\Phi$. Take the Fourier transform of both sides and boldly invoke the convolution theorem:

$$\mathcal{F}\Phi = \mathcal{F}(\varphi * \text{Ш}_p) = \mathcal{F}\varphi \cdot \mathcal{F}\text{Ш}_p = \mathcal{F}\varphi \cdot \frac{1}{p}\text{Ш}_{1/p},$$

or, at points,

$$\mathcal{F}\Phi(s) = \mathcal{F}\varphi(s)\left(\frac{1}{p}\sum_{k=-\infty}^{\infty}\delta\left(s - \frac{k}{p}\right)\right) = \frac{1}{p}\sum_{k=-\infty}^{\infty}\mathcal{F}\varphi\left(\frac{k}{p}\right)\delta\left(s - \frac{k}{p}\right).$$

Now boldly take the inverse Fourier transform:

$$\Phi(t) = \sum_{k=-\infty}^{\infty}\frac{1}{p}\mathcal{F}\varphi\left(\frac{k}{p}\right)e^{2\pi ikt/p} \quad (\text{the } \mathcal{F}\varphi\left(\frac{k}{p}\right) \text{ are constants}).$$

But

$$\begin{aligned}\frac{1}{p}\mathcal{F}\varphi\left(\frac{k}{p}\right) &= \frac{1}{p}\int_{-\infty}^{\infty} e^{-2\pi i(k/p)t}\varphi(t)\,dt \\ &= \frac{1}{p}\int_{-\infty}^{\infty} e^{-2\pi i(k/p)t}\,\Pi_p(t)\Phi(t)\,dt = \frac{1}{p}\int_{-p/2}^{p/2} e^{-2\pi i(k/p)t}\,\Phi(t)\,dt,\end{aligned}$$

[3] For example, if φ is integrable so that the coefficients c_k tend to zero. Or even less than that will do, just as long as the coefficients don't grow too rapidly.

and this is the k-th Fourier coefficient c_k of Φ. We've rederived

$$\Phi(t) = \sum_{k=-\infty}^{\infty} c_k e^{2\pi i k t/p}, \quad \text{where} \quad c_k = \frac{1}{p}\int_{-p/2}^{p/2} e^{-2\pi i (k/p)t}\,\Phi(t)\,dt\,.$$

5.5 Sampling Signals

In the previous lecture we studied three properties of Ш that make it so useful in many applications. They are:

- Periodizing
  - Convolving with Ш periodizes a function.
- Sampling
  - Multiplying by Ш samples a function.
- The Fourier transform of Ш is Ш.
  - Convolving and multiplying are themselves flip sides of the same coin via the convolution theorem for Fourier transforms.

We are now about to combine all of these ideas in a spectacular way to treat the problem of "sampling and interpolation". Let me state the problem this way:

- Given a signal $f(t)$ and a collection of *samples* of the signal, i.e., values of the signal at a set of points $f(t_0)$, $f(t_1)$, $f(t_2)$, ..., to what extent can one interpolate the values $f(t)$ at other points from the sample values?

This is an old question, and a broad one, and it would appear on the surface to have nothing to do with Ш's or Fourier transforms, or any of that. But we've already seen some clues, and the full solution is set to unfold.

5.5.1 Sampling sines and bandlimited signals

Why should we expect to be able to do interpolation at all? Imagine putting down a bunch of dots — maybe even infinitely many — and asking someone to pass a curve through them that *agrees everywhere exactly* with a predetermined mystery function passing through those dots. Ridiculous. But it's *not* ridiculous. If a relatively simple hypothesis is satisfied then interpolation can be done! Here's one way of getting some intuitive sense of the problem and what that hypothesis should be.

Suppose we *know* a signal is a single sinusoid. A sinusoid repeats, so if we have enough information to pin it down over one period, or cycle, then we know the whole thing. How many samples — how many values of the function — within one period do we need to know to know which sinusoid we have? We need three samples *strictly* within one cycle. You can think of the graph, or you can think of the equation: A general sinusoid is of the form $A\sin(2\pi\nu t + \phi)$. There are three unknowns, the amplitude A, the frequency ν and the phase ϕ. We would expect to need three equations to find the unknowns, hence we need values of the function at three points, three samples.

What if the signal is a sum of sinusoids, say

$$\sum_{n=1}^{N} A_n \sin(2\pi n\nu t + \phi_n)\,.$$

Sample points for the sum are "morally" sample points for the individual harmonics, though not explicitly. We need to take enough samples to get sufficient information to determine all of the unknowns for all of the harmonics. Now, in the time it takes for the combined signal to go through one cycle, the individual harmonics will have gone through several cycles, the lowest frequency harmonic through one cycle, the lower frequency harmonics through a few cycles, say, and the higher frequency harmonics through many. We have to take enough samples of the combined signal so that as the individual harmonics go rolling along we'll be sure to have at least three samples in *some* cycle of *every* harmonic.

To simplify and standardize we assume that we take evenly spaced samples (in t). Since we've phrased things in terms of cycles per second, to understand how many samples are enough it's then also better to think in terms of "sampling rate", i.e., samples/sec instead of "number of samples". If we are to have at least three samples strictly within a cycle then the sample points must be strictly less than a half-cycle apart. A sinusoid of frequency ν goes through a half-cycle in $1/2\nu$ seconds so we want

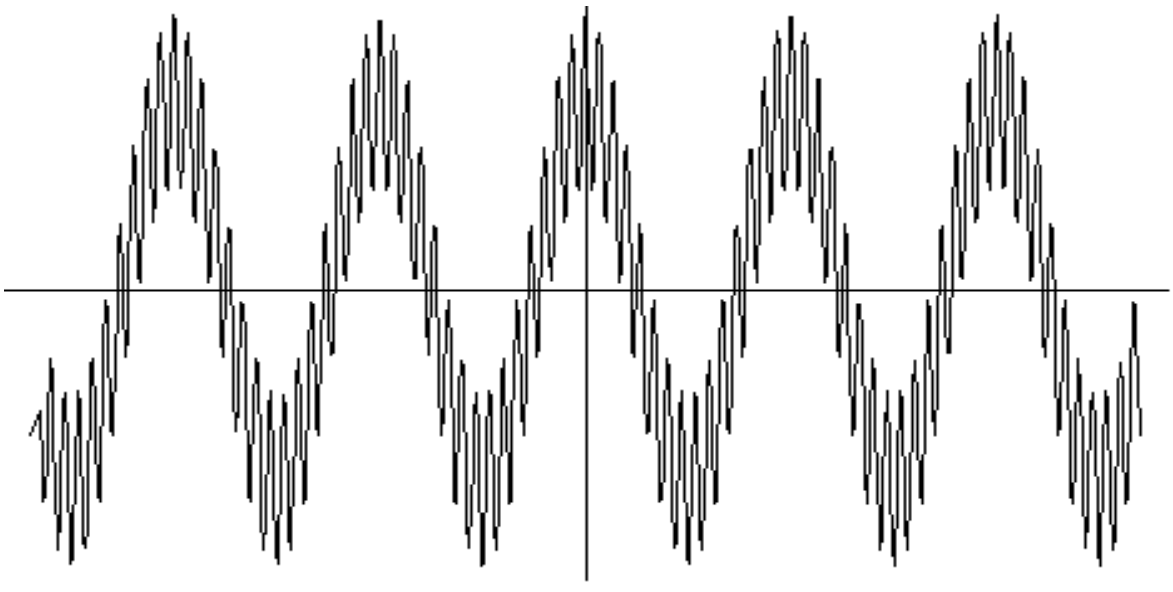
$$\text{spacing between samples} = \frac{\text{number of seconds}}{\text{number of samples}} < \frac{1}{2\nu}\,.$$

The more usual way of putting this is

$$\text{sampling rate} = \text{samples/sec} > 2\nu\,.$$

This is the rate at which we should sample a given sinusoid of frequency ν to guarantee that a single cycle will contain at least three sample points. Furthermore, if we sample at this rate for a given frequency, we will certainly have more than three sample points in some cycle of any harmonic at a *lower* frequency. Note that the sampling rate has units 1/seconds and that sample points are 1/(sampling rate) seconds apart.

For the combined signal — a sum of harmonics — the higher frequencies are driving up the sampling rate; specifically, the *highest* frequency is driving up the rate. To think of the interpolation problem geometrically, high frequencies cause more rapid oscillations, i.e., rapid changes in the function over small intervals, so to hope to interpolate such fluctuations accurately we'll need a lot of sample points and thus a high sampling rate. For example, here's a picture of the sum of two sinusoids one of low frequency and one of high frequency.



If we sample at too low rate we might miss the wiggles entirely. We might mistakenly think we had only the low frequency sinusoid, and, moreover, if all we had to go on were the samples we wouldn't even know we'd made a mistake! We'll come back to just this problem a little later.

If we sample at a rate greater than twice the highest frequency, our sense is that we will be sampling often enough for all the lower harmonics as well, and we should be able to determine everything. The problem here is if the spectrum is *unbounded*. If, as for a square wave, we have a full Fourier series and not just a finite sum of sinusoids, then we have no hope of sampling frequently enough to determine the combined signal from the samples. For a square wave, for example, there is *no* "highest frequency". That's trouble. It's time to define ourselves out of this trouble.

Bandlimited signals From the point of view of the preceding discussion, the problem for interpolation, is high frequencies, and the best thing a signal can be is a finite Fourier series. The latter is much too restrictive for applications, of course, so what's the "next best" thing a signal can be? It's one for which there *is* a highest frequency. These are the *bandlimited* signals — signals whose Fourier transforms are identically zero outside of a finite interval. Such a signal has a bounded spectrum; there is a "highest frequency".

More formally:

- A signal $f(t)$ is bandlimited if there is a finite number p such that $\mathcal{F}f(s) = 0$ for all $|s| \geq p/2$. The smallest number p for which this is true is called the *bandwidth* of $f(t)$.

There's a question about having $\mathcal{F}f$ be zero at the endpoints $\pm p/2$ as part of the definition. For the following discussion on sampling and interpolation, it's easiest to assume this is the case, and treat separately some special cases when it isn't. For those who want to know more, read the next paragraph.

Some technical remarks If $f(t)$ is an integrable function then $\mathcal{F}f(s)$ is continuous, so if $\mathcal{F}f(s) = 0$ for all $|s| > p/2$ then $\mathcal{F}f(\pm p/2) = 0$ as well. On the other hand, it's also common first to define the *support* of a function (integrable or not) as the complement of the largest open set on which the function is identically zero. (This definition can also be given for distributions.) This makes the support closed, being the complement of an open set. For example, if $\mathcal{F}f(s)$ is identically zero for $|s| > p/2$, and on no larger *open* set, then the support of $\mathcal{F}f$ is the *closed* interval $[-p/2, +p/2]$. Thus, with this definition, even if $\mathcal{F}f(\pm p/2) = 0$ the endpoints $\pm p/2$ are included in the support of $\mathcal{F}f$.

One then says, as an alternate definition, that f is bandlimited if the support of $\mathcal{F}f$ is closed and *bounded*. In mathematical terms, a closed, bounded set (in $\mathbf{R}^n$) is said to be *compact*, and so the shorthand definition of bandlimited is that $\mathcal{F}f$ has compact support. A typical compact set is a closed interval, like $[-p/2, +p/2]$, but we could also take finite unions of closed intervals. This definition is probably the one more often given, but it's a little more involved to set up, as you've just witnessed. Whichever definition of bandlimited one adopts there are always questions about what happens at the endpoints anyway, as we'll see.

5.6 Sampling and Interpolation for Bandlimited Signals

We're about to solve the interpolation problem for bandlimited signals. We'll show that interpolation is possible by finding an explicit formula that does the job. Before going through the solution, however, I want to make a general observation that's independent of the interpolation problem but is important to it.

It is unphysical to consider a signal as lasting forever in time. A physical signal $f(t)$ is naturally "time-limited", meaning that $f(t)$ is identically zero on $|t| \geq q/2$ for some q — there just isn't any signal beyond

a point. On the other hand, it is very physical to consider a bandlimited signal, one with no frequencies beyond a certain point, or at least no frequencies that our instruments can register. Well, we can't have both, at least not in the ideal world of mathematics. Here is where mathematical description meets physical expectation — and they disagree. The fact is:

- A signal cannot be both timelimited and bandlimited.

What this means in practice is that there must be inaccuracies in a mathematical model of a phenomenon that assumes a signal is both timelimited and bandlimited. Such a model can be at best an approximation, and one has to be prepared to estimate the errors as they may affect measurements and conclusions.

Here's one argument why the statement is true; I'll give a more complete proof of a more general statement in Appendix 1. Suppose f is bandlimited, say $\mathcal{F}f(s)$ is zero for $|s| \geq p/2$. Then

$$\mathcal{F}f = \Pi_p \cdot \mathcal{F}f\,.$$

Take the inverse Fourier transform of both sides to obtain

$$f(t) = p\,\mathrm{sinc}\,pt * f(t)\,.$$

Now $\mathrm{sinc}\,pt$ "goes on forever"; it decays but it has nonzero values all the way out to $\pm\infty$. Hence the convolution with f also goes on forever; it is not timelimited.

sinc as a "convolution identity" There's an interesting observation that goes along with the argument we just gave. We're familiar with δ acting as an "identity element" for convolution, meaning

$$f * \delta = f\,.$$

This important property of δ holds for *all* signals for which the convolution is defined. We've just seen for the more restricted class of bandlimited functions, with spectrum from $-p/2$ to $+p/2$, that the sinc function also has this property:

$$p\,\mathrm{sinc}\,pt * f(t) = f(t)\,.$$

The Sampling Theorem Ready to solve the interpolation problem? It uses all the important properties of Ш, but it goes so fast that you might miss the fun entirely if you read too quickly.

Suppose $f(t)$ is bandlimited with $\mathcal{F}f(s)$ identically zero for $|s| \geq p/2$. We periodize $\mathcal{F}f$ using Ш_p and then cut off to get $\mathcal{F}f$ back again:

$$\mathcal{F}f = \Pi_p(\mathcal{F}f * \text{Ш}_p)\,.$$

This is the crucial equation.

Now take the inverse Fourier transform:

$$\begin{aligned}
f(t) = \mathcal{F}^{-1}\mathcal{F}f(t) &= \mathcal{F}^{-1}(\Pi_p(\mathcal{F}f * \text{Ш}_p))(t) \\
&= \mathcal{F}^{-1}\Pi_p(t) * \mathcal{F}^{-1}(\mathcal{F}f * \text{Ш}_p)(t) \\
&\qquad \text{(taking } \mathcal{F}^{-1} \text{ turns multiplication into convolution)} \\
&= \mathcal{F}^{-1}\Pi_p(t) * (\mathcal{F}^{-1}\mathcal{F}f(t) \cdot \mathcal{F}^{-1}\text{Ш}_p(t)) \\
&\qquad \text{(ditto, except it's convolution turning into multiplication)} \\
&= p\,\text{sinc}\,pt * (f(t) \cdot \frac{1}{p}\text{Ш}_{1/p}(t)) \\
&= \text{sinc}\,pt * \sum_{k=-\infty}^{\infty} f\left(\frac{k}{p}\right)\delta\left(t - \frac{k}{p}\right) \quad \text{(the sampling property of Ш}_p\text{)} \\
&= \sum_{k=-\infty}^{\infty} f\left(\frac{k}{p}\right)\text{sinc}\,pt * \delta\left(t - \frac{k}{p}\right) \\
&= \sum_{k=-\infty}^{\infty} f\left(\frac{k}{p}\right)\text{sinc}\,p\left(t - \frac{k}{p}\right) \quad \text{(the sifting property of } \delta\text{)}
\end{aligned}$$

We've just established the classic "Sampling Theorem", though it might be better to call it the "interpolation theorem". Here it is as a single statement:

- If $f(t)$ is a signal with $\mathcal{F}f(s)$ identically zero for $|s| \geq p/2$ then

$$f(t) = \sum_{k=-\infty}^{\infty} f\left(\frac{k}{p}\right)\text{sinc}\,p\left(t - \frac{k}{p}\right).$$

Some people write the formula as

$$f(t) = \sum_{k=-\infty}^{\infty} f\left(\frac{k}{p}\right)\text{sinc}(pt - k),$$

but I generally prefer to emphasize the *sample points*

$$t_k = \frac{k}{p}$$

and then to write the formula as

$$f(t) = \sum_{k=-\infty}^{\infty} f(t_k)\,\text{sinc}\,p(t - t_k).$$

What does the formula do, once again? It computes *any* value of f in terms of *sample values*. Here are a few general comments to keep in mind:

- The sample points are spaced $1/p$ apart — the *reciprocal* of the bandwidth.[4]

[4] That sort of reciprocal phenomenon is present again in higher dimensional versions of the sampling formula. This will be a later topic for us.

- The formula involves infinitely many sample points — k/p for $k = 0, \pm 1, \pm 2, \ldots$.

  So don't think you're getting away too cheaply, and realize that any practical implementation can only involve a finite number of terms in the sum, so will necessarily be an approximation.

  - Since a bandlimited signal cannot be timelimited we should expect to have to take samples all the way out to $\pm\infty$. However, sampling a bandlimited *periodic* signal, i.e., a finite Fourier series, requires only a finite number of samples. We'll cover this, below.

Put the outline of the argument for the sampling theorem into your head — it's important. Starting with a bandlimited signal, there are three parts:

- Periodize the Fourier transform.
- Cut off this periodic function to get back where you started.
- Take the inverse Fourier transform.

Cutting off in the second step, a multiplication, exactly undoes periodizing in the first step, a convolution, *provided that* $\mathcal{F}f = \Pi_p(\mathcal{F}f * \text{Ш}_p)$. But taking the inverse Fourier transform swaps multiplication with convolution and this is why something nontrivial happens. It's almost obscene the way this works.

Sampling rates and the Nyquist frequency The bandwidth determines the minimal sampling rate we can use to reconstruct the signal from its samples. I'd almost say that the bandwidth *is* the minimal sampling rate except for the slight ambiguity about where the spectrum starts being identically zero (the "endpoint problem"). Here's the way the situation is usually expressed: If the (nonzero) spectrum runs from $-\nu_{\max}$ to $\nu_{\max}$ then we need

$$\text{sampling rate} > 2\nu_{\max}$$

to reconstruct the signal from its samples.

The number $2\nu_{\max}$ is often called the *Nyquist frequency*, after Harry Nyquist, God of Sampling, who was the first engineer to consider these problems for the purpose of communications. There are other names associated with this circle of ideas, most notably E. Whittaker, a mathematician, and C. Shannon, an all around genius and founder of Information Theory. The formula as we've given it is often referred to as the Shannon Sampling Theorem.

The derivation of the formula gives us some one-sided freedom, or rather the opportunity to do more work than we have to. We cannot take p smaller than the length of the interval where $\mathcal{F}f$ is supported, the bandwidth, but we can take it larger. That is, if p is the bandwidth and $q > p$ we can periodize $\mathcal{F}f$ to have period q by convolving with Ш_q and we still have the fundamental equation

$$\mathcal{F}f = \Pi_q(\mathcal{F}f * \text{Ш}_q)\,.$$

(Draw a picture.) The derivation can then proceed exactly as above and we get

$$f(t) = \sum_{k=-\infty}^{\infty} f(\tau_k)\,\text{sinc}\,q(t - \tau_k)$$

where the sample points are

$$\tau_k = \frac{k}{q}\,.$$

These sample points are spaced closer together than the sample points $t_k = k/p$. The sampling rate is higher than we need. We're doing more work than we have to.

5.7 Interpolation a Little More Generally

Effective approximation and interpolation of signals raises a lot of interesting and general questions. One approach that provides a good framework for many such questions is to bring in orthogonality. It's very much analogous to the way we looked at Fourier series.

Interpolation and orthogonality We begin with still another amazing property of sinc functions — they form an orthonormal collection. Specifically, the family of sinc functions $\{\text{sinc}(t-n) : n = 0, \pm 1, \pm 2, \ldots\}$ is orthonormal with respect to the usual inner product on $L^2(\mathbf{R})$. Recall that the inner product is

$$(f, g) = \int_{-\infty}^{\infty} f(t)\overline{g(t)}\, dt\,.$$

The calculation to establish the orthonormality property of the sinc functions uses the general Parseval identity,

$$\int_{-\infty}^{\infty} f(t)\overline{g(t)}\, dt = \int_{-\infty}^{\infty} \mathcal{F}f(s)\overline{\mathcal{F}g(s)}\, ds\,.$$

We then have

$$\begin{aligned}\int_{-\infty}^{\infty} \text{sinc}(t-n)\ \text{sinc}(t-m)\, dt &= \int_{-\infty}^{\infty} (e^{-2\pi isn}\Pi(s))\,\overline{(e^{-2\pi ism}\Pi(s))}\, ds \\ &= \int_{-\infty}^{\infty} e^{2\pi is(m-n)}\Pi(s)\Pi(s)\, ds = \int_{-1/2}^{1/2} e^{2\pi is(m-n)}\, ds\end{aligned}$$

From here direct integration will give you that this is 1 when $n = m$ and 0 when $n \neq m$.

In case you're fretting over it, the sinc function *is* in $L^2(\mathbf{R})$ and the product of two sinc functions is integrable. Parseval's identity holds for functions in $L^2(\mathbf{R})$, though we did not establish this.

Now let's consider bandlimited signals $g(t)$, and to be definite let's suppose the spectrum is contained in $-1/2 \leq s \leq 1/2$. Then the Nyquist sampling rate is 1, i.e., we sample at the integer points, and the interpolation formula takes the form

$$g(t) = \sum_{n=-\infty}^{\infty} g(n)\,\text{sinc}(t-n)\,.$$

Coupled with the result on orthogonality, this formula suggest that the family of sinc functions forms an orthonormal *basis* for the space of bandlimited signals with spectrum in $[-1/2, 1/2]$, and that we're expressing $g(t)$ in terms of this basis. To see that this really is the case, we interpret the coefficients (the

sample values $g(n)$) as the inner product of $g(t)$ with $\mathrm{sinc}(t-n)$. We have, again using Parseval,

$$\begin{aligned}
(g(t), \mathrm{sinc}(t-n)) &= \int_{-\infty}^{\infty} g(t)\ \mathrm{sinc}(t-n)\, dt \\
&= \int_{-\infty}^{\infty} \mathcal{F}g(s)\mathcal{F}(\mathrm{sinc}(t-n))\, ds \quad \text{(by Parseval)} \\
&= \int_{-\infty}^{\infty} \mathcal{F}g(s)\overline{(e^{-2\pi isn}\Pi(s))}\, ds \\
&= \int_{-1/2}^{1/2} \mathcal{F}g(s)e^{2\pi ins}\, ds \\
&= \int_{-\infty}^{\infty} \mathcal{F}g(s)e^{2\pi ins}\, ds \quad \text{(because } g \text{ is bandlimited)} \\
&= g(n) \quad \text{(by Fourier inversion)}
\end{aligned}$$

It's perfect! The interpolation formula says that $g(t)$ is written in terms of an orthonormal basis, and the coefficient $g(n)$, the n-th sampled value of $g(t)$, is exactly the projection of $g(t)$ onto the n-th basis element:

$$g(t) = \sum_{n=-\infty}^{\infty} g(n)\,\mathrm{sinc}(t-n) = \sum_{n=-\infty}^{\infty} \big(g(t), \mathrm{sinc}(t-n)\big)\,\mathrm{sinc}(t-n)\,.$$

Lagrange interpolation Certainly for computational questions, going way back, it is desirable to find reasonably simple *approximations* of complicated functions, particularly those arising from solutions to differential equations.[5] The classic way to approximate is to interpolate. That is, to find a simple function that, at least, assumes the same values as the complicated function at a given finite set of points. Curve fitting, in other words. The classic way to do this is via polynomials. One method, presented here just for your general background and know-how, is due to Lagrange.

Suppose we have n points $t_1, t_2, \ldots, t_n$. We want a polynomial of degree $n-1$ that assumes given values at the n sample points. (Why degree $n-1$?)

For this, we start with an n-th degree polynomial that vanishes exactly at those points. This is given by

$$p(t) = (t-t_1)(t-t_2)\cdots(t-t_n)\,.$$

Next put

$$p_k(t) = \frac{p(t)}{t-t_k}\,.$$

Then $p_k(t)$ is a polynomial of degree $n-1$; we divide out the factor $(t-t_k)$ and so $p_k(t)$ vanishes at the same points as $p(t)$ except at t_k. Next consider the quotient

$$\frac{p_k(t)}{p_k(t_k)}\,.$$

This is again a polynomial of degree $n-1$. The key property is that $p_k(t)/p_k(t_k)$ vanishes at the sample points t_j *except* at the point t_k where the value is 1; i.e.,

$$\frac{p_k(t_j)}{p_k(t_k)} = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases}$$

[5] The sinc function may not really qualify as an "easy approximation". How is it computed, really?

To interpolate a function by a polynomial (to fit a curve through a given set of points) we just scale and add. That is, suppose we have a function $g(t)$ and we want a *polynomial* that has values $g(t_1)$, $g(t_2), \dots,$ $g(t_n)$ at the points t_1, $t_2, \dots,$ t_n. We get this by forming the sum

$$p(t) = \sum_{k=1}^{n} g(t_k) \frac{p_k(t)}{p_k(t_k)} .$$

This does the trick. It is known as the *Lagrange Interpolation Polynomial*. Remember, unlike the sampling formula we're not reconstructing *all* the values of $g(t)$ from a set of sample values. We're approximating $g(t)$ by a polynomial that has the same values as $g(t)$ at a prescribed set of points.

The sinc function is an analog of the $p_k(t)/p_k(t_k)$ for "Fourier interpolation", if we can call it that. With

$$\operatorname{sinc} t = \frac{\sin \pi t}{\pi t} .$$

we recall some properties, analogous to the polynomials we built above:

- $\operatorname{sinc} t = 1$ when $t = 0$
- $\operatorname{sinc} t = 0$ at nonzero integer points $t = \pm 1, \pm 2, \dots$.

Now shift this and consider

$$\operatorname{sinc}(t - k) = \frac{\sin \pi (t - k)}{\pi (t - k)} .$$

This has the value 1 at $t = k$ and is zero at the other integers.

Suppose we have our signal $g(t)$ and the sample points $\dots, g(-2)$, $g(-1)$, $g(0)$, $g(1)$, $g(2), \dots$. So, again, we're sampling at evenly spaced points, and we've taken the sampling rate to be 1 just to simplify. To interpolate these values we would then form the sum

$$\sum_{n=-\infty}^{\infty} g(k) \operatorname{sinc}(t - k) .$$

There it is again — the general interpolation formula. In the case that $g(t)$ is bandlimited (bandwidth 1 in this example) we know we recover all values of $g(t)$ from the sample values.

5.8 Finite Sampling for a Bandlimited Periodic Signal

We started this whole discussion of sampling and interpolation by arguing that one ought to be able to interpolate the values of a finite sum of sinusoids from knowledge of a finite number of samples. Let's see how this works out, but rather than starting from scratch let's use what we've learned about sampling for general bandlimited signals.

As always, it's best to work with the complex form of a sum of sinusoids, so we consider a real signal given by

$$f(t) = \sum_{k=-N}^{N} c_k e^{2\pi i k t/q}, \quad c_{-k} = \overline{c_k} .$$

$f(t)$ is periodic of period q. Recall that $c_{-k} = \overline{c_k}$. Some of the coefficients may be zero, but we assume that $c_N \neq 0$.

There are $2N + 1$ terms in the sum (don't forget $k = 0$) and it should take $2N + 1$ sampled values over one period to determine $f(t)$ completely. You might think it would take twice this many sampled values

because the values of $f(t)$ are real and we have to determine *complex* coefficients. But remember that $c_{-k} = \overline{c_k}$, so if we know c_k we know c_{-k}. Think of the $2N + 1$ sample values as enough information to determine the real number c_0 and the N complex numbers c_1, c_2, ..., c_N.

The Fourier transform of f is

$$\mathcal{F}f(s) = \sum_{k=-N}^{N} c_k \delta\left(s - \frac{k}{q}\right)$$

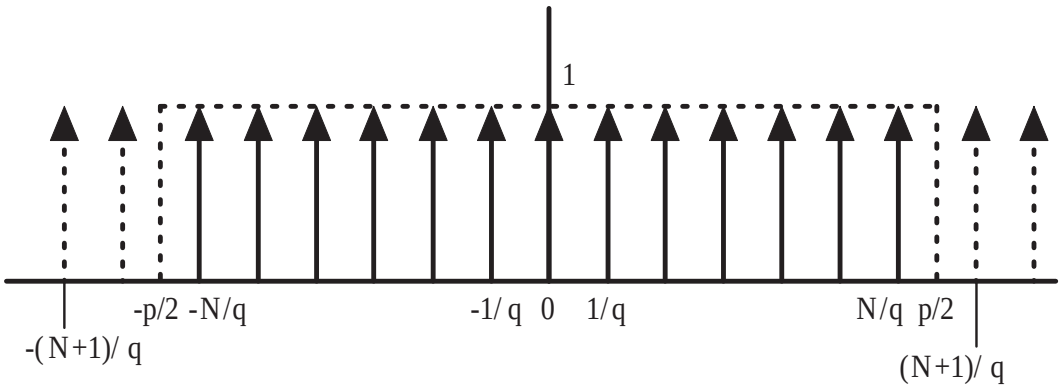
and the spectrum goes from $-N/q$ to N/q. The sampling formula applies to $f(t)$, and we can write an equation of the form

$$f(t) = \sum_{k=-\infty}^{\infty} f(t_k)\,\mathrm{sinc}\,p(t - t_k)\,,$$

but it's a question of what to take for the sampling rate, and hence how to space the sample points.

We want to make use of the known periodicity of $f(t)$. If the sample points t_k are a fraction of a period apart, say q/M for an M to be determined, then the values $f(t_k)$ with $t_k = kq/M$, $k = 0, \pm 1, \pm 2, \ldots$ will repeat after M samples. We'll see how this collapses the interpolation formula.

To find the right sampling rate, p, think about the derivation of the sampling formula, the first step being: "periodize $\mathcal{F}f$". The Fourier transform $\mathcal{F}f$ is a bunch of δ's spaced $1/q$ apart (and scaled by the coefficients c_k). The natural periodization of $\mathcal{F}f$ is to keep the spacing $1/q$ in the periodized version, essentially making the periodized $\mathcal{F}f$ a scaled version of $\text{Ш}_{1/q}$. We do this by convolving $\mathcal{F}f$ with Ш_p where $p/2$ is the midpoint between N/q, the last point in the spectrum of $\mathcal{F}f$, and the point $(N + 1)/q$, which is the next point $1/q$ away. Here's a picture.



Thus we find p from

$$\frac{p}{2} = \frac{1}{2}\left(\frac{N}{q} + \frac{N+1}{q}\right) = \frac{(2N+1)}{2q}, \quad \text{or} \quad p = \frac{2N+1}{q}\,.$$

We periodize $\mathcal{F}f$ by Ш_p (draw yourself a picture of this!), cut off by Π_p, then take the inverse Fourier transform. The sampling formula back in the time domain is

$$f(t) = \sum_{k=-\infty}^{\infty} f(t_k)\,\mathrm{sinc}\,p(t - t_k)$$

with

$$t_k = \frac{k}{p}\,.$$

With our particular choice of p let's now see how the q-periodicity of $f(t)$ comes into play. Write

$$M = 2N + 1\,,$$

so that

$$t_k = \frac{k}{p} = \frac{kq}{M}\,.$$

Then, to repeat what we said earlier, the sample points are spaced a fraction of a period apart, q/M, and after $f(t_0)$, $f(t_1)$, ..., $f(t_{M-1})$ the sample values repeat, e.g., $f(t_M) = f(t_0)$, $f(t_{M+1}) = f(t_1)$ and so on. More succinctly,

$$t_{k+k'M} = t_k + k'q\,,$$

and so

$$f(t_{k+k'M}) = f(t_k + k'q) = f(t_k)\,,$$

for any k and k'. Using this periodicity of the coefficients in the sampling formula, the single sampling sum splits into M sums as:

$$\begin{aligned}
&\sum_{k=-\infty}^{\infty} f(t_k)\,\mathrm{sinc}\,p(t-t_k) \\
&\quad = f(t_0)\sum_{m=-\infty}^{\infty}\mathrm{sinc}(pt - mM) \;+\; f(t_1)\sum_{m=-\infty}^{\infty}\mathrm{sinc}(pt - (1+mM)) + \\
&\qquad f(t_2)\sum_{m=-\infty}^{\infty}\mathrm{sinc}(pt - (2+mM)) \;+\cdots+\; f(t_{M-1})\sum_{m=-\infty}^{\infty}\mathrm{sinc}(pt - (M-1+mM))
\end{aligned}$$

Those sums of sincs on the right are periodizations of $\mathrm{sinc}\,pt$ and, remarkably, they have a simple closed form expression. The k-th sum is

$$\sum_{m=-\infty}^{\infty}\mathrm{sinc}(pt - k - mM) = \mathrm{sinc}(pt-k) * \text{Ш}_{M/p}(t) = \frac{\mathrm{sinc}(pt-k)}{\mathrm{sinc}(\frac{1}{M}(pt-k))} = \frac{\mathrm{sinc}(p(t-t_k))}{\mathrm{sinc}(\frac{1}{q}(t-t_k))}\,.$$

(I'll give a derivation of this at the end of this section.) Using these identities, we find that the sampling formula to interpolate

$$f(t) = \sum_{k=-N}^{N} c_k e^{2\pi i k t/q}$$

from $2N + 1 = M$ sampled values is

$$f(t) = \sum_{k=0}^{2N} f(t_k)\frac{\mathrm{sinc}(p(t-t_k))}{\mathrm{sinc}(\frac{1}{q}(t-t_k))}, \quad \text{where } p = \frac{2N+1}{q},\ t_k = \frac{k}{p} = \frac{kq}{2N+1}\,.$$

This is the "finite sampling theorem" for periodic functions.

It might also be helpful to write the sampling formula in terms of frequencies. Thus, if the lowest frequency is $\nu_{\min} = 1/q$ and the highest frequency is $\nu_{\max} = N\nu_{\min}$ then

$$f(t) = \sum_{k=0}^{2N} f(t_k)\frac{\mathrm{sinc}((2\nu_{max} + \nu_{min})(t-t_k))}{\mathrm{sinc}(\nu_{\min}(t-t_k))}, \quad \text{where } t_k = \frac{kq}{2N+1}\,.$$

The sampling rate is

$$\text{sampling rate} = 2\nu_{\max} + \nu_{min}\,.$$

Compare this to

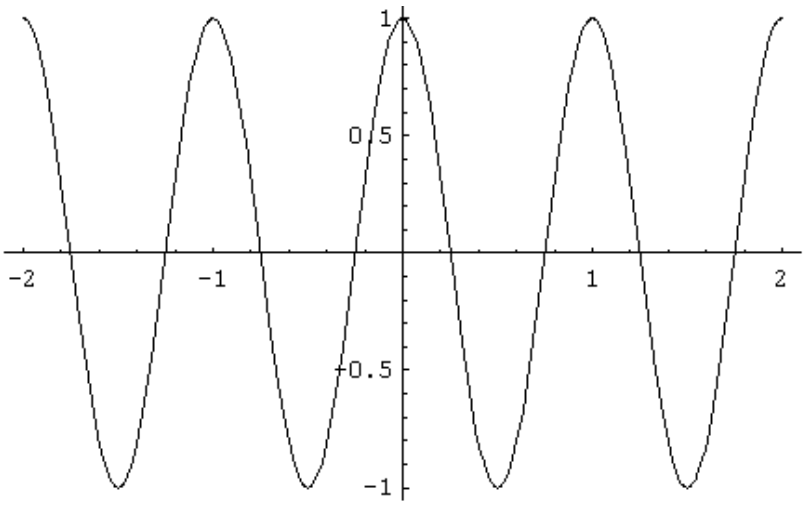
$$\text{sampling rate} > 2\nu_{\max}$$

for a general bandlimited function.

Here's a simple example of the formula. Take $f(t) = \cos 2\pi t$. There's only one frequency, and $\nu_{\min} = \nu_{\max} = 1$. Then $N = 1$, the sampling rate is 3 and the sample points are $t_0 = 0$, $t_1 = 1/3$, and $t_2 = 2/3$. The formula says

$$\cos 2\pi t = \frac{\operatorname{sinc} 3t}{\operatorname{sinc} t} + \cos\left(\frac{2\pi}{3}\right)\frac{\operatorname{sinc}(3(t-\frac{1}{3}))}{\operatorname{sinc}(t-\frac{1}{3})} + \cos\left(\frac{4\pi}{3}\right)\frac{\operatorname{sinc}(3(t-\frac{2}{3}))}{\operatorname{sinc}(t-\frac{2}{3})}\,.$$

Does this really work? I'm certainly not going to plow through the trig identities needed to check it! However, here's a plot of the right hand side.



Any questions? Ever thought you'd see such a complicated way of writing $\cos 2\pi t$?

Periodizing sinc Functions In applying the general sampling theorem to the special case of a periodic signal, we wound up with sums of sinc functions which we recognized (sharp-eyed observers that we are) to be periodizations. Then, out of nowhere, came a closed form expression for such periodizations as a ratio of sinc functions. Here's where this comes from, and here's a fairly general result that covers it.

Lemma Let p, $q > 0$ and let N be the largest integer strictly less than $pq/2$. Then

$$\sum_{k=-\infty}^{\infty} \operatorname{sinc}(pt - kpq) = \operatorname{sinc}(pt) * \text{Ш}_q(t) = \frac{1}{pq}\frac{\sin((2N+1)\pi t/q)}{\sin(\pi t/q)}\,.$$

There's a version of this lemma with $N \le pq/2$, too, but that's not important for us. In terms of sinc functions the formula is

$$\operatorname{sinc}(pt) * \text{Ш}_q(t) = \frac{2N+1}{pq}\frac{\operatorname{sinc}((2N+1)t/q)}{\operatorname{sinc}(t/q)}\,.$$

It's then easy to extend the lemma slightly to include periodizing a shifted sinc function, $\mathrm{sinc}(pt+b)$, namely

$$\sum_{k=-\infty}^{\infty} \mathrm{sinc}(pt+b-kpq) = \mathrm{sinc}(pt+b) * \text{Ш}_q(t) = \frac{2N+1}{pq}\frac{\mathrm{sinc}\left(\frac{2N+1}{pq}(pt+b)\right)}{\mathrm{sinc}\left(\frac{1}{pq}(pt+b)\right)}$$

This is what is needed in the last part of the derivation of the finite sampling formula.

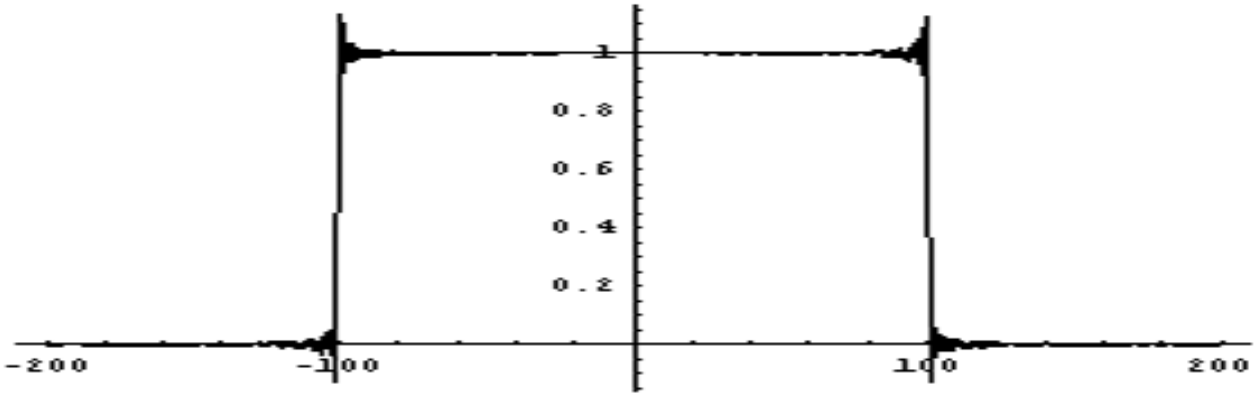
Having written this lemma down so grandly I now have to admit that it's really only a special case of the general sampling theorem as we've already developed it, though I think it's fair to say that this is only "obvious" in retrospect. The fact is that the ratio of sine functions on the right hand side of the equation is a bandlimited signal (we've seen it before, see below) and the sum for $\mathrm{sinc}(pt) * \text{Ш}_q(t)$ is just the sampling formula applied to that function. One usually thinks of the sampling theorem as going *from* the signal *to* the series of sampled values, but it can also go the other way. This admission notwithstanding, I still want to go through the derivation, from scratch.

One more thing before we do that. If $p = q = 1$, so that $N = 0$, the formula in the lemma gives

$$\sum_{n=-\infty}^{\infty} \mathrm{sinc}(t-n) = \mathrm{sinc}\, t * \text{Ш}_1(t) = 1\,.$$

Striking. Still don't believe it? Here's a plot of

$$\sum_{n=-100}^{100} \mathrm{sinc}(t-n)\,.$$



Note the Gibbs-like phenomena at the edges. This means there's some issue with what kind of convergence is involved, which is the last thing I want to worry about.

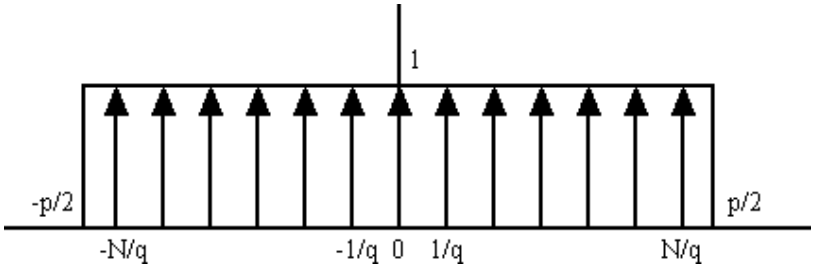
We proceed with the derivation of the formula

$$\mathrm{sinc}(pt) * \text{Ш}_q(t) = \frac{1}{pq}\frac{\sin((2N+1)\pi t/q)}{\sin(\pi t/q)}$$

This will look awfully familiar; indeed I'll really just be repeating the derivation of the general sampling formula for this special case. Take the Fourier transform of the convolution:

$$\mathcal{F}(\mathrm{sinc}(pt) * \text{Ш}_q(t)) = \mathcal{F}(\mathrm{sinc}(pt)) \cdot \mathcal{F}\text{Ш}_q(t) = \frac{1}{p}\Pi_p(s) \cdot \frac{1}{q}\text{Ш}_{1/q}(s) = \frac{1}{pq}\sum_{n=-N}^{N} \delta(s-\frac{n}{q})$$

See the figure below.



And now take the inverse Fourier transform:

$$\mathcal{F}^{-1}\left(\frac{1}{pq}\sum_{n=-N}^{N}\delta\Big(s-\frac{n}{q}\Big)\right)=\frac{1}{pq}\sum_{n=-N}^{N}e^{2\pi int/q}=\frac{1}{pq}\frac{\sin(\pi(2N+1)t/q))}{\sin(\pi t/q)}.$$

There it is. One reason I wanted to go through this is because it is another occurrence of the sum of exponentials and the identity

$$\sum_{n=-N}^{N}e^{2\pi int/q}=\frac{\sin(\pi(2N+1)t/q))}{\sin(\pi t/q)}\,,$$

which we've now seen on at least two other occasions. Reading the equalities backwards we have

$$\mathcal{F}\left(\frac{\sin(\pi(2N+1)t/q))}{\sin(\pi t/q)}\right)=\mathcal{F}\Big(\sum_{n=-N}^{N}e^{2\pi int/q}\Big)=\sum_{n=-N}^{N}\delta\Big(s-\frac{n}{q}\Big)\,.$$

This substantiates the earlier claim that the ratio of sines is bandlimited, and hence we could have appealed to the sampling formula directly instead of going through the argument we just did. But who would have guessed it?

5.9 Troubles with Sampling

In Section 5.6 we established a remarkable result on sampling and interpolation for bandlimited functions:

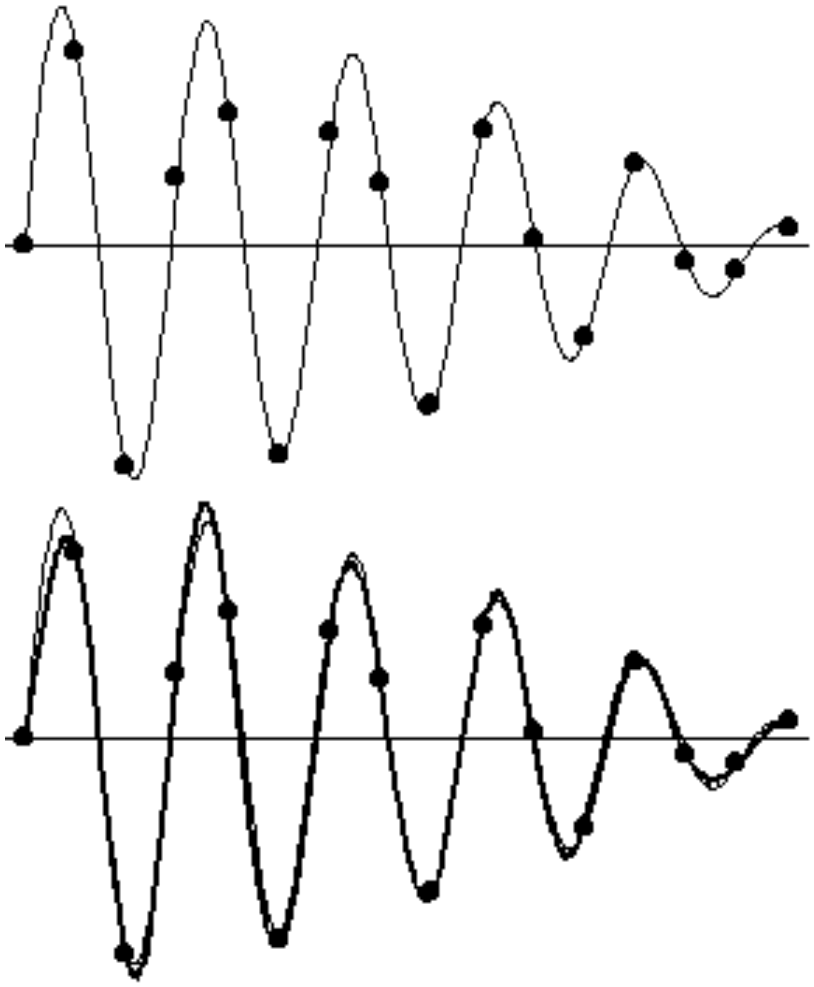
- If $f(t)$ is a bandlimited signal whose Fourier transform is identically zero for $|s|\geq p/2$ then

$$f(t)=\sum_{k=-\infty}^{\infty}f(t_k)\,\mathrm{sinc}\,p(t-t_k),\quad\text{where }\ t_k=\frac{k}{p}\,.$$

The bandwidth, a property of the signal in the frequency domain, is the minimal sampling rate and is the reciprocal of the spacing of the sample points, a construction in the time domain.

We have had our day of triumph. Now, we'll be visited by troubles. Actually we'll study just one type of trouble and the havoc it can wreak with our wondrous formula. This is meant to be a brief encounter. Any one of these examples can be treated in much greater depth depending on the particular area where they typically arise, e.g., digital audio and computer music, computer graphics, imaging and image compression.

Before we get into things, here's a picture of the sampling formula in action. The first figure shows a function and a set of evenly spaced sample points. The second figure is the function together with the sinc interpolation based on these samples (plotted as a thinner curve).



Of course, the fit is not exact because we're only working with a finite set of samples and the sampling formula asks for the sample values at *all* the points k/p, $k = 0, \pm 1, \pm 2, \ldots$. But it's pretty close.

Think about the trade-offs here. If the signal is timelimited, as in the above graph, then it cannot be bandlimited and so the sampling theorem doesn't even apply. At least it doesn't apply perfectly — it may be that the spectrum decays to a small enough level that the sinc interpolation is extremely accurate. On the other hand, if a signal is bandlimited then it cannot be timelimited, but any interpolation for real-world, computational purposes has to be done with a finite set of samples, so that interpolation must be only an approximation. These problems are absolutely inevitable. The approaches are via filters, first low pass filters done *before* sampling to force a signal to be bandlimited, and then other kinds of filters (smoothing) following whatever reconstruction is made from the samples. Particular kinds of filters are designed for particular kinds of signals, e.g., sound or images.

5.9.1 The trouble with undersampling — aliasing

What if we work a little less hard than dictated by the bandwidth. What if we "undersample" a bit and try to apply the interpolation formula with a little lower sampling rate, with the sample points spaced a little farther apart. Will the interpolation formula produce "almost" a good fit, good enough to hear or to see? Maybe yes, maybe no. A disaster is a definite possibility.

Sampling sines, redux Let's revisit the question of sampling and interpolation for a simple sine function and let's work with an explicit example. Take the signal given by

$$f(t) = \cos \tfrac{9\pi}{2} t \,.$$

The frequency of this signal is $9/4$ Hz. If we want to apply our formula for finite sampling we should take a sampling rate of $2 \times (9/4) + 9/4 = 27/4 = 6.75$ samples/sec. (If we want to apply the general sampling formula we can take the sampling rate to be anything $> 9/2 = 4.5$.) Suppose our sampler is stuck in low and we can take only one sample every second. Then our samples have values

$$\cos \tfrac{9\pi}{2} n, \quad n = 0, 1, 2, 3, \ldots \,.$$

There is another, lower frequency signal that has the same samples. To find it, take away from $9\pi/2$ the largest multiple of 2π that leaves a remainder of less than π in absolute value (so there's a spread of less than 2π — one full period — to the left and right). You'll see what I mean as the example proceeds. Here we have

$$\tfrac{9\pi}{2} = 4\pi + \tfrac{\pi}{2}$$

Then

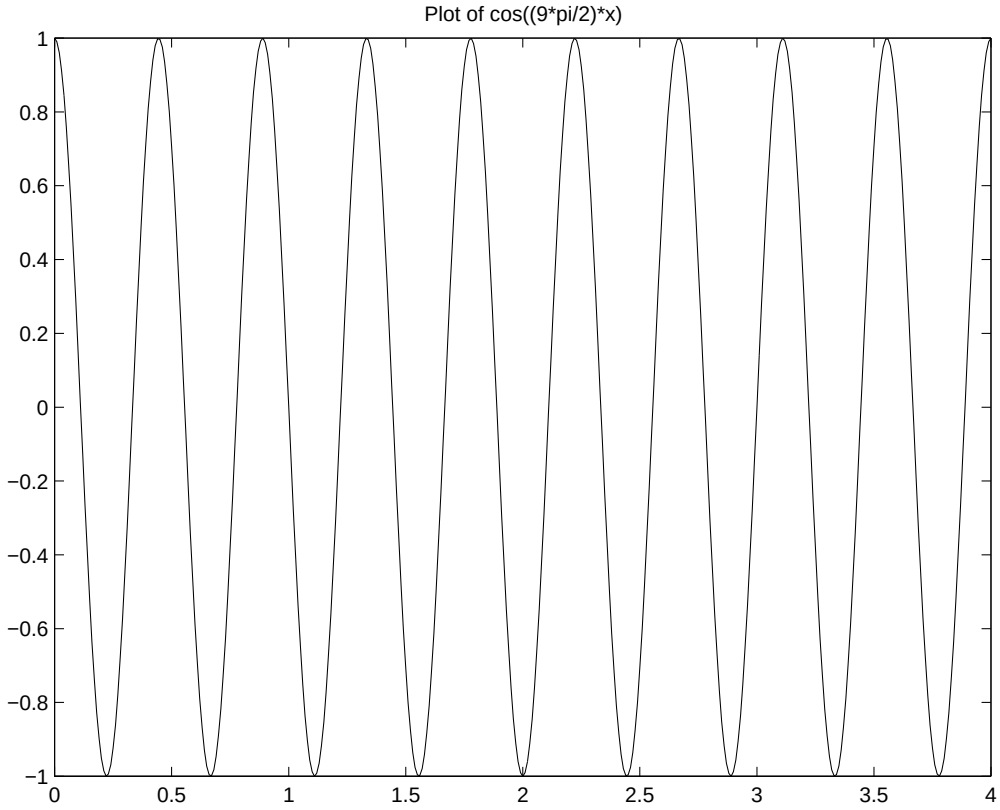
$$\cos \tfrac{9\pi}{2} n = \cos \left(\left(4\pi + \tfrac{\pi}{2} \right) n \right) = \cos \tfrac{\pi}{2} n \,.$$

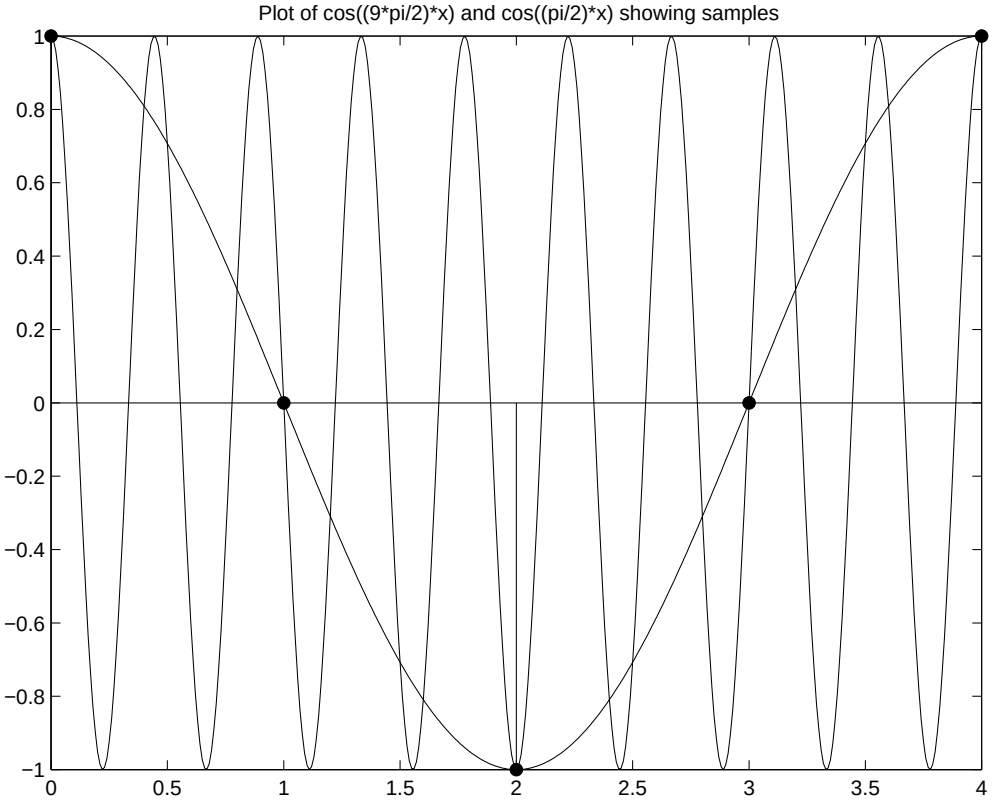
So the signal $f(t)$ has the same samples at 0, 1, 2, and so on, as the signal

$$g(t) = \cos \tfrac{\pi}{2} t$$

whose frequency is only $1/4$. The two functions *are not* the same everywhere, but their samples at the integers are equal.

Here are plots of the original signal $f(t)$ and of $f(t)$ and $g(t)$ plotted together, showing how the curves match up at the sample points. The functions $f(t)$ and $g(t)$ are called *aliases* of each other. They are indistinguishable *as far as their sample values go.*





You have no doubt seen this phenomenon illustrated with a strobe light flashing on and off on a moving fan, for example. Explain that illustration to yourself.

Now let's analyze this example in the frequency domain, essentially repeating the derivation of the sampling formula for this particular function at the particular sampling rate of 1 Hz. The Fourier transform of $f(t) = \cos 9\pi t/2$ is

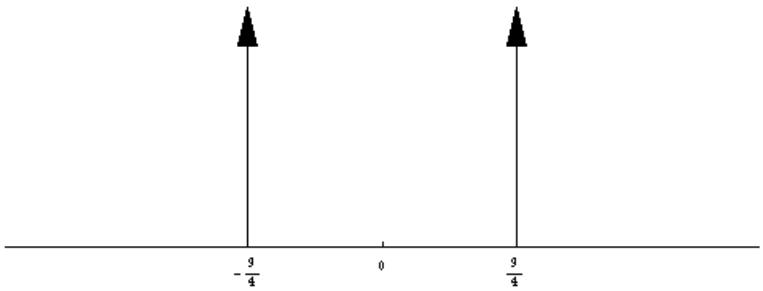
$$\mathcal{F}f(s) = \tfrac{1}{2}\left(\delta\left(s - \tfrac{9}{4}\right) + \delta\left(s + \tfrac{9}{4}\right)\right) .$$

To "sample at $p = 1$Hz" means, first off, that in the frequency domain we:

- Periodize $\mathcal{F}f$ by Ш_1
- Cut off by Π_1

After that we take the inverse Fourier transform and, *by definition*, this gives the interpolation to $f(t)$ using the sample points $f(0), f(\pm 1), f(\pm 2), \ldots$. The question is whether this interpolation gives back $f(t)$ — we know it doesn't, but what goes wrong?

The Fourier transform of $\cos 9\pi t/2$ looks like



For the periodization step we have by direct calculation,

$$\begin{aligned}\mathcal{F}f(s) * \text{Ш}_1(s) &= \tfrac{1}{2}\left[\delta\left(s-\tfrac{9}{4}\right)+\delta\left(s+\tfrac{9}{4}\right)\right] * \sum_{k=-\infty}^{\infty}\delta(s-k)\\ &= \tfrac{1}{2}\sum_{k=-\infty}^{\infty}\left(\delta\left(s-\tfrac{9}{4}\right)*\delta(s-k)+\delta\left(s+\tfrac{9}{4}\right)*\delta(s-k)\right)\\ &= \tfrac{1}{2}\sum_{k=-\infty}^{\infty}\left(\delta\left(s-\tfrac{9}{4}-k\right)+\delta\left(s+\tfrac{9}{4}-k\right)\right) \quad \text{(remember the formula } \delta_a * \delta_b = \delta_{a+b})\end{aligned}$$

Multiplying by Π_1 cuts off outside $(-1/2, +1/2)$, and we get δ's within $-1/2 < s < 1/2$ if, working separately with $\delta\left(s-\frac{9}{4}-k\right)$ and $\delta\left(s+\frac{9}{4}-k\right)$, we have:

$$-\tfrac{1}{2} < -\tfrac{9}{4}-k < \tfrac{1}{2} \qquad -\tfrac{1}{2} < \tfrac{9}{4}-k < \tfrac{1}{2}$$

$$\tfrac{7}{4} < -k < \tfrac{11}{4} \qquad -\tfrac{11}{4} < k < -\tfrac{7}{4}$$

$$-\tfrac{11}{4} < k < -\tfrac{7}{4} \qquad \tfrac{7}{4} < k < \tfrac{11}{4}$$

Thus we get δ's within $-1/2 < s < 1/2$ if

$$k = -2 \quad \text{and the term} \quad \delta\left(s-\tfrac{9}{4}-(-2)\right) = \delta\left(s+\tfrac{1}{4}\right),$$

and

$$k = 2 \quad \text{and the term} \quad \delta\left(s+\tfrac{9}{4}-2\right) = \delta\left(s-\tfrac{1}{4}\right).$$

All other δ's in $\mathcal{F}f(s) * \text{Ш}_1(s)$ will be outside the range $-1/2 < s < 1/2$ and so

$$\Pi(s)(\mathcal{F}f(s) * \text{Ш}_1(s)) = \tfrac{1}{2}\left(\delta\left(s+\tfrac{1}{4}\right)+\delta\left(s-\tfrac{1}{4}\right)\right)$$

We *do not have*

$$\Pi_1(\mathcal{F}f * \text{Ш}_1) = \mathcal{F}f\,.$$

So if we take the inverse Fourier transform of $\Pi_1(\mathcal{F}f * \text{Ш}_1)$ we *do not get* f back. We can take the inverse Fourier transform of $\Pi_1(\mathcal{F}f * \text{Ш}_1)$ anyway, and this produces

$$\mathcal{F}^{-1}\left(\tfrac{1}{2}\left(\delta\left(s-\tfrac{1}{4}\right)+\delta\left(s+\tfrac{1}{4}\right)\right)\right) = \tfrac{1}{2}(e^{\pi i t/2}+e^{-\pi i t/2}) = \cos\tfrac{\pi}{2}t$$

There's the aliased signal!

$$g(t) = \cos\tfrac{\pi}{2}t$$

Why *is* $g(t)$ an "alias" of the original signal $f(t)$? It's still quite right to think of $\mathcal{F}^{-1}(\Pi_1(\mathcal{F}f * \text{Ш}_1)$ as an interpolation based on sampling $f(t)$ at 1 Hz. That's exactly what it is, it's just not a good one. The sampling formula is

$$\begin{aligned}\mathcal{F}^{-1}(\Pi_1(\mathcal{F}f * \text{Ш}_1))(t) &= \operatorname{sinc} t * (f(t)\cdot \text{Ш}_1(t)) = \operatorname{sinc} t * \sum_{k=-\infty}^{\infty} f(t)\,\delta(t-k)\\ &= \operatorname{sinc} t * \sum_{k=-\infty}^{\infty} f(k)\delta(t-k) = \sum_{k=-\infty}^{\infty} f(k)\operatorname{sinc}(t-k) = \sum_{k=-\infty}^{\infty}\cos\tfrac{9\pi k}{2}\operatorname{sinc}(t-k)\,.\end{aligned}$$

But this sum of sincs provided by the sampling formula isn't $f(t) = \cos\frac{9\pi}{2}t$. It's $g(t) = \cos\frac{\pi}{2}t$ (though you'd never know that just from the formula). Interpolating the samples of f according to the formula at

the sampling rate of 1 Hz — *too low a sampling rate* — has not produced f it has produced g, an alias of f. Cool.

Before we leave this example let's look a little more at

$$\mathcal{F}f(s) * \text{Ш}_1(s) = \tfrac{1}{2} \sum_{k=-\infty}^{\infty} \left(\delta\left(s - \tfrac{9}{4} - k\right) + \delta\left(s + \tfrac{9}{4} - k\right)\right) .$$

Being a convolution with Ш_1 this is periodic of period 1, but, actually, it has a *smaller* period. To find it, write

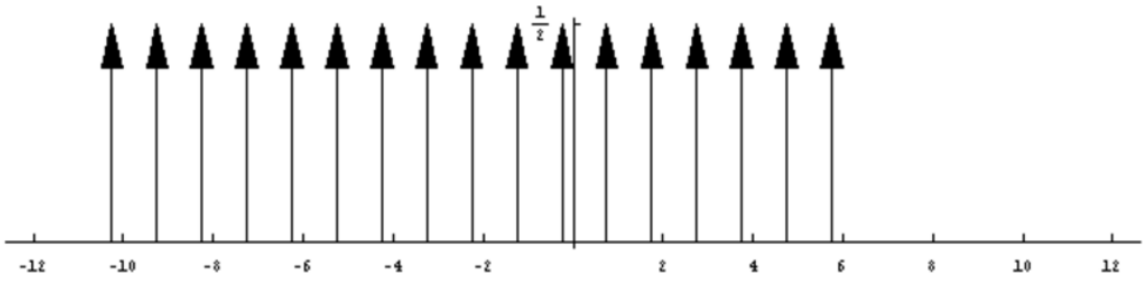
$$\begin{aligned}
\tfrac{1}{2} \sum_{k=-\infty}^{\infty} \left(\delta\left(s - \tfrac{9}{4} - k\right) + \delta\left(s + \tfrac{9}{4} - k\right)\right) &= \tfrac{1}{2}\left(\text{Ш}_1\left(s - \tfrac{9}{4}\right) + \text{Ш}_1\left(s + \tfrac{9}{4}\right)\right) \\
&= \tfrac{1}{2}\left(\text{Ш}_1\left(s - 2 - \tfrac{1}{4}\right) + \text{Ш}_1\left(s + 2 + \tfrac{1}{4}\right)\right) \\
&= \tfrac{1}{2}\left(\text{Ш}_1\left(s - \tfrac{1}{4}\right) + \text{Ш}_1\left(s + \tfrac{1}{4}\right)\right) \\
&\qquad \text{(because } \text{Ш}_1 \text{ is periodic of period 1)}
\end{aligned}$$

This last expression is periodic of period 1/2, for

$$\begin{aligned}
\tfrac{1}{2}\left(\text{Ш}_1\left(s - \tfrac{1}{4} + \tfrac{1}{2}\right) + \text{Ш}_1\left(s + \tfrac{1}{4} + \tfrac{1}{2}\right)\right) &= \tfrac{1}{2}\left(\text{Ш}_1\left(s + \tfrac{1}{4}\right) + \text{Ш}_1\left(s + \tfrac{3}{4}\right)\right) \\
&= \tfrac{1}{2}\left(\text{Ш}_1\left(s + \tfrac{1}{4}\right) + \text{Ш}_1\left(s + 1 - \tfrac{1}{4}\right)\right) \\
&= \tfrac{1}{2}\left(\text{Ш}_1\left(s + \tfrac{1}{4}\right) + \text{Ш}_1\left(s - \tfrac{1}{4}\right)\right) \\
&\qquad \text{(using the periodicity of } \text{Ш}_1\text{)}
\end{aligned}$$

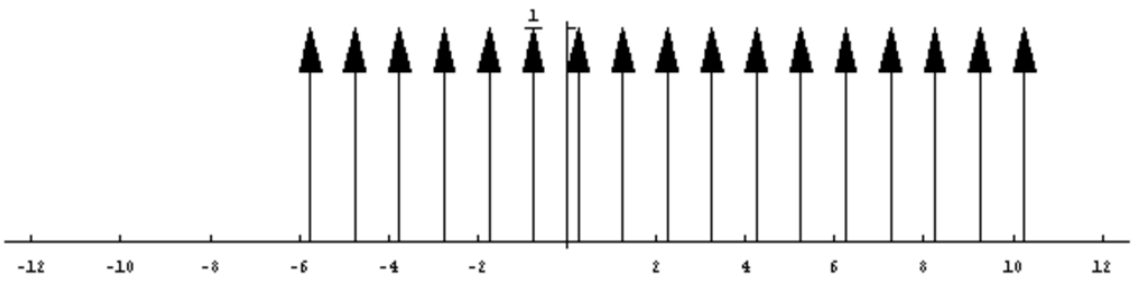
You can also see the "reduced" periodicity of $(1/2)(\text{Ш}(s - 9/4) + \text{Ш}(s + 9/4))$ graphically from the way $\text{Ш}_1(s - 9/4)$ and $\text{Ш}_1(s + 9/4)$ line up, but this is a little trickier. Here's a plot of

$$\tfrac{1}{2} \sum_{k=-8}^{8} \delta\left(s - k - \tfrac{9}{4}\right) .$$

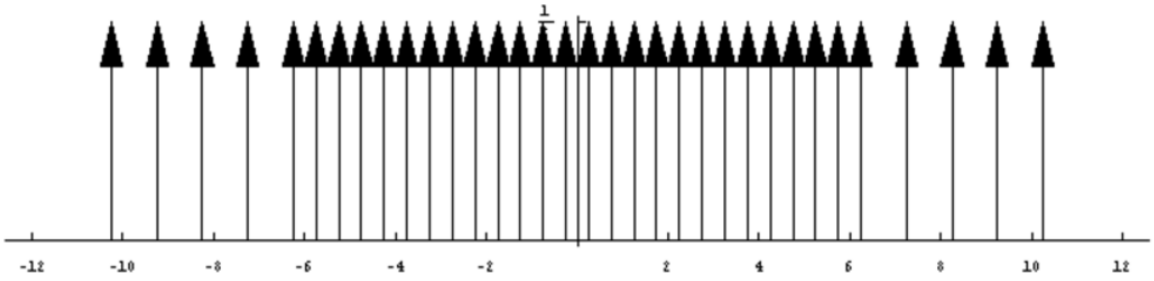


Here's a plot of

$$\tfrac{1}{2} \sum_{k=-8}^{8} \delta\left(s - k + \tfrac{9}{4}\right) .$$



Here's a plot of the sum of the two:



You see the periodicity starting to show up, but it's only there completely when we take the sum of the two, full, shifted Ш's.

I encourage you to work with $\cos 9\pi/2$ in the frequency domain using other sampling rates — see what periodizations look like and what happens when you cut off.

5.9.2 Trouble at the endpoints?

When talking about the technical definition of what it means for a signal to be bandlimited I mentioned "the endpoint problem", whether or not the Fourier transform is zero there, and how this sometimes requires special consideration. Here's what I had in mind. Take two very simple signals,

$$f(t) = \sin 2\pi t, \quad \text{and} \quad g(t) = \cos 2\pi t\,,$$

each of period 1. The Fourier transforms are

$$\mathcal{F}f(s) = \frac{1}{2i}\left(\delta(s-1) - \delta(s+1)\right) \quad \text{and} \quad \mathcal{F}g(s) = \frac{1}{2}\left(\delta(s-1) + \delta(s+1)\right)\,.$$

The bandwidth is 2 for each signal and the Fourier transforms are not zero at the endpoints $\pm p/2 = \pm 1$.

If we apply the sampling formula with $p = 2$ to $\sin 2\pi t$ we get the upsetting news that

$$\sin 2\pi t = \sum_{k=-\infty}^{\infty} \sin\tfrac{2\pi k}{2}\,\mathrm{sinc}(2t-k) = \sum_{k=-\infty}^{\infty} \sin k\pi\,\mathrm{sinc}(2t-k) = 0\,.$$

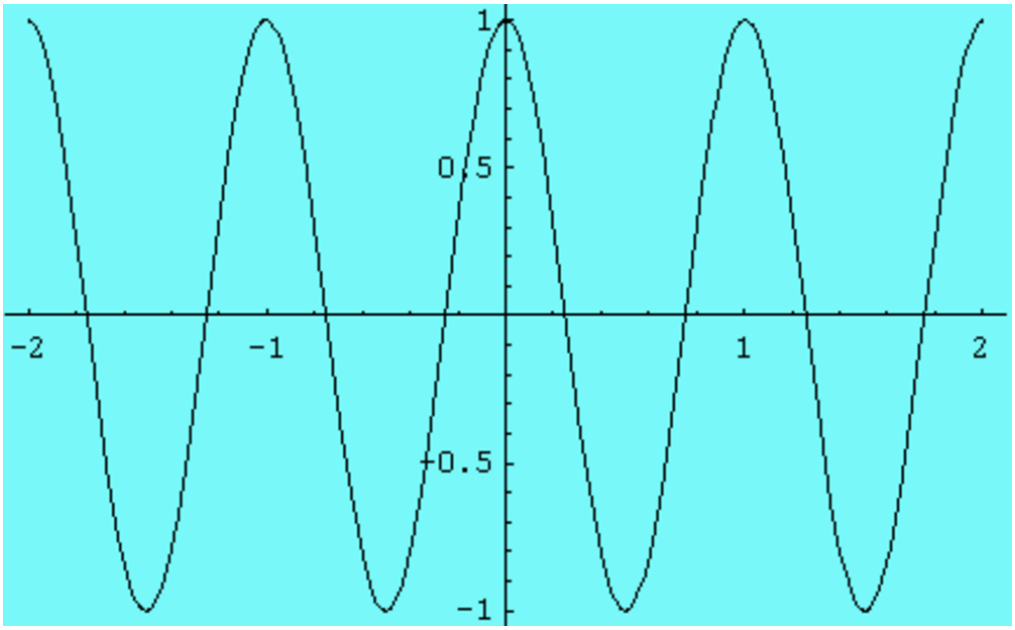
On the other hand, for $\cos 2\pi t$ the formula gives, again with $p = 2$,

$$\begin{aligned}\cos 2\pi t &= \sum_{k=-\infty}^{\infty} \cos\tfrac{2\pi k}{2}\,\mathrm{sinc}(2t-k) \\ &= \sum_{k=-\infty}^{\infty} \cos k\pi\,\mathrm{sinc}(2t-k) = \sum_{k=-\infty}^{\infty} \mathrm{sinc}(2t-2k) - \sum_{k=-\infty}^{\infty} \mathrm{sinc}(2t-2k-1)\,,\end{aligned}$$

which we might like to believe — at least both the series of sinc functions have period 1. Here's a plot of

$$\sum_{k=-50}^{50} \cos k\pi\,\mathrm{sinc}(2t-k)$$

for some further encouragement.



It's easy to see what goes wrong with the sampling formula in the example of $\sin 2\pi t$. The first step in the derivation is to periodize the Fourier transform, and for $\sin 2\pi t$ this results in

$$\begin{aligned}\mathcal{F}f * \text{Ш}_2 &= \frac{1}{2i}(\delta_1 - \delta_{-1}) * \sum_{k=-\infty}^{\infty} \delta_{2k} \\ &= \frac{1}{2i} \sum_{k=-\infty}^{\infty} (\delta_{2k+1} - \delta_{2k-1}) \;=\; 0\end{aligned}$$

because the series telescopes and the terms cancel. (I'm using the notation $\delta_a = \delta(t-a)$ here.)

For $\cos 2\pi t$ we find something different:

$$\begin{aligned}\mathcal{F}g * \text{Ш}_2 &= \frac{1}{2}(\delta_1 + \delta_{-1}) * \sum_{k=-\infty}^{\infty} \delta_{2k} \\ &= \frac{1}{2} \sum_{k=-\infty}^{\infty} (\delta_{2k+1} + \delta_{2k-1}) = \sum_{k=-\infty}^{\infty} \delta_{2k+1}\end{aligned}$$

because the series telescopes and the terms add. So far so good, but is it true that $\mathcal{F}g = \Pi_2(\mathcal{F}g * \text{Ш}_2)$ as needed in the second step of the derivation of the sampling formula? We are asking whether

$$\Pi_2 \cdot \text{Ш}_{2n+1} = \Pi_2 \cdot \sum_{n=-\infty}^{\infty} \delta_{2n+1} = \tfrac{1}{2}(\delta_1 + \delta_{-1})\,,$$

This is correct — cutting off a δ "at the edge" by a Π results in half the δ, as in the above. Stated generally,

$$\Pi_1 \delta_{1/2} = \tfrac{1}{2}\delta_{1/2}\,.$$

This property of cutting off δ's requires some extra justification and we won't go into it. But it works.

I chose $\sin 2\pi t$ and $\cos 2\pi t$ as simple examples illustrating the problems with the sampling rate right at the bandwidth. This is the extreme case. The signal $\sin 2\pi t$ is aliased to zero while $\cos 2\pi t$ is reconstructed

without a problem.[6] I'm hesitant to formulate a general principle here. I think it's best to say, as I did earlier, that any particular "endpoint problem" should call for special considerations.

Aliasing in general Aliasing is the "natural phenomenon" associated with the sampling formula. "Aliasing" is a somewhat generic term that refers to the following situation. Let $f(t)$ be a given signal. We sample $f(t)$ at a rate q samples/sec and write down the sampling series based on the sample values $f(t_k)$, $t_k = k/q$:

$$g(t) = \sum_{k=-\infty}^{\infty} f(t_k)\,\operatorname{sinc} q(t - t_k)\,.$$

I'm calling this signal g because, while the series produces some signal, if the sampling rate q is too slow (less than the bandwidth) it *will not* be the original signal f. However, remember that the sinc function satisfies

$$\begin{aligned}\operatorname{sinc} q(t_j - t_k) &= \frac{\sin q\pi(t_j - t_k)}{q\pi(t_j - t_k)} \\ &= \frac{\sin \pi(j-k)}{\pi(j-k)} \quad (\text{using } t_j = \frac{j}{q},\ t_k = \frac{k}{q}) \\ &= \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases}\end{aligned}$$

and hence $g(t)$ does have the same sample values as $f(t)$:

$$g(t_j) = \sum_{k=-\infty}^{\infty} f(t_k)\,\operatorname{sinc} q(t_j - t_k) = f(t_j)\,.$$

In this case we say that f and g are aliases. They cannot be distinguished based on their sampled values at the points $t_k = k/q$.

We saw some examples of how aliasing comes about for a signal of a single frequency. How does it come about in general? Just as we analyzed the phenomenon in the preceding examples, it's best to understand aliasing in general as a breakdown in the derivation of the sampling formula.

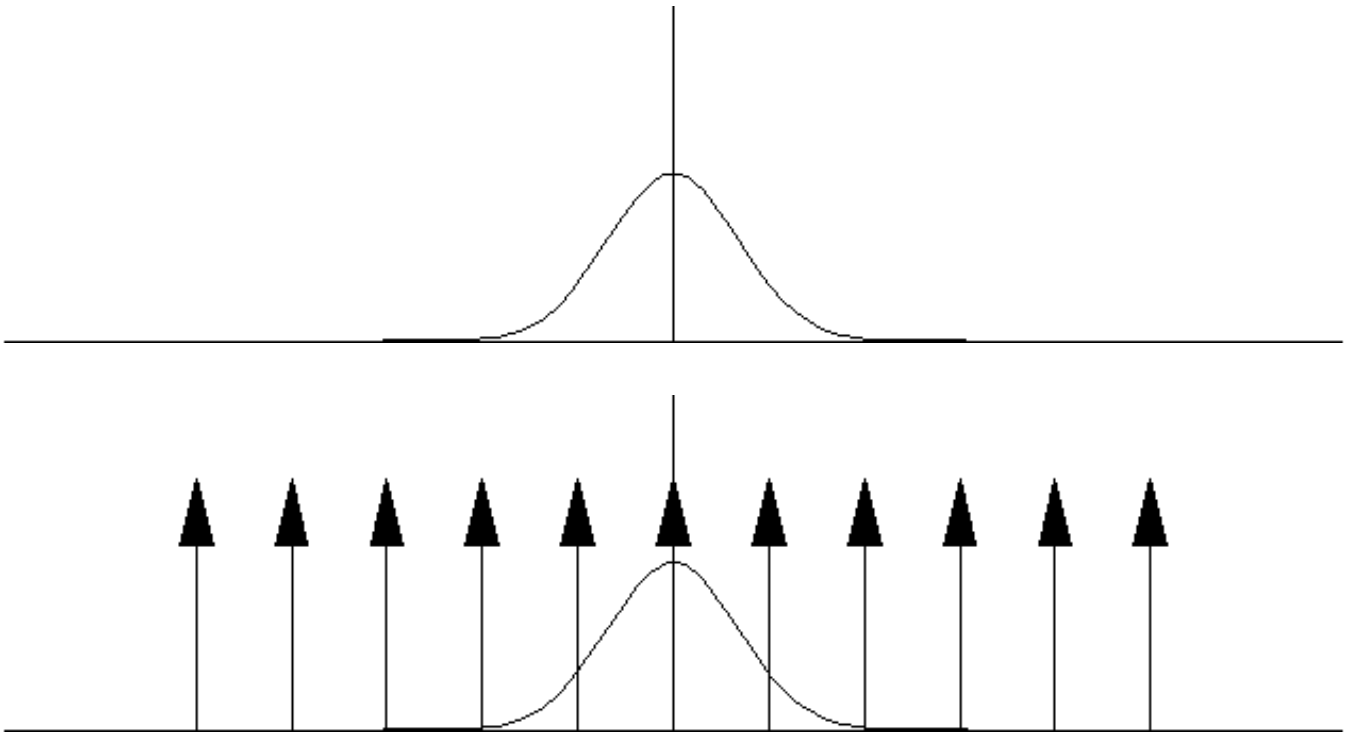
Here's what goes wrong if the sampling rate q is too low. Given a signal $f(t)$ the first step is to periodize $\mathcal{F}f$ by convolving it with Ш_q. To say the sampling rate is too slow is to say that q is less than the bandwidth, so here's a picture of the Fourier transform $\mathcal{F}f$ of a generic signal and a superimposed Ш_q — no convolution yet, just $\mathcal{F}f$ and Ш_q. The arrows are spaced q apart. The bandwidth is about $4q$ (about 4 arrows worth).

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[6] As an exercise, you can show that the result produced for the latter by the interpolation formula works:

$$\sum_{k=-\infty}^{\infty} \operatorname{sinc}(2t - 2k) - \sum_{k=-\infty}^{\infty} \operatorname{sinc}(2t - 2k - 1) = \cos 2\pi t, \quad \text{really}\,.$$

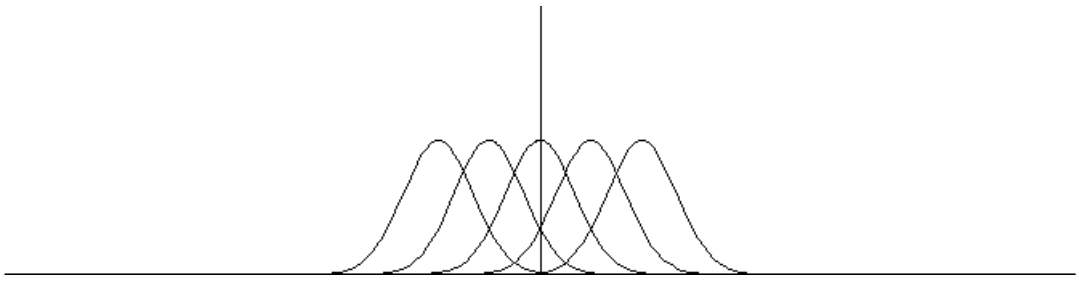
You can apply the sinc periodization lemma from the previous lecture.



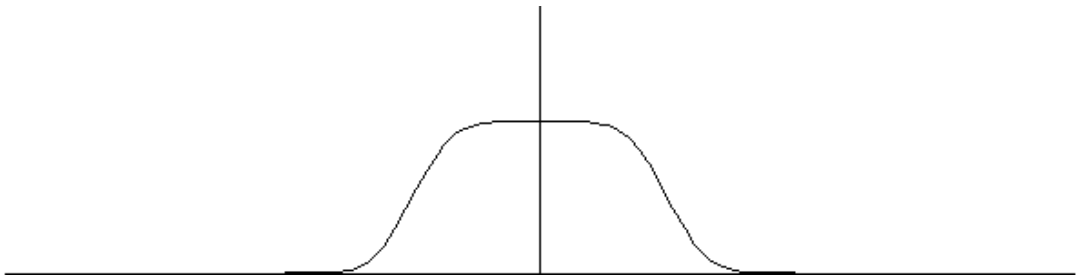
Now

$$(\mathcal{F}f * \text{Ш}_q)(s) = \sum_{k=-\infty}^{\infty} \mathcal{F}f(s - kq)$$

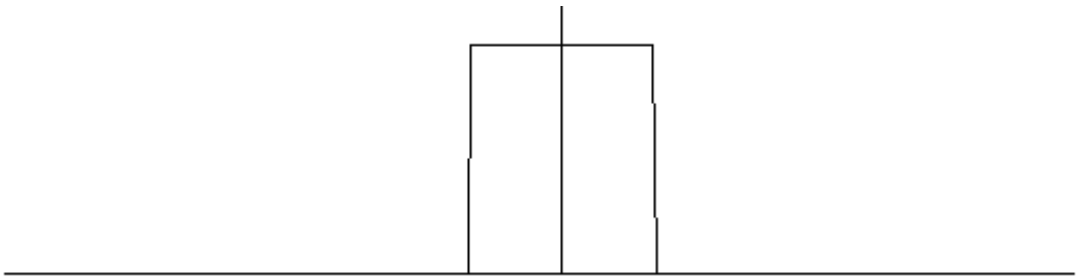
and here's a plot of $\mathcal{F}f(s+2q)$, $\mathcal{F}f(s+q)$, $\mathcal{F}f(s)$, $\mathcal{F}f(s-q)$, $\mathcal{F}f(s-2q)$.



Here's a plot of the sum:



Finally, here's a plot of the cut-off $\Pi_q(\mathcal{F}f * \text{Ш}_q)$:



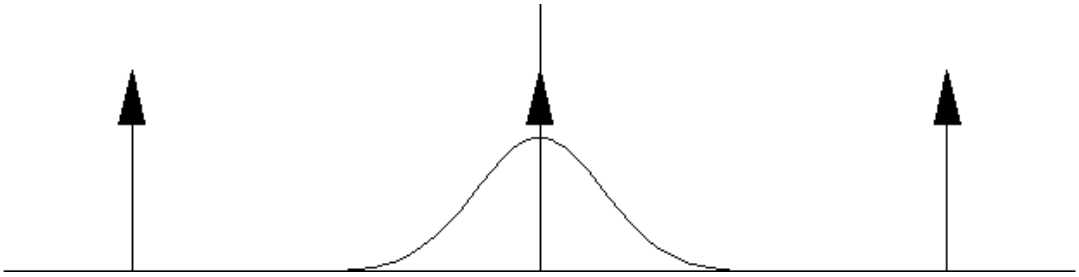
That's right, it looks pretty much like Π_q itself, because for $-q/2 < s < q/2$ the convolution $\mathcal{F}f * \text{Ш}_q$ is pretty much equal to 1. It sure doesn't look like the original $\mathcal{F}f$. We sure *don't* have

$$\mathcal{F}f = \Pi_q(\mathcal{F}f * \text{Ш}_q)\,,$$

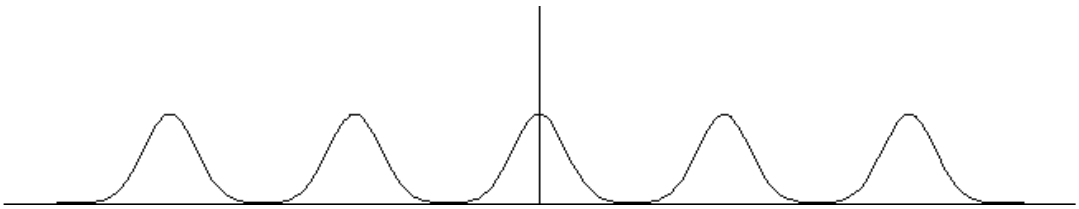
so taking the inverse Fourier transform sure *isn't* going to give back f. (It will give something that looks more like $q\,\text{sinc}\,qt$, in fact.)

What went wrong? You saw it in one of the pictures, above. See that little tail of $\mathcal{F}f$ roughly between the δ's at $\pm q$ and $\pm 2q$. Parts of that tail get shifted into the interval between $-q/2$ and $q/2$ by convolving $\mathcal{F}f$ with shifted δ's, and convolving $\mathcal{F}f$ with Ш_q adds up all those shifts. This is how aliasing happens.

Just so you have another picture of when sampling works, here's $\mathcal{F}f$ with Ш_p superimposed where p is the bandwidth. (Remember, the support of $\mathcal{F}f$ is between $-p/2$ and $+p/2$ — that's $\pm p$ *over* 2 — while the δ's in Ш_p are spaced p apart:



And here's the convolution $\mathcal{F}f * \text{Ш}_p$:



Cutting this off by Π_p will give back $\mathcal{F}f$. Taking the inverse Fourier transform will give back f in terms of its sampling series.

5.10 Appendix: How Special is Ш?

I'd like to elaborate on Ш as a kind of "fundamental object" in discussing periodicity. To begin with, let's show:

- If S is a periodic, tempered distribution of period 1 with $\mathcal{F}S = S$ then $S = c\text{Ш}$ for a constant $c \in \mathbf{R}$.

The argument goes like this. First we observe that S is even, for

$$\mathcal{F}S^- = \mathcal{F}^{-1}S = S = \mathcal{F}S\,,$$

whence

$$S^- = S\,.$$

Now expand S in a Fourier series

$$S = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n t}\,.$$

This is perfectly legit for periodic tempered distributions. Since S is even the coefficients c_n are real, and moreover $c_{-n} = c_n$. Using $\mathcal{F}S = S$ we may write

$$\sum_{n=-\infty}^{\infty} c_n e^{2\pi i n t} = S = \mathcal{F}S = \sum_{n=-\infty}^{\infty} c_n \delta_n\,.$$

We'll prove by induction that

$$c_n = c_0 \quad \text{for all } n \geq 0\,,$$

and hence that

$$S = c_0 \sum_{n=-\infty}^{\infty} \delta_n = c_0 \text{Ш}_1\,.$$

The assertion is trivial for $n = 0$. Suppose it has been proved for $n \leq N-1$. Let $p = 2N+1$ and consider the cut-off $\Pi_p \mathcal{F}S$. Then on the one hand

$$\Pi_p \mathcal{F}S = \sum_{n=-(N-1)}^{N-1} c_n\delta_n + (c_N\delta_N + c_{-N}\delta_{-N}) = c_0 \sum_{n=-(N-1)}^{N-1} \delta_n + c_N(\delta_N + \delta_{-N})$$

using the induction hypothesis. On the other hand,

$$\Pi_p \mathcal{F}S = \Pi_p S = \sum_{n=-\infty}^{\infty} c_n \Pi_p e^{2\pi i n s}.$$

Integrate both sides from $-\infty$ to ∞ (or pair with an appropriate test function — however you'd like to say it.) This results in

$$(2N-1)c_0 + 2c_N = \sum_{n=-\infty}^{\infty} c_n \int_{-N-\frac{1}{2}}^{N+\frac{1}{2}} e^{2\pi i n s}\,ds = (2N+1)c_0 = (2N-1)c_0 + 2c_0\,.$$

Thus

$$c_N = c_0\,,$$

completing the induction.

Remark The hypothesis that the period is 1 is necessary, since $\text{Ш}(2x) + \frac{1}{2}\text{Ш}(x/2)$ is periodic of period two and is its own Fourier transform. (In general, if f is even, then $f + \mathcal{F}f$ is its own Fourier transform.)

The preceding result coupled with the existence result $\mathcal{F}\text{Ш} = \text{Ш}$ provided by the Poisson summation formula, and with a normalization tacked on, results in:

> **Theorem** There is exactly one periodic, tempered distribution of period 1 that is equal to its Fourier transform and has Fourier coefficient $c_0 = 1$.

This is one of those happy "there is exactly one of these" theorems that indicate what one might take to be the essential building blocks of a subject. It's a little late in the day to decide to base all discussion of periodic functions on Ш, but there might be some things to do.

5.11 Appendix: Timelimited vs. Bandlimited Signals

Here's a more careful treatment of the result that a bandlimited signal cannot be timelimited. We'll actually prove a more general statement and perhaps I should have said that no *interesting* signal can be both timelimited and bandlimited, because here's what we'll show precisely:

- Suppose $f(t)$ is a bandlimited signal. If there is some interval $a < t < b$ on which $f(t)$ is identically zero, then $f(t)$ is identically zero for all t.

This is a tricky argument. f is bandlimited so $\mathcal{F}f(s)$ is zero, say, for $|s| \geq p/2$. The Fourier inversion formula says

$$f(t) = \int_{-\infty}^{\infty} \mathcal{F}f(s)e^{2\pi ist}\,ds = \int_{-p/2}^{p/2} \mathcal{F}f(s)e^{2\pi ist}\,ds\,.$$

(We assume the signal is such that Fourier inversion holds. You can take f to be a Schwartz function, but some more general signals will do.) Suppose $f(t)$ is zero for $a < t < b$. Then for t in this range,

$$\int_{-p/2}^{p/2} \mathcal{F}f(s)e^{2\pi ist}\,ds = 0\,.$$

Differentiate with respect to t under the integral. If we do this n-times we get

$$0 = \int_{-p/2}^{p/2} \mathcal{F}f(s)(2\pi is)^n e^{2\pi ist}\,ds = (2\pi i)^n \int_{-p/2}^{p/2} \mathcal{F}f(s)s^n e^{2\pi ist}\,ds\,,$$

so that

$$\int_{-p/2}^{p/2} \mathcal{F}f(s)s^n e^{2\pi ist}\,ds = 0\,.$$

Again, this holds for all t with $a < t < b$; pick one, say t_0. Then

$$\int_{-p/2}^{p/2} \mathcal{F}f(s)s^n e^{2\pi ist_0}\,ds = 0\,.$$

But now for *any* t (anywhere, not just between a and b) we can write

$$\begin{aligned} f(t) &= \int_{-p/2}^{p/2} \mathcal{F}f(s) e^{2\pi i s t}\, ds = \int_{-p/2}^{p/2} \mathcal{F}f(s) e^{2\pi i s(t-t_0)} e^{2\pi i s t_0}\, ds \\ &= \int_{-p/2}^{p/2} \sum_{n=0}^{\infty} \frac{(2\pi i(t-t_0))^n}{n!} s^n e^{2\pi i s t_0} \mathcal{F}f(s)\, ds \\ &\qquad \text{(using the Taylor series expansion for } e^{2\pi i s(t-t_0)}\text{)} \\ &= \sum_{n=0}^{\infty} \frac{(2\pi i(t-t_0))^n}{n!} \int_{-p/2}^{p/2} s^n e^{2\pi i s t_0} \mathcal{F}f(s)\, ds = \sum_{n=0}^{\infty} \frac{(2\pi i(t-t_0))^n}{n!} 0 \;=\; 0\,. \end{aligned}$$

Hence $f(t)$ is zero for all t.

The same argument *mutatis mutandis* will show:

- If $f(t)$ is timelimited and if $\mathcal{F}f(s)$ is identically zero on any interval $a < s < b$ then $\mathcal{F}f(s)$ is identically zero for all s.

Then $f(t)$ is identically zero, too, by Fourier inversions.

Remark 1, for eager seekers of knowledge This bandlimited vs. timelimited result is often proved by establishing a relationship between timelimited signals and analytic functions (of a complex variable), and then appealing to results from the theory of analytic functions. That connection opens up an important direction for applications of the Fourier transform, but we can't go there and the direct argument we just gave makes this approach unnecessary.

Remark 2, for overwrought math students and careful engineers Where in the preceding argument did we use that $p < \infty$? It's needed in switching integration and summation, in the line

$$\int_{-p/2}^{p/2} \sum_{n=0}^{\infty} \frac{(2\pi i(t-t_0))^n}{n!} s^n e^{2\pi i s t_0} \mathcal{F}f(s)\, ds = \sum_{n=0}^{\infty} \frac{(2\pi i(t-t_0))^n}{n!} \int_{-p/2}^{p/2} s^n e^{2\pi i s t_0} \mathcal{F}f(s)\, ds$$

The theorems that tell us "the integral of the sum is the sum of the integral" require as an essential hypothesis that the series converges *uniformly*. "Uniformly" means, loosely, that if we plug a particular value into the converging series we can estimate the rate at which the series converges *independent* of that particular value.[7] In the sum-and-integral expression, above, the variable s ranges over a finite interval, from $-p/2$ to $+p/2$. Over such a finite interval the series for the exponential converges uniformly, essentially because the terms can only get so big — so they can be estimated uniformly — when s can only get so big. We can switch integration and summation in this case. If, however, we had to work with

$$\int_{-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{(2\pi i(t-t_0))^n}{n!} s^n e^{2\pi i s t_0} \mathcal{F}f(s)\, ds\,,$$

i.e., if we did not have the assumption of bandlimitedness, then we could not make uniform estimates for the convergence of the series and switching integration and summation is not justified.

It's not only unjustified, it's really wrong. If we could drop the assumption that the signal is bandlimited we'd be "proving" the statement: If $f(t)$ is identically zero on an interval then it's identically zero. Think

[7] We can make "uniform" estimates, in other words. We saw this sort of thing in the notes on convergence of Fourier series.

of the implications of such a dramatic statement. In a phone conversation if you paused for a few seconds to collect your thoughts your signal would be identically zero on that interval of time, and therefore you would have nothing to say at all, ever again. Be careful.[8]

[8] However, if $f(t)$ is a real analytic signal, that is if it is given by a convergent power series at each point in its domain, then the implication: "$f(t)$ identically zero on an interval $\Rightarrow$ $f(t)$ identically zero everywhere" is true.

Chapter 6

Discrete Fourier Transform

The Modern World

According to some, the modern world began in 1965 when J. Cooley and J. Tukey published their account of an efficient method for numerical computation of the Fourier transform.[1] According to some others, the method was known to Gauss in the mid 1800s; the idea that lies at the heart of the algorithm is clearly present in an unpublished paper that appeared posthumously in 1866. Take your pick.

Whatever the history, the present and future demands are that we process continuous signals by discrete methods. Computers and digital processing systems can work with finite sums only. To turn the continuous into the discrete and finite requires that a signal be both time-limited and band-limited, something we know cannot be true, and that we take a finite number of samples, something we know cannot suffice. But it works. At least such approximations work to the extent that a large fraction of the world's economy depends upon them, and that's not a bad measure of success.

Some would argue that one shouldn't think in terms of "turning the continuous into the discrete", but rather that measurements and data in the real world come to us in discrete form, and that's how we should understand the world and work with it. Period. For some, "discrete" versus "continuous" rises to the level of a religious war, and this battle can be fought out in the different approaches to the discrete Fourier transform. I'm taking sides, at least initially, in favor of "from continuous to discrete" as a way of motivating the definition. For one thing, we have built up a lot of intuition and understanding for the Fourier transform, its properties, and its uses, and I hope some of that intuition can transfer as we now work with the discrete Fourier transform. My choice for now is to make the discrete look as similar to the continuous as possible.

6.1 From Continuous to Discrete

Start with a signal $f(t)$ and its Fourier transform $\mathcal{F}f(s)$, both functions of a continuous variable. We want to:

- Find a discrete version of $f(t)$ that's a reasonable approximation of $f(t)$.
- Find a discrete version of $\mathcal{F}f(s)$ that's a reasonable approximation of $\mathcal{F}f(s)$.

[1] Incidentally, Tukey is also credited with coining the term "bit" as an abbreviation for "binary digit" — how about that for immortality!

- Find a way that the discrete version of $\mathcal{F}f(s)$ is related to the discrete version of $f(t)$ that's a reasonable approximation to the way $\mathcal{F}f(s)$ is related to $f(t)$.

Good things to try for, but it's not quite straightforward. Here's the setup.

We suppose that $f(t)$ is zero outside of $0 \le t \le L$. We also suppose that the Fourier transform $\mathcal{F}f(s)$ is zero, or effectively zero (beyond our ability to measure, negligible energy, whatever) outside of $0 < s < 2B$. I'm taking the support of $\mathcal{F}f$ to be the interval from 0 to $2B$ instead of $-B$ to B because it will make the initial indexing of the sample points easier; this will not be an issue in the end. We'll also take L and B to both be integers so we don't have to round up or down in any of the considerations that follow; you can think of that as our first concession to the discrete.

Thus we are regarding $f(t)$ as both time-limited and band-limited, with the knowledge that this can only be approximately true. Remember, however, that we're ultimately going to come up with a definition of a discrete Fourier transform that will make sense in and of itself regardless of these shaky initial assumptions. After the definition is written down we could erase all that came before it, or merely cast a brief glance backwards from the discrete to the continuous with a few comments on how the former approximates the latter. Many treatments of the discrete Fourier transform that start with the discrete and stay with the discrete do just that. We're trying not to do that.

According to the sampling theorem (misapplied here, but play along), we can reconstruct $f(t)$ perfectly from its samples if we sample at the rate $2B$ samples per second. Since $f(t)$ is defined on an interval of length L and the samples are $1/2B$ apart, that means that we want a total of

$$N = \frac{L}{1/2B} = 2BL \quad \text{(note that } N \text{ is therefore even)}$$

evenly spaced samples, at the points

$$t_0 = 0,\, t_1 = \frac{1}{2B},\, t_2 = \frac{2}{2B}, \ldots,\, t_{N-1} = \frac{N-1}{2B}\,.$$

To know the values $f(t_k)$ is to know $f(t)$, reasonably well. Thus we state:

- The discrete version of $f(t)$ is the list of sampled values $f(t_0), f(t_1), \ldots, f(t_{N-1})$.

Next, represent the discrete version of $f(t)$ (the list of sampled values) "continuously" with the aid of a finite impulse train (a finite Ш-function) at the sample points:

$$\sum_{n=0}^{N-1} \delta(t - t_n)$$

that is,

$$f_{\text{discrete}}(t) = f(t) \sum_{n=0}^{N-1} \delta(t - t_n) = \sum_{n=0}^{N-1} f(t_n)\delta(t - t_n)\,.$$

This is what we have considered previously as the sampled form of $f(t)$. The Fourier transform of f_{discrete} is

$$\mathcal{F}f_{\text{discrete}}(s) = \sum_{n=0}^{N-1} f(t_n)\mathcal{F}\delta(t - t_n) = \sum_{n=0}^{N-1} f(t_n)e^{-2\pi i s t_n}\,.$$

This is close to what we want — it's the continuous Fourier transform of the sampled form of $f(t)$.

Now let's change perspective and look at things in the frequency domain. The function $f(t)$ is limited to $0 \le t \le L$, and this determines a sampling rate for reconstructing $\mathcal{F}f(s)$ from *its* samples in the *frequency* domain. The sampling rate is $1/L$. (*Not* $2/L$: think about how you would derive the sampling formula when the function is nonzero over an interval from 0 to $p/2$ rather than $-p/2$ to $+p/2$.) We sample $\mathcal{F}f(s)$ over the interval from 0 to $2B$ in the frequency domain at points spaced $1/L$ apart. The number of sample points is

$$\frac{2B}{1/L} = 2BL = N$$

the same number of sample points as for $f(t)$. The sample points for $\mathcal{F}f(s)$ are of the form m/L, and there are N of them:

$$s_0 = 0,\ s_1 = \frac{1}{L},\ \ldots,\ s_{N-1} = \frac{N-1}{L}\,.$$

The discrete version of $\mathcal{F}f(s)$ that we take is *not* $\mathcal{F}f(s)$ evaluated at these sample points s_m. Rather, it is $\mathcal{F}f_{\text{discrete}}(s)$ evaluated at the sample points. We base the approximation of $\mathcal{F}f(s)$ on the discrete version of $f(t)$. To ease the notation write $F(s)$ for $\mathcal{F}f_{\text{discrete}}(s)$. Then:

- The discrete version of $\mathcal{F}f(s)$ is the list of values

$$F(s_0) = \sum_{n=0}^{N-1} f(t_n)e^{-2\pi i s_0 t_n},\ F(s_1) = \sum_{n=0}^{N-1} f(t_n)e^{-2\pi i s_1 t_n},\ \ldots,\ F(s_{N-1}) = \sum_{n=0}^{N-1} f(t_n)e^{-2\pi i s_{N-1} t_n}\,.$$

By this definition, we now have a way of going from the discrete version of $f(t)$ to the discrete version of $\mathcal{F}f(s)$, namely,

$$F(s_m) = \sum_{n=0}^{N-1} f(t_n)e^{-2\pi i s_m t_n}\,.$$

These sums, one for each m from $m = 0$ to $m = N-1$, are supposed to be an approximation to the Fourier transform going from $f(t)$ to $\mathcal{F}f(s)$. In what sense is this a discrete approximation to the Fourier transform? Here's one way of looking at it. Since $f(t)$ is timelimited to $0 \le t \le L$, we have

$$\mathcal{F}f(s) = \int_0^L e^{-2\pi i s t}\, f(t)\, dt\,.$$

Thus at the sample points s_m,

$$\mathcal{F}f(s_m) = \int_0^L e^{-2\pi i s_m t}\, f(t)\, dt\,.$$

and to know the values $\mathcal{F}f(s_m)$ is to know $\mathcal{F}f(s)$ reasonably well. Now use the sample points t_k for $f(t)$ to write a Riemann sum approximation for the integral. The spacing Δt of the points is $1/2B$, so

$$\mathcal{F}f(s_m) = \int_0^L f(t)e^{-2\pi i s_n t}\, dt \approx \sum_{n=0}^{N-1} f(t_n)e^{-2\pi i s_m t_n}\Delta t = \frac{1}{2B}\sum_{n=0}^{N-1} f(t_n)e^{-2\pi i s_m t_n} = \frac{1}{2B}F(s_m)\,.$$

This is the final point:

- Up to the factor $1/2B$, the values $F(s_m)$ provide an approximation to the values $\mathcal{F}f(s_m)$.

Writing a Riemann sum as an approximation to the integral defining $\mathcal{F}f(s_m)$ essentially discretizes the integral, and this is an alternate way of getting to the expression for $F(s_n)$, up to the factor $2B$. We short-circuited this route by working directly with $\mathcal{F}f_{\text{discrete}}(s)$.

You may find the "up to the factor $1/2B$" unfortunate in this part of the discussion, but it's in the nature of the subject. In fact, back in Chapter 2 we encountered a similar kind of "up to the factor ..." phenomenon when we obtained the Fourier transform as a limit of the Fourier coefficients for a Fourier series.

We are almost ready for a definition, but there's one final comment to clear the way for that. Use the definition of the sample points

$$t_n = \frac{n}{2B}\,, \quad s_m = \frac{m}{L}$$

to write

$$F(s_m) = \sum_{n=0}^{N-1} f(t_n)e^{-2\pi i s_m t_n} = \sum_{n=0}^{N-1} f(t_n)e^{-2\pi i nm/2BL} = \sum_{n=0}^{N-1} f(t_n)e^{-2\pi i nm/N}\,.$$

This form of the exponential, $e^{-2\pi i nm/N}$, puts more emphasis on the index of the inputs (n) and outputs (m) and on the number of points (N) and "hides" the sample points themselves. That's the last step toward the discrete.

6.2 The Discrete Fourier Transform (DFT)

This development in the previous section suggests a general definition. Instead of thinking in terms of sampled values of a continuous signal and sampled value of its Fourier transform, we may think of the discrete Fourier transform as an operation that accepts as input a list of N numbers and returns as output a list of N numbers.

There are actually a number of things to say about the inputs and outputs of this operation, and we won't try to say them all at once. For the present, suffice it to say that we'll generally use the vector and "discrete signal" notation and write N-tuples as

$$\mathbf{f} = (\mathbf{f}[0], \mathbf{f}[1], \ldots, \mathbf{f}[N-1])\,.$$

I'll write vectors in boldface. If you want to use another notation, that's fine, but pick something — for much of this chapter you really will need a notation to distinguish a vector from a scalar. Note that the indexing goes from 0 to $N-1$ rather than from 1 to N. This is one of the things we'll comment on later.

Here's the definition of the discrete Fourier transform.

- Let $\mathbf{f} = (\mathbf{f}[0], \mathbf{f}[1], \ldots, \mathbf{f}[N-1])$ be an N-tuple. The *discrete Fourier transform (DFT)* of $\mathbf{f}$ is the N-tuple $\mathbf{F} = (\mathbf{F}[0], \mathbf{F}[1], \ldots, \mathbf{F}[N-1])$ defined by

$$\mathbf{F}[m] = \sum_{n=0}^{N-1} \mathbf{f}[n]e^{-2\pi i mn/N}\,, \quad m = 0, 1, \ldots, N-1\,.$$

It's perfectly legitimate to let the inputs $\mathbf{f}[n]$ be complex numbers, though for applications to signals they'll typically be real. The computed values $\mathbf{F}[m]$ are complex, being sums of complex exponentials.

6.2.1 Notation and conventions 1

I said that I wanted to set things up to look as much like the continuous case as possible. A little notation can go a long way here.

First off, I want to take Matlab seriously. Part of Matlab's usefulness is to formulate operations and commands in terms of vectors and to operate componentwise on vectors with many of the ordinary rules of arithmetic. We'll do the same here. So, for example, if

$$\mathbf{x} = (\mathbf{x}[0], \mathbf{x}[1], \dots, \mathbf{x}[N-1]) \quad \text{and} \quad \mathbf{y} = (\mathbf{y}[0], \mathbf{y}[1], \dots, \mathbf{y}[N-1])$$

then, by definition,

$$\mathbf{x}\,\mathbf{y} = (\mathbf{x}[0]\mathbf{y}[0], \mathbf{x}[1]\mathbf{y}[1], \dots, \mathbf{x}[N-1]\mathbf{y}[N-1])$$

(componentwise product of the vectors, not to be confused with the dot product)

$$\frac{\mathbf{x}}{\mathbf{y}} = \left(\frac{\mathbf{x}[0]}{\mathbf{y}[0]}, \frac{\mathbf{x}[1]}{\mathbf{y}[1]}, \dots, \frac{\mathbf{x}[N-1]}{\mathbf{y}[N-1]}\right) \quad \text{when the individual quotients make sense}$$

$$\mathbf{x}^p = (\mathbf{x}[0]^p, \mathbf{x}[1]^p, \dots, \mathbf{x}[N-1]^p) \quad \text{when the individual powers make sense}$$

and so on. These operations are all standard in Matlab. We even allow a function of one variable (think sine or cosine, for example) to operate *on a vector* componentwise via

$$f\big((\mathbf{x}[0], \mathbf{x}[1], \dots, \mathbf{x}[N-1])\big) = \big(f(\mathbf{x}[0]), f(\mathbf{x}[1]), \dots, f(\mathbf{x}[N-1])\big)\,.$$

We also use the Matlab notation $[r\,{:}\,s]$ for the tuple of numbers $(r, r+1, r+2, \dots, s)$ —very useful for indexing. Finally, we'll also write

$$\mathbf{0} = (0, 0, \dots, 0)$$

for the zero vector.

Vector complex exponential The definition of the discrete Fourier transform — like that of the continuous Fourier transform — involves a complex exponential. We'll write

$$\omega = e^{2\pi i/N}$$

and occasionally we'll decorate this to

$$\omega_N = e^{2\pi i/N}$$

when we want to emphasize the N in a particular formula. Note that

$$\operatorname{Re}\omega_N = \cos 2\pi/N\,, \quad \operatorname{Im}\omega_N = \sin 2\pi/N\,.$$

We have seen ω_N in various places in our work. It's an N-th root of unity, meaning

$$\omega_N^N = e^{2\pi i N/N} = e^{2\pi i} = 1\,.$$

Then for any integer n

$$\omega_N^{Nn} = 1$$

and in general, for any integers n and k

$$\omega_N^{Nn+k} = \omega_N^k\,.$$

This comes up often enough that it's worth pointing out. Also note that when N is even

$$\omega_N^{N/2} = e^{2\pi i N/2N} = e^{i\pi} = -1 \quad \text{and hence} \quad \omega_N^{kN/2} = (-1)^k\,.$$

Finally, some people write $\omega_N = e^{-2\pi i n/N}$ (minus instead of plus in the exponential) and $W = e^{2\pi i n/N}$. Be aware of that if you peruse the literature.

For the discrete Fourier transform, where vectors are naturally involved, it's helpful to introduce a complex exponential vector. There are N distinct N-th roots of unity, corresponding to the N powers of ω_N:

$$1 = \omega_N^0, \omega_N^1, \omega_N^2, \dots, \omega_N^{N-1}\,.$$

We let

$$\boldsymbol{\omega} = (1, \omega, \omega^2, \dots, \omega^{N-1})$$

be the vector in $\mathbf{C}^N$ consisting of the N distinct powers of ω. (Drop the subscript N now — it's understood.) Be careful to keep the underline notation here (or whatever notation you like).

The vector real and imaginary parts of $\boldsymbol{\omega}$ are

$$\operatorname{Re}\boldsymbol{\omega} = \cos\left(\frac{2\pi}{N}[0 : N-1]\right), \quad \operatorname{Im}\boldsymbol{\omega} = \sin\left(\frac{2\pi}{N}[0 : N-1]\right).$$

See how that works:

$$\cos\left(\frac{2\pi}{N}[0 : N-1]\right) \quad \text{is short for} \quad \left(1, \cos\frac{2\pi}{N}, \cos\frac{4\pi}{N}, \dots, \cos\frac{2\pi(N-1)}{N}\right).$$

Also important are the powers of $\boldsymbol{\omega}$. We write

$$\boldsymbol{\omega}^k = (1, \omega^k, \omega^{2k}, \dots, \omega^{(N-1)k})$$

for the vector of k-th powers. Then also

$$\boldsymbol{\omega}^{-k} = (1, \omega^{-k}, \omega^{-2k}, \dots, \omega^{-(N-1)k})\,.$$

Note how we write the components:

$$\boldsymbol{\omega}^k[m] = \omega^{km}\,, \quad \boldsymbol{\omega}^{-k}[m] = \omega^{-km}\,.$$

(You can see why it's important here to use notations that, while similar, can distinguish a vector from a scalar.)

Taking powers of $\boldsymbol{\omega}$ is cyclic of order N, meaning that

$$\boldsymbol{\omega}^N = (1^N, e^{2\pi i N/N}, e^{4\pi i N/N}, \dots, e^{2\pi i(N-1)N/N}) = (1, 1, 1, \dots, 1)$$

For shorthand, we'll write

$$\mathbf{1} = (1, 1, \dots, 1)$$

for the vector of all 1's. Then, compactly,

$$\boldsymbol{\omega}^N = \mathbf{1}\,.$$

and

$$\boldsymbol{\omega}^{nN} = \mathbf{1} \quad \text{and} \quad \boldsymbol{\omega}^{nN+k} = \boldsymbol{\omega}^k$$

for any integers n and k.

Along with making the discrete case look like the continuous case goes making vector calculations look like scalar calculations.

The DFT in vector form Introducing the vector complex exponential allows us to write the formula defining the discrete Fourier transform in a way, and with a notation, that really looks like a discrete version of the continuous transform. The DFT is given by (defined by)

$$\underline{\mathcal{F}}\,\mathbf{f} = \sum_{k=0}^{N-1} \mathbf{f}[k]\boldsymbol{\omega}^{-k}\,.$$

I've underlined the $\mathcal{F}$ to show its vector character. We'll write $\underline{\mathcal{F}}_N$ if we need to call attention to the N.

To emphasize once again its nature, *the DFT of a vector is another vector.* The components of $\underline{\mathcal{F}}\,\mathbf{f}$ are the values of $\underline{\mathcal{F}}\,\mathbf{f}$ at the points $m = 0, 1, \ldots, N-1$:

$$\underline{\mathcal{F}}\,\mathbf{f}[m] = \sum_{k=0}^{N-1} \mathbf{f}[k]\boldsymbol{\omega}^{-k}[m] = \sum_{k=0}^{N-1} \mathbf{f}[k]\omega^{-km} = \sum_{k=0}^{N-1} \mathbf{f}[k]e^{-2\pi ikm/N}\,.$$

We note one special value:

$$\underline{\mathcal{F}}\,\mathbf{f}[0] = \sum_{k=0}^{N-1} \mathbf{f}[k]\boldsymbol{\omega}^{-k}[0] = \sum_{k=0}^{N-1} \mathbf{f}[k]$$

the sum of the components of the input $\mathbf{f}$. For this reason some people define the DFT with a $1/N$ in front, so that the zeroth component of the output is the average of the components of the input, like the zeroth Fourier coefficient of a periodic function is the average of the function over one period. We're not doing this.

The DFT in matrix form The DFT takes vectors to vectors, and it does so *linearly.* To state this formally as a property:

$$\underline{\mathcal{F}}\,(\mathbf{f}_1 + \mathbf{f}_2) = \underline{\mathcal{F}}\,\mathbf{f}_1 + \underline{\mathcal{F}}\,\mathbf{f}_2 \quad \text{and} \quad \underline{\mathcal{F}}\,(\alpha\mathbf{f}) = \alpha\underline{\mathcal{F}}\,\mathbf{f}\,.$$

Showing this is easy, and make sure you see why it's easy; e.g.,

$$\begin{aligned}\underline{\mathcal{F}}\,(\mathbf{f}_1 + \mathbf{f}_2) = \sum_{k=0}^{N-1} (\mathbf{f}_1 + \mathbf{f}_2)[k]\boldsymbol{\omega}^{-k} &= \sum_{k=0}^{N-1} (\mathbf{f}_1[k] + \mathbf{f}_2[k])\boldsymbol{\omega}^{-k} \\ &= \sum_{k=0}^{N-1} \mathbf{f}_1[k]\boldsymbol{\omega}^{-k} + \sum_{k=0}^{N-1} \mathbf{f}_2[k]\boldsymbol{\omega}^{-k} = \underline{\mathcal{F}}\,\mathbf{f}_1 + \underline{\mathcal{F}}\,\mathbf{f}_2\,.\end{aligned}$$

As a linear transformation from $\mathbf{C}^N$ to $\mathbf{C}^N$, the DFT, $\mathbf{F} = \underline{\mathcal{F}}\,\mathbf{f}$, is exactly the matrix equation

$$\begin{pmatrix} F[0] \\ F[1] \\ F[2] \\ \vdots \\ F[N-1] \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^{-1\cdot 1} & \omega^{-1\cdot 2} & \cdots & \omega^{-(N-1)} \\ 1 & \omega^{-2\cdot 1} & \omega^{-2\cdot 2} & \cdots & \omega^{-2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{-(N-1)\cdot 1} & \omega^{-(N-1)\cdot 2} & \cdots & \omega^{-(N-1)^2} \end{pmatrix} \begin{pmatrix} f[0] \\ f[1] \\ f[2] \\ \vdots \\ f[N-1] \end{pmatrix}.$$

That is, the discrete Fourier transform is the big old $N \times N$ matrix

$$\underline{\mathcal{F}} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \cdots & \omega^{-(N-1)} \\ 1 & \omega^{-2} & \omega^{-4} & \cdots & \omega^{-2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{-(N-1)} & \omega^{-2(N-1)} & \cdots & \omega^{-(N-1)^2} \end{pmatrix}.$$

Again notice that we take the indices for $\mathbf{f}$ and $\mathbf{F}$ to go from 0 to $N-1$ instead of from 1 to N. This is a standard convention in the subject, but it clashes with the conventions of matrix terminology, where for an $N \times N$ matrix we usually write the indices from 1 to N. Tough break, but it seems to be etched in stone — *be careful.*

Positive and negative frequencies The heading of this section may seem a little odd, since for $\mathbf{F} = \underline{\mathcal{F}}\,\mathbf{f}$ we index the output $\mathbf{F} = (\mathbf{F}[0], \mathbf{F}[1], \ldots, \mathbf{F}[N-1])$, and hence the points in the spectrum, from 0 to $N-1$ — no negative indices in sight. It will seem less odd when we discuss reindexing, but this requires a little more preparation. For now, and for use in the example to follow, there is something important to point out about the values of $\underline{\mathcal{F}}\,\mathbf{f}$ that is analogous to $\mathcal{F}f(-s) = \overline{\mathcal{F}f(s)}$ in the continuous case when $f(t)$ is real.

Suppose that N is even. (This was the case in our derivation of the formula for the DFT, and it's often assumed, though it's not necessary for the ultimate definition of $\underline{\mathcal{F}}$.) Suppose also that we consider real inputs $\mathbf{f} = (\mathbf{f}[0], \mathbf{f}[1], \ldots, \mathbf{f}[N-1])$. Something special happens at the midpoint, $N/2$, of the spectrum. We find

$$\begin{aligned}\underline{\mathcal{F}}\,\mathbf{f}[N/2] = \sum_{k=0}^{N-1} \mathbf{f}[k]\boldsymbol{\omega}^{-k}[N/2] &= \sum_{k=0}^{N-1} \mathbf{f}[k]\omega^{-kN/2} \\ &= \sum_{k=0}^{N-1} \mathbf{f}[k]e^{-\pi ik} = \sum_{k=0}^{N-1}(-1)^{-k}\mathbf{f}[k] \quad (\text{using } \omega^{N/2} = -1)\end{aligned}$$

The value of the transform at $N/2$ is $\underline{\mathcal{F}}\,\mathbf{f}[N/2]$ and is an alternating sum of the components of the input vector $\mathbf{f}$. In particular, $\underline{\mathcal{F}}\,\mathbf{f}[N/2]$ is *real.*

More than $\underline{\mathcal{F}}\,\mathbf{f}[N/2]$ being real, though, is that the spectrum "splits" at $N/2$. For a start, look at $\underline{\mathcal{F}}\,\mathbf{f}[(N/2)+1]$ and $\underline{\mathcal{F}}\,\mathbf{f}[(N/2)-1]$:

$$\underline{\mathcal{F}}\,\mathbf{f}[\frac{N}{2}+1] = \sum_{k=0}^{N-1} \mathbf{f}[k]\boldsymbol{\omega}^{-k}[\frac{N}{2}+1] = \sum_{k=0}^{N-1} \mathbf{f}[k]\omega^{-k}\omega^{-Nk/2} = \sum_{k=0}^{N-1} \mathbf{f}[k]\omega^{-k}(-1)^{-k}$$

$$\underline{\mathcal{F}}\,\mathbf{f}[\frac{N}{2}-1] = \sum_{k=0}^{N-1} \mathbf{f}[k]\boldsymbol{\omega}^{-k}[\frac{N}{2}-1] = \sum_{k=0}^{N-1} \mathbf{f}[k]\omega^{k}\omega^{-Nk/2} = \sum_{k=0}^{N-1} \mathbf{f}[k]\omega^{k}(-1)^{-k}$$

Comparing the two calculations we see that

$$\underline{\mathcal{F}}\,\mathbf{f}[\frac{N}{2}+1] = \overline{\underline{\mathcal{F}}\,\mathbf{f}[\frac{N}{2}-1]}\,.$$

Similarly, we get

$$\underline{\mathcal{F}}\,\mathbf{f}[\frac{N}{2}+2] = \sum_{k=0}^{N-1} \mathbf{f}[k]\omega^{-2k}(-1)^{-k}\,, \qquad \underline{\mathcal{F}}\,\mathbf{f}[\frac{N}{2}-2] = \sum_{k=0}^{N-1} \mathbf{f}[k]\omega^{2k}(-1)^{-k}$$

so that

$$\underline{\mathcal{F}}\,\mathbf{f}[\frac{N}{2}+2] = \overline{\underline{\mathcal{F}}\,\mathbf{f}[\frac{N}{2}-2]}\,.$$

This pattern persists down to

$$\underline{\mathcal{F}}\,\mathbf{f}[1] = \sum_{k=0}^{N-1} \mathbf{f}[k]\omega^{-k}$$

and

$$\underline{\mathcal{F}}\,\mathbf{f}[N-1] = \sum_{k=0}^{N-1} \mathbf{f}[k]\omega^{-k(N-1)} = \sum_{k=0}^{N-1} \mathbf{f}[k]\omega^{k}\omega^{-kN} = \sum_{k=0}^{N-1} \mathbf{f}[k]\omega^{k}$$

i.e., to

$$\underline{\mathcal{F}}\,\mathbf{f}[1] = \overline{\underline{\mathcal{F}}\,\mathbf{f}[N-1]}\,.$$

Here's where it stops; recall that $\underline{\mathcal{F}}\,\mathbf{f}[0]$ is the sum of the components of $\mathbf{f}$.

Because of this result (with an alternate explanation later in this chapter), when the spectrum is indexed from 0 to $N-1$ the convention is to say that the frequencies from $m=1$ to $m=N/2-1$ are the positive frequencies and those from $N/2+1$ to $N-1$ are the negative frequencies. Whatever adjectives one uses, the important upshot is that, for a real input $\mathbf{f}$, all the information in the spectrum is in the first component $\underline{\mathcal{F}}\,\mathbf{f}[0]$ (the "DC" component, the sum of components of the input), the components $\underline{\mathcal{F}}\,\mathbf{f}[1], \underline{\mathcal{F}}\,\mathbf{f}[2], \ldots, \underline{\mathcal{F}}\,\mathbf{f}[N/2-1]$, and the special value $\underline{\mathcal{F}}\,\mathbf{f}[N/2]$ (the alternating sum of the components of the input). The remaining components of $\underline{\mathcal{F}}\,\mathbf{f}$ are just the complex conjugates of those from 1 to $N/2-1$. As we'll see, this has practical importance.

6.3 Two Grids, Reciprocally Related

Refer back to our understanding that the DFT finds the sampled Fourier transform of a sampled signal. We have a grid of points in the time domain and a grid of points in the frequency domain where the discrete version of the signal and the discrete version of its Fourier transform are known. More precisely, shifting to the discrete point of view, the values of the signal at the points in the time domain are all we know about the signal and the values we compute according to the DFT formula are all we know about its transform.

In the time domain the signal is limited to an interval of length L. In the frequency domain the transform is limited to an interval of length $2B$. When you plot the discrete signal and its DFT (or rather, e.g., the magnitude of its DFT since the DFT is complex), you should (probably) plot over these intervals (but your software might not give you this for the DFT). The grid points in the time domain are spaced $1/2B$ apart. The grid points in the frequency domain are spaced $1/L$ apart, so note (again) that the spacing in one domain is determined by properties of the function in the other domain. The two grid spacings are related to the third quantity in the setup, the number of sample points, N. The equation is

$$N = 2BL$$

Any two of these quantities — B, L, or N — determine the third via this relationship. The equation is often written another way, in terms of the grid spacings. If $\Delta t = 1/2B$ is the grid spacing in the time domain and $\Delta\nu = 1/L$ is the grid spacing in the frequency domain, then

$$\frac{1}{N} = \Delta t \Delta\nu\,.$$

These two (equivalent) equations are referred to as the *reciprocity relations*. A thing to put into your head is that for a fixed number of sample points N making Δt small means making $\Delta\nu$ large, and vice versa.

Here's why all this is important. For a given problem you want to solve — for a given signal you want to analyze by taking the Fourier transform — you typically either *know* or *choose* two of:

- How long the signal lasts, i.e., how long you're willing to sit there taking measurements — that's L.
- How many measurements you make — that's N.
- How often you make a measurement — that's Δt.

Once two of these are determined everything else is set.

6.4 Appendix: Gauss's Problem

Finally, here's the problem Gauss considered on representing the orbit of an asteroid by a finite Fourier series. Gauss was interested in astronomy, as he was in everything else, and occupied himself for a period in calculating orbits of recently discovered asteroids. This led to two great computational techniques. One was taken up right away (least squares curve fitting) and the other was forgotten (the algorithm for efficient computation of Fourier coefficients — the fast Fourier transform — as mentioned above).

It was in calculating the orbit of Pallas that Gauss introduced methods that became the FFT algorithm. He had twelve data points for the orbit, relating the "ascension" θ, measured in degrees, and the "declination" X, measured in minutes of arc. It appears from the data that X depends periodically on θ, so the problem is to interpolate a finite Fourier series based on the twelve samples. Gauss considered a sum of the form

$$X = f(\theta) = a_0 + \sum_{k=1}^{5} \left[a_k \cos\left(\frac{2\pi k\theta}{360}\right) + b_k \sin\left(\frac{2\pi k\theta}{360}\right)\right] + a_6 \cos\left(\frac{12\pi\theta}{360}\right).$$

Here's the data:

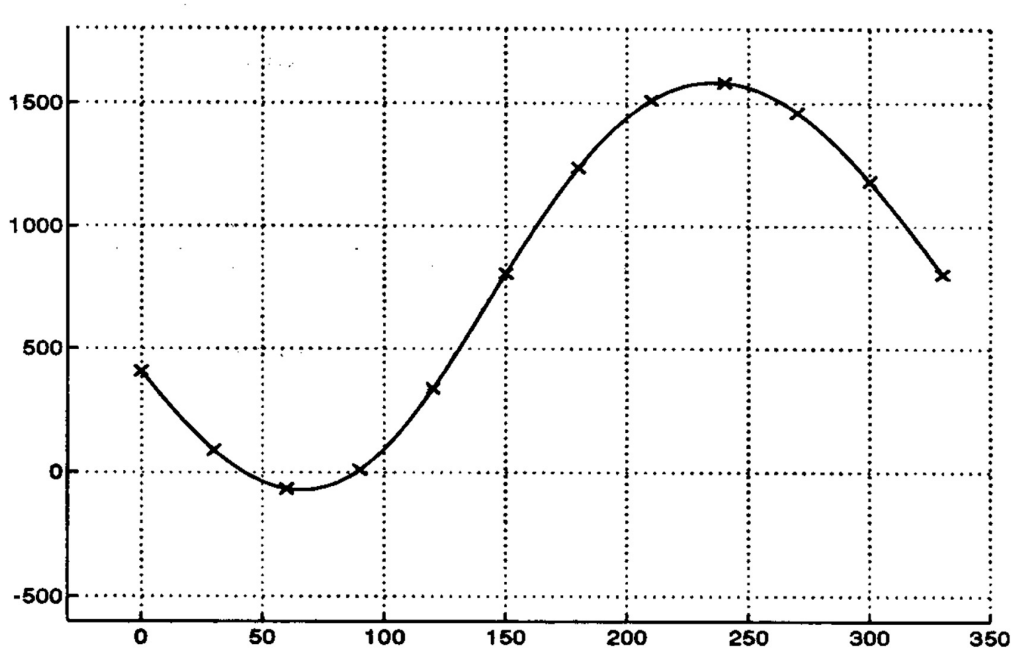
θ	0	30	60	90	120	150	180	210	240	270	300	330
X	408	89	−66	10	338	807	1238	1511	1583	1462	1183	804

How to determine the 12 unknown coefficients based on the samples $X_n = f(\theta_n)$? Want to take a whack at it? It requires solving 12 linear equations in 12 unknowns, something Gauss could have done by hand. Nevertheless, he was enough taken by the symmetries inherent in using sines and cosines to devise a scheme that rearranges the algebra in a way to reduce the total number of steps — essentially introducing a collection of easier subproblems whose solutions are later recombined to give the desired grand solutions. That rearrangement is what we'll talk about later, and I won't attempt to reconstruct Gauss's solution.

Here are the coefficients, as found by a modern FFT routine:

k	0	1	2	3	4	5	6
a_k	780.6	−411.0	43.4	−4.3	−1.1	0.3	0.1
b_k	–	−720.2	−2.2	5.5	−1.0	−0.3	–

More impressive is the graph, shown in the following figure.



6.5 Getting to Know Your Discrete Fourier Transform

We introduced the discrete Fourier transform (DFT) as a discrete approximation of the usual Fourier transform. The DFT takes an N-tuple $\mathbf{f} = (\mathbf{f}[0], \mathbf{f}[1], \ldots, \mathbf{f}[N-1])$ (the input) and returns an N-tuple $\mathbf{F} = (\mathbf{F}[0], \mathbf{F}[1], \ldots, \mathbf{F}[N-1])$ (the output) via the formula

$$\mathbf{F} = \underline{\mathcal{F}}\,\mathbf{f} = \sum_{k=0}^{N-1} \mathbf{f}[k]\boldsymbol{\omega}^{-k}$$

where $\boldsymbol{\omega}$ is the vector complex exponential,

$$\boldsymbol{\omega} = (1, \omega, \omega^2, \ldots, \omega^{N-1}), \quad \text{where} \quad \omega = e^{2\pi i/N}.$$

Evaluating $\underline{\mathcal{F}}\,\mathbf{f}$ at an index m gives the m-th component of the output

$$\mathbf{F}[m] = \sum_{k=0}^{N-1} \mathbf{f}[k]\omega^{-km} = \sum_{k=0}^{N-1} \mathbf{f}[k]e^{-2\pi ikm/N}$$

We also write $\underline{\mathcal{F}}$ as the $N \times N$ matrix,

$$\underline{\mathcal{F}} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \cdots & \omega^{-(N-1)} \\ 1 & \omega^{-2} & \omega^{-4} & \cdots & \omega^{-2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{-(N-1)} & \omega^{-2(N-1)} & \cdots & \omega^{-(N-1)^2} \end{pmatrix}.$$

The (m, n) entry is just ω^{-mn} where the rows and columns are indexed from 0 to $N-1$.

We want to develop some general properties of the DFT, much as we did when we first introduced the continuous Fourier transform. Most properties of the DFT correspond pretty directly to properties of the

continuous Fourier transform, though there are differences. You should try to make the most of these correspondences, if only to decide for yourself when they are close and when they are not. Use what you know!

Derivations using the DFT are often easier than those for the Fourier transform because there are no worries about convergence, but discrete derivations can have their own complications. The skills you need are in manipulating sums, particularly sums of complex exponentials, and in manipulating matrices. In both instances it's often a good idea to first calculate a few examples, say for DFTs of size three or four, to see what the patterns are.[2] We'll find formulas that are interesting and that are very important in applications.

6.6 Periodicity, Indexing, and Reindexing

Let me begin by highlighting a *difference* between the discrete and continuous cases rather than a similarity. The definition of the DFT suggests, even compels, some additional structure to the outputs and inputs. The output values $\mathbf{F}[m]$ are defined initially only for $m = 0$ to $m = N - 1$, but their definition as

$$\mathbf{F}[m] = \sum_{k=0}^{N-1} \mathbf{f}[k]\omega^{-km}$$

implies a periodicity property. Since

$$\omega^{-k(m+N)} = \omega^{-km}$$

we have

$$\sum_{k=0}^{N-1} \mathbf{f}[k]\omega^{-k(m+N)} = \sum_{k=0}^{N-1} \mathbf{f}[k]\omega^{-km} = \mathbf{F}[m]\,.$$

If we consider the left hand side as the DFT formula producing an output, then that output would be $\mathbf{F}[m + N]$. More generally, and by the same kind of calculation, we would have

$$\mathbf{F}[m + nN] = \mathbf{F}[m]$$

for any integer n. Thus, instead of just working with $\mathbf{F}$ as an N-tuple it's natural to "extend" it to be a periodic sequence of period N. For example, if we start off with $N = 4$ and the values $(\mathbf{F}[0], \mathbf{F}[1], \mathbf{F}[2], \mathbf{F}[3])$ then, by definition, the periodic extension of $\mathbf{F}$ has $\mathbf{F}[4] = F[0]$, $\mathbf{F}[5] = \mathbf{F}[1]$, $\mathbf{F}[6] = \mathbf{F}[2]$, and so on, and going in the other direction, $\mathbf{F}[-1] = \mathbf{F}[3]$, $\mathbf{F}[-2] = \mathbf{F}[2]$, and so on. In general,

$$\mathbf{F}[p] = \mathbf{F}[q] \quad \text{if } p - q \text{ is a multiple of } N, \text{ positive or negative.}$$

or put another way

$$\mathbf{F}[p] = \mathbf{F}[q] \quad \text{if } p \equiv q \bmod N.$$

We then have the formula

$$\mathbf{F}[m] = \sum_{k=0}^{N-1} \mathbf{f}[k]\omega^{-km}$$

for *all* integers m.

Because of these observations, and unless instructed otherwise:

[2] One thing to be mindful of in deriving formulas in the general case, in particular when working with sums, is what you call the "index of summation" (analogous to the "variable of integration") and not to get it confused or in conflict with other variables that are in use. Derivations might also involve "changing the variable of summation", analogous to "changing the variable of integration", a procedure that seems easier in the case of integrals than in sums, maybe just because of all the practice we've had with integrals.

- We will always assume that $\mathbf{F}$ is a periodic sequence of period N.

Once we define the inverse DFT it will emerge that an input $\mathbf{f}$ to the DFT also extends naturally to be a periodic sequence of period N. We'll also assume that, starting now.

- We will always assume that $\mathbf{f}$ is a periodic sequence of period N.

So, briefly, we assume that *all* our discrete signals are periodic. To take an important example, if, according to this dictum, we consider the vector complex exponential not just as a vector but as a periodic discrete signal then we can define it simply by

$$\boldsymbol{\omega}[n] = \omega^n, \quad n \text{ an integer.}$$

As for how this differs from the continuous case, we certainly *can* consider periodicity — that's what the subject of Fourier series is all about, after all — but when working with the Fourier transform we don't *have* to consider periodicity. In the discrete case we really do. Some things just don't work (e.g., convolution) if we don't work with periodic inputs and outputs.

If we were developing the DFT from a purely mathematical point of view, we would probably incorporate periodicity as part of the initial definition, and this is sometimes done. It would make some parts of the mathematical development a little smoother (though no different in substance), but I think on balance it's a mistake. It's extra baggage early on and can make the tie in with physical applications more awkward.

6.6.1 Notation and conventions 2

Having extended the inputs and outputs to be periodic sequences, it's mostly a matter of taste, or a preferred point of view, whether one then wants to think of an input to the DFT as a vector in $\mathbf{C}^N$ that is to be "extended periodically", as a discrete signal $\mathbf{f}\colon \mathbf{Z} \to \mathbf{C}$ that is periodic of period N, where $\mathbf{Z}$ stands for the integers. Each of these viewpoints can be helpful. For example, vectors are helpful if one wants to think of the DFT as a matrix, or when inner products are involved.

When confusion arises — and it does arise — it usually comes from the vector or N-tuple stance, and from questions and conventions on how vectors are indexed, whether from 0 to $N-1$ or from 1 to N, or other choices. In fact, indexing and reindexing the components of a DFT is something that just seems to come up — it certainly comes up in varied implementations of the DFT, and it's something you *have* to be able to handle if you use different packages or program any of these formulas.

The definition of the DFT that we've given is pretty standard, and it's the one we'll use. One sometimes finds an alternate definition of the DFT, used especially in imaging problems, where N is assumed to be even and the index set for *both* the inputs $\mathbf{f}$ and the outputs $\mathbf{F}$ is taken to be $[-(N/2)+1 : N/2] = (-(N/2)+1, -(N/2)+2, \ldots, -1, 0, 1, \ldots, N/2)$. The definition of the DFT is then:

$$\underline{\mathcal{F}}\,\mathbf{f} = \sum_{k=-N/2+1}^{N/2} \mathbf{f}[k]\boldsymbol{\omega}^{-k} \quad \text{or in components} \quad \mathbf{F}[m] = \sum_{k=-N/2+1}^{N/2} \mathbf{f}[k]\omega^{-km}\,.$$

We would be led to this indexing of the inputs and outputs if, in the sampling-based derivation we gave of the DFT in the previous lecture, we sampled on the time interval from $-L/2$ to $L/2$ and on the frequency interval from $-B$ to B. Then, using the index set $[-(N/2)+1 : N/2]$, the sample points in the time domain would be of the form

$$t_{-N/2+1} = -\frac{L}{2} + \frac{1}{2B} = \frac{-N/2+1}{2B}\,,\; t_{-N/2+2} = \frac{-N/2+2}{2B}\,, \ldots,\; t_{N/2} = \frac{-N/2+N}{2B} = \frac{N/2}{2B} = \frac{L}{2}\,,$$

and in the frequency domain of the form

$$s_{-N/2+1} = -B + \frac{1}{L} = \frac{-N/2+1}{L}\,,\ s_{-N/2+2} = \frac{-N/2+2}{L}\,,\ldots,\ s_{N/2} = \frac{-N/2+N}{L} = \frac{N/2}{L} = B\,.$$

(Maybe you can see why I didn't want to set things up this way for our first encounter.)

The "new" definition, above, of the DFT is completely equivalent to the first definition *because* of periodicity. There's a phrase one sees that's supposed to alleviate the tension over this and other similar sorts of things. It goes something like:

> "The DFT can be defined over any set of N consecutive indices."

What this means most often in practice is that we can write

$$\underline{\mathcal{F}}\,\mathbf{f} = \sum_{k=p}^{p+N-1} \mathbf{f}[k]\boldsymbol{\omega}^{-k}\,.$$

We'll explain this thoroughly in a later section. It's tedious, but not difficult. If you think of an input (or output) $\mathbf{f}$ as a periodic discrete signal (something your software package can't really do) then you don't have to worry about "how it's indexed". It goes on forever, and any block of N consecutive values, $\mathbf{f}[p], \mathbf{f}[p+1], \ldots, \mathbf{f}[p+N-1]$, should be as good as any other because the values of $\mathbf{f}$ repeat. You still have to establish the quoted remark, however, to be assured that finding the DFT gives the same result on any such block you need. This is essentially a discrete form of the statement for continuous periodic functions that the Fourier coefficients can be calculated by integrating over any period.

Positive and negative frequencies, again Let's tie up a loose end and see how periodicity makes honest negative frequencies correspond to negative frequencies "by convention". Suppose we have a periodic input $\mathbf{f}$ and output $\mathbf{F} = \underline{\mathcal{F}}\,\mathbf{f}$ indexed from $-(N/2)+1$ to $N/2$. We would certainly say in this case that the negative frequencies go from $-(N/2)+1$ to -1, with corresponding outputs $\mathbf{F}[-(N/2)+1]$, $\mathbf{F}[-(N/2)+2]$, $\ldots$ $\mathbf{F}[-1]$. Where do these frequencies go if we "reindex" from 0 to $N-1$? Using periodicity,

$$\begin{aligned}
\mathbf{F}[-\frac{N}{2}+1] &= \mathbf{F}[-\frac{N}{2}+1+N] = \mathbf{F}[\frac{N}{2}+1] \\
\mathbf{F}[-\frac{N}{2}+2] &= \mathbf{F}[-\frac{N}{2}+2+N] = \mathbf{F}[\frac{N}{2}+2] \\
&\quad \text{and so on up to} \\
\mathbf{F}[-1] &= \mathbf{F}[-1+N]
\end{aligned}$$

The "honest" negative frequencies at $-(N/2)+1, \ldots, -1$, are by periodicity the "negative frequencies by convention" at $N/2+1, \ldots, N-1$.

6.7 Inverting the DFT and Many Other Things Along the Way

By now it should be second nature to you to expect that any (useful) transform ought to have an inverse transform. The DFT is no exception. The DFT does have an inverse, and the formula for the inverse is quite simple and is very similar to the formula for the DFT itself, (almost) just like the continuous case. The key to inversion is the "discrete orthogonality" of the complex exponentials. We're going to look at the problem of finding $\underline{\mathcal{F}}^{-1}$ from both the vector point of view and the matrix point of view, with more emphasis on the former. You can take your pick which you prefer, but it's helpful to know both.

6.7.1 The discrete δ

Good news! No need to go through the theory of distributions to define a δ in the discrete case. We can do it directly and easily by setting

$$\boldsymbol{\delta}_0 = (1, 0, \dots, 0)\,.$$

In words, there's a 1 in the zeroth slot and 0's in the remaining $N-1$ slots. $\boldsymbol{\delta}_0$ is really just the first basis vector of $\mathbf{C}^N$ under an assumed name (the way we're indexing vectors from 0 to $N-1$), but to make comparisons to the continuous case we prefer to accord it independent status. We didn't specify N, and so, strictly speaking, there's a $\boldsymbol{\delta}_0$ for each N, but since $\boldsymbol{\delta}_0$ will always arise in a context where the N is otherwise specified we'll set aside that detail. As a periodic signal the definition of $\boldsymbol{\delta}_0$ is

$$\boldsymbol{\delta}_0[m] = \begin{cases} 1 & m \equiv 0 \bmod N, \\ 0 & \text{otherwise} \end{cases}$$

For the DFT of $\boldsymbol{\delta}_0$ we have

$$\underline{\mathcal{F}}\boldsymbol{\delta}_0 = \sum_{k=0}^{N-1} \boldsymbol{\delta}_0[k]\boldsymbol{\omega}^{-k} = \boldsymbol{\omega}^0 = \mathbf{1}\,.$$

Great — just like $\mathcal{F}\delta = 1$, and no tempered distributions in sight!

If we think of applying the DFT as matrix multiplication, then $\underline{\mathcal{F}}\boldsymbol{\delta}_0$ pulls out the first column, which is $\mathbf{1}$. (We may drop the subscript 0 on $\boldsymbol{\delta}_0$ if it's clear from the context.)

The shifted discrete $\boldsymbol{\delta}$ is just what you think it is,

$$\boldsymbol{\delta}_k = (0, \dots, 0, 1, 0, \dots, 0)$$

with a 1 in the k-th slot and zeros elsewhere. That is, the lowly k-th natural basis vector of $\mathbf{C}^N$ is now masquerading as the important $\boldsymbol{\delta}_k$ and we can write for an arbitrary vector $\mathbf{f}$,

$$\mathbf{f} = \sum_{k=0}^{N-1} \mathbf{f}[k]\boldsymbol{\delta}_k\,.$$

As a periodic discrete signal

$$\boldsymbol{\delta}_k[m] = \begin{cases} 1 & m \equiv k \mod N \\ 0 & \text{otherwise}\,. \end{cases}$$

Note, then, that if $\mathbf{f}$ is a periodic discrete signal we can still write

$$\mathbf{f} = \sum_{k=0}^{N-1} \mathbf{f}[k]\boldsymbol{\delta}_k\,.$$

The $\boldsymbol{\delta}_k$'s can be viewed as a basis for $\mathbf{C}^N$ and also as a basis for the N-periodic discrete signals.

For the DFT of $\boldsymbol{\delta}_k$ we have

$$\underline{\mathcal{F}}\boldsymbol{\delta}_k = \sum_{n=0}^{N-1} \boldsymbol{\delta}_k[n]\boldsymbol{\omega}^{-n} = \boldsymbol{\omega}^{-k}$$

From the matrix point of view, taking the DFT of $\boldsymbol{\delta}_k$ pulls out the k-th column of the DFT matrix, and that's $\boldsymbol{\omega}^{-k}$.

These are our first explicit transforms, and if we believe that the discrete case can be made to look like the continuous case, the results are encouraging. We state them again.

- The DFTs of the discrete $\boldsymbol{\delta}$ and shifted $\boldsymbol{\delta}$ are:

$$\underline{\mathcal{F}}\boldsymbol{\delta}_0 = \mathbf{1} \quad \text{and} \quad \underline{\mathcal{F}}\boldsymbol{\delta}_k = \boldsymbol{\omega}^{-k}\,.$$

We'll establish other properties of discrete $\boldsymbol{\delta}_k$'s (convolution, sampling) later.

To know $\underline{\mathcal{F}}$ on a basis Notice that the linearity of $\underline{\mathcal{F}}$ and the knowledge that $\underline{\mathcal{F}}\boldsymbol{\delta}_k = \boldsymbol{\omega}^{-k}$ recovers the general formula for $\underline{\mathcal{F}}$. Indeed,

$$\mathbf{f} = \sum_{k=0}^{N-1} \mathbf{f}[k]\boldsymbol{\delta}_k \;\Rightarrow\; \underline{\mathcal{F}}\mathbf{f} = \sum_{k=0}^{N-1} \mathbf{f}[k]\underline{\mathcal{F}}\boldsymbol{\delta}_k = \sum_{k=0}^{N-1} \mathbf{f}[k]\boldsymbol{\omega}^{-k}\,.$$

6.7.2 Orthogonality of the vector complex exponentials

Having introduced the vector complex exponential

$$\boldsymbol{\omega} = (1, \omega, \omega^2, \ldots, \omega^{N-1})$$

and its k-th power

$$\boldsymbol{\omega}^k = (1, \omega^k, \omega^{2k}, \ldots, \omega^{(N-1)k})$$

it is easy to formulate a key property:

If k and ℓ are any integers then

$$\boldsymbol{\omega}^k \cdot \boldsymbol{\omega}^\ell = \begin{cases} 0 & k \not\equiv \ell \bmod N \\ N & k \equiv \ell \bmod N \end{cases}$$

Thus the powers of the vector complex exponentials are "almost" orthonormal. We could make them orthonormal by considering instead

$$\frac{1}{\sqrt{N}}(1, \omega^k, \omega^{2k}, \ldots, \omega^{(N-1)k})\,,$$

but we won't do this.

In the continuous case the analogous result is that the family of functions $(1/\sqrt{T})e^{2\pi int/T}$ are orthonormal with respect to the inner product on $L^2([0,T])$:

$$\int_0^T \frac{1}{\sqrt{T}} e^{2\pi imt/T} \frac{1}{\sqrt{T}} e^{-2\pi int/T}\, dt = \frac{1}{T}\int_0^T e^{2\pi i(m-n)t/T}\, dt = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$$

Orthogonality in the continuous case is actually easier to establish than in the discrete case, because, sometimes, integration is easier than summation. However, you pretty much established the result in the

discrete case on your first problem set this quarter: the problem of adding up roots of unity is exactly what's involved, which is why I asked you to do that question way back then.

There are two things we'll need for the derivation. The first is that $\omega^{kN} = 1$ where k is any integer. The second is the sum of a finite geometric series, something we have used repeatedly:

$$1 + z + z^2 + \cdots + z^{N-1} = \begin{cases} \dfrac{1 - z^N}{1 - z} & z \neq 1 \\ N & z = 1 \end{cases}$$

We'll use this formula for $z = \omega$. In that case,

$$1 + \omega + \omega^2 + \ldots + \omega^{N-1} = \frac{1 - \omega^N}{1 - \omega} = \frac{0}{1 - \omega} = 0$$

which is the exercise you did on the first problem set. More generally, if k is not an integer multiple of N, so that $\omega^k \neq 1$ while $\omega^{kN} = 1$, then

$$1 + \omega^k + \omega^{2k} + \ldots \omega^{(N-1)k} = \frac{1 - \omega^{kN}}{1 - \omega^k} = 0\,,$$

while if k is an integer multiple of N then $\omega^k = 1$ and

$$1 + \omega^k + \omega^{2k} + \ldots \omega^{(N-1)k} = 1 + 1 + \ldots + 1 = N\,.$$

Succinctly,

$$1 + \omega^k + \omega^{2k} + \cdots + \omega^{(N-1)k} = \begin{cases} 0 & k \not\equiv 0 \bmod N \\ N & k \equiv 0 \bmod N \end{cases}$$

Let's compute the inner product $\boldsymbol{\omega}^k \cdot \boldsymbol{\omega}^\ell$:

$$\begin{aligned} \boldsymbol{\omega}^k \cdot \boldsymbol{\omega}^\ell &= \sum_{n=0}^{N-1} \boldsymbol{\omega}^k[n]\overline{\boldsymbol{\omega}^\ell[n]} \\ &= \sum_{n=0}^{N-1} \boldsymbol{\omega}^k[n]\boldsymbol{\omega}^{-\ell}[n] \quad \text{(taking Matlab seriously)} \\ &= \sum_{n=0}^{N-1} \boldsymbol{\omega}^{k-\ell}[n] \quad \text{(ditto)} \\ &= \sum_{n=0}^{N-1} \omega^{(k-\ell)n} \;= \begin{cases} 0 & k - \ell \not\equiv 0 \bmod N \\ N & k - \ell \equiv 0 \bmod N \end{cases} = \begin{cases} 0 & k \not\equiv \ell \bmod N \\ N & k \equiv \ell \bmod N \end{cases} \end{aligned}$$

Done.

Remark From this result we conclude that the N distinct vectors $\mathbf{1}$, $\boldsymbol{\omega}^{-1}$, $\boldsymbol{\omega}^{-2}$, $\ldots$, $\boldsymbol{\omega}^{-(N-1)}$ are a basis of $\mathbf{C}^N$ (and so are $\mathbf{1}$, $\boldsymbol{\omega}$, $\boldsymbol{\omega}^2$, $\ldots$, $\boldsymbol{\omega}^{N-1}$). From the earlier result that $\underline{\mathcal{F}}\,\boldsymbol{\delta}_k = \boldsymbol{\omega}^{-k}$ we then know that $\underline{\mathcal{F}}$ is invertible. This doesn't tell us what the inverse is, however. We have to work a little harder for that.

The DFT of the vector complex exponential With the orthogonality of the vector complex exponentials established, a number of other important results are now within easy reach. For example, we can now find $\underline{\mathcal{F}}\,\boldsymbol{\omega}^k$.

By definition,

$$\underline{\mathcal{F}}\boldsymbol{\omega}^k = \sum_{n=0}^{N-1} \boldsymbol{\omega}^k[n]\boldsymbol{\omega}^{-n}$$

and its ℓ-th component is then

$$\begin{aligned}\underline{\mathcal{F}}\boldsymbol{\omega}^k[\ell] &= \sum_{n=0}^{N-1} \boldsymbol{\omega}^k[n]\boldsymbol{\omega}^{-n}[\ell] \\ &= \sum_{n=0}^{N-1} \omega^{kn}\omega^{-n\ell} = \boldsymbol{\omega}^k \cdot \boldsymbol{\omega}^\ell \;=\; \begin{cases} 0 & k \not\equiv \ell \bmod N \\ N & k \equiv \ell \bmod N \end{cases}\end{aligned}$$

We recognize this, and we are pleased.

- The discrete Fourier transform of $\boldsymbol{\omega}^k$ is

$$\underline{\mathcal{F}}\boldsymbol{\omega}^k = N\boldsymbol{\delta}_k$$

Perhaps, we are *almost* pleased. There's a factor of N that comes in that we don't see, in any way, in the continuous case. Here it traces back, ultimately, to $||\boldsymbol{\omega}||^2 = N$.

The appearance of a factor N or $1/N$ in various formulas, *always* wired somehow to $||\boldsymbol{\omega}||^2 = N$, is one thing that makes the discrete case appear different from the continuous case, and it's a pain in the neck to keep straight. Be careful.

6.7.3 Reversed signals and their DFTs

For a discrete signal, $\mathbf{f}$, defined on the integers, periodic or not, the corresponding reversed signal, $\mathbf{f}^-$, is defined by

$$\mathbf{f}^-[m] = \mathbf{f}[-m]\,.$$

If $\mathbf{f}$ is periodic of period N, as we henceforth again assume, and we write it as the vector

$$\mathbf{f} = (\mathbf{f}[0], \mathbf{f}[1], \ldots, \mathbf{f}[N-1])\,.$$

Then

$$\mathbf{f}^- = (\mathbf{f}[N], \mathbf{f}[N-1], \ldots, \mathbf{f}[1]) \quad (\text{using } \mathbf{f}[N] = \mathbf{f}[0])$$

which makes the description of $\mathbf{f}^-$ as "reversed" even more apt (though, as in many irritating instances, the indexing is a little off). Defined directly in terms of its components this is

$$\mathbf{f}^-[n] = \mathbf{f}[N-n]$$

and this formula is good for all integers n. This description of $\mathbf{f}^-$ is often quite convenient. Note that reversing a signal satisfies the principle of superposition (is linear as an operation on signals):

$$(\mathbf{f}+\mathbf{g})^- = \mathbf{f}^- + \mathbf{g}^- \quad \text{and} \quad (\alpha\mathbf{f})^- = \alpha\mathbf{f}^-\,.$$

It's even more than that — we have

$$(\mathbf{f}\,\mathbf{g})^- = (\mathbf{f}^-)(\mathbf{g}^-)\,.$$

Let's consider two special cases of reversed signals. First, clearly

$$\boldsymbol{\delta}_0^- = \boldsymbol{\delta}_0$$

and though we'll pick up more on evenness and oddness later, this says that $\boldsymbol{\delta}_0$ is even. For the shifted $\boldsymbol{\delta}$,

$$\boldsymbol{\delta}_k^- = \boldsymbol{\delta}_{-k}\,.$$

I'll let you verify that. With this result we can write

$$\mathbf{f}^- = \left(\sum_{k=0}^{N-1} \mathbf{f}[k]\boldsymbol{\delta}_k\right)^- = \sum_{k=0}^{N-1} \mathbf{f}[k]\boldsymbol{\delta}_{-k}\,.$$

One might say that the $\boldsymbol{\delta}_k$ are a basis for the forward signals and the $\boldsymbol{\delta}_{-k}$ are a basis for the reversed signals.

Next let's look at $\boldsymbol{\omega}$. First we have,

$$\boldsymbol{\omega}^- = (\boldsymbol{\omega}[N], \boldsymbol{\omega}[N-1], \boldsymbol{\omega}[N-2], \ldots, \boldsymbol{\omega}[1]) = (1, \omega^{N-1}, \omega^{N-1}, \omega^{N-2}, \ldots, \omega)\,.$$

But now notice (as we could have noticed earlier) that

$$\omega^{N-1}\omega = \omega^N = 1 \;\Rightarrow\; \omega^{N-1} = \omega^{-1}\,.$$

Likewise

$$\omega^{N-2}\omega^2 = \omega^N = 1 \;\Rightarrow\; \omega^{N-2} = \omega^{-2}\,.$$

Continuing in this way we see, very attractively,

$$\boldsymbol{\omega}^- = (1, \omega^{-1}, \omega^{-2}, \ldots, \omega^{-(N-1)}) = \boldsymbol{\omega}^{-1}\,.$$

In the same way we find, equally attractively,

$$(\boldsymbol{\omega}^k)^- = \boldsymbol{\omega}^{-k}\,.$$

Of course then also

$$(\boldsymbol{\omega}^{-k})^- = \boldsymbol{\omega}^k\,.$$

This has an important consequence for the DFT — our first discrete "duality result". (Though we haven't yet introduced the inverse DFT, which is how one usually thinks about duality. It's the inverse DFT that we're pointing toward). Let's consider $\underline{\mathcal{F}}\,\mathbf{f}^-$, the DFT of the reversed signal. To work with the expression for $\underline{\mathcal{F}}\,\mathbf{f}^-$ we'll need to use periodicity of $\mathbf{f}$ and do a little fancy foot work changing the variable of summation in the definition of the DFT. Here's how it goes

$$\begin{aligned}
\underline{\mathcal{F}}\,\mathbf{f}^- &= \sum_{k=0}^{N-1} \mathbf{f}^-[k]\boldsymbol{\omega}^{-k} \\
&= \sum_{k=0}^{N-1} \mathbf{f}\,[N-k]\,\boldsymbol{\omega}^{-k} \quad \text{(reversing } \mathbf{f}\text{)} \\
&= \sum_{\ell=N}^{1} \mathbf{f}\,[\ell]\,\boldsymbol{\omega}^{\ell-N} \quad \text{(letting } \ell = N-k\text{)} \\
&= \sum_{\ell=N}^{1} \mathbf{f}\,[\ell]\,\boldsymbol{\omega}^{\ell} \quad \text{(since } \boldsymbol{\omega}^{-N} = 1\text{)}
\end{aligned}$$

But using $\mathbf{f}[N] = \mathbf{f}[0]$ and $\boldsymbol{\omega}^N = \boldsymbol{\omega}^0 = \mathbf{1}$ we can clearly write

$$\begin{aligned}\sum_{\ell=N}^{1} \mathbf{f}[\ell]\boldsymbol{\omega}^{\ell} &= \sum_{\ell=0}^{N-1} \mathbf{f}[\ell]\boldsymbol{\omega}^{\ell} \\ &= \left(\sum_{\ell=0}^{N-1} \mathbf{f}[\ell]\boldsymbol{\omega}^{-\ell}\right)^{-} = (\underline{\mathcal{F}}\,\mathbf{f})^{-}\,.\end{aligned}$$

We have shown $\underline{\mathcal{F}}\,\mathbf{f}^{-} = (\underline{\mathcal{F}}\,\mathbf{f})^{-}$ Cool. A little drawn out, but cool.

This then tells us that

$$\underline{\mathcal{F}}\,\boldsymbol{\omega}^{-k} = (\underline{\mathcal{F}}\,\boldsymbol{\omega}^{k})^{-} = (N\boldsymbol{\delta}_k)^{-} = N\boldsymbol{\delta}_{-k}\,.$$

In turn, from here we get a second duality result. Start with

$$\underline{\mathcal{F}}\,\mathbf{f} = \sum_{k=0}^{N-1} \mathbf{f}[k]\boldsymbol{\omega}^{-k}$$

and apply $\underline{\mathcal{F}}$ *again*. This produces

$$\underline{\mathcal{F}}\,\underline{\mathcal{F}}\,\mathbf{f} = \sum_{k=0}^{N-1} \mathbf{f}[k]\underline{\mathcal{F}}\,\boldsymbol{\omega}^{-k} = N\sum_{k=0} \mathbf{f}[k]\boldsymbol{\delta}_{-k} = N\mathbf{f}^{-}$$

To give the two results their own display:

- Duality relations for the DFT are:

$$\underline{\mathcal{F}}\,\mathbf{f}^{-} = (\underline{\mathcal{F}}\,\mathbf{f})^{-} \quad \text{and} \quad \underline{\mathcal{F}}\,\underline{\mathcal{F}}\,\mathbf{f} = N\mathbf{f}^{-}\,.$$

We'll do more on reversed signals, evenness and oddness, etc., but we've waited long enough for the big moment.

6.7.4 The inverse DFT

We take our cue for finding $\underline{\mathcal{F}}^{-1}$ from the duality results in the continuous case that say

$$\mathcal{F}^{-1}f = \mathcal{F}f^{-} = (\mathcal{F}f)^{-}\,.$$

The only thing we don't have in the discrete case is the definition of $\underline{\mathcal{F}}^{-1}$, and this equation tells us how we might try defining it. There's actually a factor of N coming in, and because I know what's going to happen I'll put it in now and *define*

$$\underline{\mathcal{F}}^{-1}\mathbf{f} = \frac{1}{N}\underline{\mathcal{F}}\,\mathbf{f}^{-}$$

and so equivalently

$$\underline{\mathcal{F}}^{-1}\mathbf{f} = \frac{1}{N}(\underline{\mathcal{F}}\,\mathbf{f})^{-} \quad \text{and also} \quad \underline{\mathcal{F}}^{-1}\mathbf{f} = \frac{1}{N}\sum_{n=0}^{N-1} \mathbf{f}[n]\boldsymbol{\omega}^{n}\,.$$

Let's see why this really does give us an inverse of $\underline{\mathcal{F}}$.

It's clear that $\underline{\mathcal{F}}^{-1}$ as defined this way is linear. We also need to know,

$$\begin{aligned}\underline{\mathcal{F}}^{-1}\boldsymbol{\omega}^{-k} &= \frac{1}{N}\underline{\mathcal{F}}(\boldsymbol{\omega}^{-k})^- \quad \text{(definition of } \underline{\mathcal{F}}^{-1}\text{)} \\ &= \frac{1}{N}\underline{\mathcal{F}}\boldsymbol{\omega}^{k} \quad \text{(using } (\boldsymbol{\omega}^{-k})^- = \boldsymbol{\omega}^{k}\text{)} \\ &= \frac{1}{N}N\boldsymbol{\delta}_k = \boldsymbol{\delta}_k\,.\end{aligned}$$

With this,

$$\begin{aligned}\underline{\mathcal{F}}^{-1}\underline{\mathcal{F}}\,\mathbf{f} &= \underline{\mathcal{F}}^{-1}\left(\sum_{k=0}^{N-1}\mathbf{f}[k]\boldsymbol{\omega}^{-k}\right) \\ &= \sum_{k=0}^{N-1}\mathbf{f}[k]\underline{\mathcal{F}}^{-1}\boldsymbol{\omega}^{-k} = \sum_{k=0}^{N-1}\mathbf{f}[k]\boldsymbol{\delta}_k = \mathbf{f}\,.\end{aligned}$$

A similar calculation shows that

$$\underline{\mathcal{F}}\,\underline{\mathcal{F}}^{-1}\mathbf{f} = \mathbf{f}\,.$$

Good show. We have shown that $\underline{\mathcal{F}}^{-1}$ really does give an inverse to $\underline{\mathcal{F}}$.

Most developments introduce the definition

$$\underline{\mathcal{F}}^{-1}\mathbf{f}[n] = \frac{1}{N}\sum_{k=0}^{N-1}\mathbf{f}[k]\omega^{kn}$$

directly, as a *deus ex machina*, and then proceed to discover the discrete orthogonality of the complex exponentials along the way to showing that this definition does give an inverse to $\underline{\mathcal{F}}$. There's nothing wrong with that — it sure saves time — but we took the longer route to see all those signposts marking the way that said "The discrete corresponds to the continuous. Use what you know."

Periodicity of the inputs We can now resolve the issue of the periodicity of the inputs to the DFT, something we left hanging. Remember that if $\mathbf{F} = \underline{\mathcal{F}}\,\mathbf{f}$ then the formula for the DFT pretty much forces us to extend $\mathbf{F}$ to be an N-periodic sequence. But now, since

$$\mathbf{f} = \underline{\mathcal{F}}^{-1}\mathbf{F}$$

it's clear that we *must* also consider the inputs $\mathbf{f}$ to be periodic. Which we've been doing. Enough said.

6.7.5 The inverse DFT in matrix form

It's possible to develop the properties of the DFT, and to obtain the inverse DFT, from the point of view of matrices. We won't do that — once is enough! — but it's worth pointing out what happens for $\underline{\mathcal{F}}^{-1}$ as a matrix.

We need to recall some notions from linear algebra. To give the definitions it's first necessary to remember that the transpose of a matrix A, denoted by A^T, is obtained by interchanging the rows and columns of A. If A is an $M \times N$ matrix then A^T is an $N \times M$ matrix, and in the case of a square matrix ($M = N$) taking the transpose amounts to reflecting the entries across the main diagonal. As a linear transformation, if $A\colon \mathbf{R}^N \to \mathbf{R}^M$ then $A^T\colon \mathbf{R}^M \to \mathbf{R}^N$. If, for shorthand, we write A generically in terms of its entries, as

in $A = (a_{ij})$, then we write $A^T = (a_{ji})$; note that the diagonal entries a_{ii}, where $i = j$, are unaffected by taking the transpose.

Square matrices can have a special property with respect to taking the transpose — they get to be symmetric: A square matrix A is *symmetric* if

$$A^T = A\,.$$

In words, interchanging the rows and columns gives back the same matrix — it's symmetric across the main diagonal. (The diagonal entries need not be equal to each other!)

A different notion also involving a matrix and its transpose is orthogonality. A square matrix is *orthogonal* if

$$A^T A = I\,,$$

where I is the identity matrix. Now, be careful; "symmetric" and "orthogonal" are *independent notions* for matrices. A matrix can be one and not the other.

For matrices with complex entries (operating on real or complex vectors) the more appropriate notion corresponding to simple symmetry in the real case is *Hermitian symmetry*. For this we form the transpose and take the complex conjugate of the entries. If A is a complex matrix then we use A^* to denote the conjugate transpose. A square matrix A is *Hermitian* if

$$A^* = A\,.$$

Finally, a square matrix is *unitary* if

$$A^* A = I\,.$$

Once again, "Hermitian" and "unitary" are independent notions for complex matrices. A matrix is unitary if and only if its columns (or rows) are orthonormal with respect to the complex dot product, and hence form an orthonormal basis.

It's this last point that is of relevance to the DFT. The DFT as a matrix is

$$\underline{\mathcal{F}} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \cdots & \omega^{-(N-1)} \\ 1 & \omega^{-2} & \omega^{-4} & \cdots & \omega^{-2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{-(N-1)} & \omega^{-2(N-1)} & \cdots & \omega^{-(N-1)^2} \end{pmatrix}$$

and the columns (and rows) are just the vectors $\boldsymbol{\omega}^k$ for k from 0 to $N-1$. But we know that the powers of the vector complex exponential are orthogonal, and nearly orthonormal up to the factor N. Thus $\underline{\mathcal{F}}$, as a matrix, is "nearly unitary", meaning that

$$\underline{\mathcal{F}}^*\,\underline{\mathcal{F}} = NI\,.$$

This identifies the inverse DFT as $(1/N)$ times the conjugate transpose of the DFT:

$$\underline{\mathcal{F}}^{-1} = \frac{1}{N}\,\underline{\mathcal{F}}^*\,.$$

As a matrix,

$$\frac{1}{N}\underline{\mathcal{F}}^* = \frac{1}{N}\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \bar{\omega}^{-1} & \bar{\omega}^{-2} & \cdots & \bar{\omega}^{-(N-1)} \\ 1 & \bar{\omega}^{-2} & \bar{\omega}^{-4} & \cdots & \bar{\omega}^{-2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \bar{\omega}^{-(N-1)} & \bar{\omega}^{-2(N-1)} & \cdots & \bar{\omega}^{-(N-1)^2} \end{pmatrix}^*$$
$$= \frac{1}{N}\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^{1} & \omega^{2} & \cdots & \omega^{(N-1)} \\ 1 & \omega^{2} & \omega^{4} & \cdots & \omega^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{(N-1)} & \omega^{2(N-1)} & \cdots & \omega^{(N-1)^2} \end{pmatrix}.$$

And if we look at the matrix product

$$\frac{1}{N}\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^{1} & \omega^{2} & \cdots & \omega^{(N-1)} \\ 1 & \omega^{2} & \omega^{4} & \cdots & \omega^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{(N-1)} & \omega^{2(N-1)} & \cdots & \omega^{(N-1)^2} \end{pmatrix}\begin{pmatrix} F[0] \\ F[1] \\ F[2] \\ \vdots \\ F[N-1] \end{pmatrix} = \begin{pmatrix} f[0] \\ f[1] \\ f[2] \\ \vdots \\ f[N-1] \end{pmatrix}$$

we get back the earlier formula for the inverse,

$$\mathbf{f}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \mathbf{F}[k]\omega^{kn}\,.$$

6.8 Properties of the DFT

We now want to go through a series of basic properties and formulas for the DFT that are analogous to those in the continuous case. This will mostly be a listing of results, often without much additional discussion. Use it as a reference.

Parseval's Identity There's a version of Parseval's identity for the DFT, featuring an extra factor of N that one has to keep track of:

$$\underline{\mathcal{F}}\,\mathbf{f}\cdot\underline{\mathcal{F}}\,\mathbf{g} = N(\mathbf{f}\cdot\mathbf{g})\,.$$

The derivation goes like this, using properties of the complex inner product and the orthogonality of the vector exponentials.

$$\underline{\mathcal{F}}\,\mathbf{f}\cdot\underline{\mathcal{F}}\,\mathbf{g} = \left(\sum_{k=0}^{N-1}\mathbf{f}[k]\boldsymbol{\omega}^{-k}\right)\cdot\left(\sum_{\ell=0}^{N-1}\mathbf{g}[\ell]\boldsymbol{\omega}^{-\ell}\right)$$
$$= \sum_{k=0}^{N-1}\sum_{\ell=0}^{N-1}\mathbf{f}[k]\,\overline{\mathbf{g}[\ell]}(\boldsymbol{\omega}^{-k}\cdot\boldsymbol{\omega}^{-\ell}) = N\sum_{k=0}^{N-1}\mathbf{f}[k]\overline{\mathbf{g}[k]} = N(\mathbf{f}\cdot\mathbf{g})\,.$$

If $\mathbf{f}=\mathbf{g}$ then the identity becomes

$$||\underline{\mathcal{F}}\,\mathbf{f}||^2 = N||\mathbf{f}||^2\,.$$

Parseval's identity is still another way of saying that $\underline{\mathcal{F}}$ is almost unitary as a matrix. A unitary matrix A has the property that

$$A\mathbf{f} \cdot A\mathbf{g} = \mathbf{f} \cdot \mathbf{g}\,,$$

that is, it *preserves* the inner product. $\underline{\mathcal{F}}$ almost does this.

Shifts and the shift theorem In formulating the shift theorem for the DFT it's helpful to introduce the delay operator for a discrete signal $\mathbf{f}$. For an integer p we define the signal $\tau_p\mathbf{f}$ by

$$\tau_p\mathbf{f}[n] = \mathbf{f}[n-p]\,.$$

The version of the shift theorem for the DFT looks just like its continuous cousin:

$$\underline{\mathcal{F}}\,(\tau_p\mathbf{f}) = \boldsymbol{\omega}^{-p}\underline{\mathcal{F}}\,\mathbf{f}\,.$$

The verification of this is a homework problem. Note that we need $\mathbf{f}$ to be periodic for shifting and the shift theorem to make sense.

The modulation theorem Modulation also works as in the continuous case. The modulation of a discrete signal $\mathbf{f}$ is, by definition, a signal

$$\begin{aligned}\boldsymbol{\omega}^n\mathbf{f} &= (1, \omega^n, \omega^{2n}, \ldots, \omega^{(N-1)n})\,(\mathbf{f}[0], \mathbf{f}[1], \mathbf{f}[2], \ldots, \mathbf{f}[N-1]) \quad \text{(componentwise product!)}\\ &= (\mathbf{f}[0], \omega^n\mathbf{f}[1], \omega^{2n}\mathbf{f}[2], \ldots, \omega^{(N-1)n}\mathbf{f}[N-1])\,.\end{aligned}$$

We can find $\underline{\mathcal{F}}\,(\boldsymbol{\omega}^n\,\mathbf{f})$ directly from the definition:

$$\underline{\mathcal{F}}\,(\boldsymbol{\omega}^n\mathbf{f}) = \sum_{k=0}^{N-1}\mathbf{f}[k]\omega^{kn}\boldsymbol{\omega}^{-k}$$

and so the m-th component is

$$\underline{\mathcal{F}}\,(\boldsymbol{\omega}^n\mathbf{f})[m] = \sum_{k=0}^{N-1}\mathbf{f}[k]\omega^{kn}\boldsymbol{\omega}^{-km} = \sum_{k=0}^{N-1}\mathbf{f}[k]\omega^{-k(m-n)}\,.$$

But if we shift $\underline{\mathcal{F}}\,\mathbf{f}$ by n we obtain

$$\tau_n(\underline{\mathcal{F}}\,\mathbf{f}) = \tau_n\left(\sum_{k=0}^{N-1}\mathbf{f}[k]\boldsymbol{\omega}^{-k}\right) = \sum_{k=0}^{N-1}\mathbf{f}[k]\,\tau_n\boldsymbol{\omega}^{-k}\,,$$

and the m-th component of the right hand side is

$$\left(\sum_{k=0}^{N-1}\mathbf{f}[k]\,\tau_n\boldsymbol{\omega}^{-k}\right)[m] = \sum_{k=0}^{N-1}\mathbf{f}[k]\,(\tau_n\boldsymbol{\omega}^{-k})[m] = \sum_{k=0}^{N-1}\mathbf{f}[k]\omega^{-k(m-n)}\,,$$

just as we had above. We conclude that

$$\underline{\mathcal{F}}\,(\boldsymbol{\omega}^n\mathbf{f}) = \tau_n(\underline{\mathcal{F}}\,\mathbf{f})\,.$$

This is the modulation theorem.

6.8.1 Convolution

Convolution and its use with the DFT is the basis of digital filtering. In considering how things should be defined, let me ask again the same question we asked in the continuous case: How can we use one signal to modify another? In the continuous case we discovered convolution in the time domain by looking at multiplication in the frequency domain, and we'll do the same thing now.

Given $\mathbf{F}$ and $\mathbf{G}$, we can consider their componentwise product $\mathbf{F}\,\mathbf{G}$. The question is:

- If $\mathbf{F} = \underline{\mathcal{F}}\,\mathbf{f}$ and $\mathbf{G} = \underline{\mathcal{F}}\,\mathbf{g}$ is there an $\mathbf{h}$ so that $\mathbf{F}\,\mathbf{G} = \underline{\mathcal{F}}\,\mathbf{h}$?

The technique to analyze this is to "interchange the order of summation", much as we often interchanged the order of integration (e.g., $dx\,dy$ instead of $dy\,dx$) in deriving formulas for Fourier integrals. We did exactly that in the process of coming up with convolution in the continuous setting.

For the DFT and the question we have posed:

$$
\begin{aligned}
(\underline{\mathcal{F}}^{-1}(\mathbf{F}\,\mathbf{G}))[m] &= \frac{1}{N}\sum_{n=0}^{N-1}\mathbf{F}[n]\,\mathbf{G}[n]\,\omega^{mn} \\
&= \frac{1}{N}\sum_{n=0}^{N-1}\left[\sum_{k=0}^{N-1}\mathbf{f}[k]\omega^{-kn}\right]\left[\sum_{\ell=0}^{N-1}\mathbf{g}[\ell]\omega^{-\ell n}\right]\omega^{mn} \\
&\qquad \text{(we collect the powers of } \omega\text{, all of which have an } n\text{)} \\
&= \sum_{k=0}^{N-1}\mathbf{f}[k]\sum_{\ell=0}^{N-1}\mathbf{g}[\ell]\left[\frac{1}{N}\sum_{n=0}^{N-1}\omega^{-kn}\omega^{-\ell n}\omega^{mn}\right] = \sum_{k=0}^{N-1}\mathbf{f}[k]\sum_{\ell=0}^{N-1}\mathbf{g}[\ell]\left[\frac{1}{N}\sum_{n=0}^{N-1}\omega^{n(m-k-\ell)}\right]
\end{aligned}
$$

Now look at the final sum in brackets. As in earlier calculations, this is a finite geometric series whose sum is N when $m-k-\ell \equiv 0 \bmod N$ and is zero if $m-k-\ell \not\equiv 0 \bmod N$. This takes the periodicity of the inputs and outputs into account, and we really must work modulo N in order to do that because $m-k-\ell$ could be less than 0 or bigger than $N-1$. Thus the final line above becomes

$$\sum_{k=0}^{N-1}\mathbf{f}[k]\,\mathbf{g}[m-k]\,.$$

Therefore, if

$$\mathbf{h}[m] = \sum_{k=0}^{N-1}\mathbf{f}[k]\,\mathbf{g}[m-k]\,, \quad m = 0,\ldots,N-1\,,$$

then $\underline{\mathcal{F}}\,\mathbf{h} = \mathbf{F}\,\mathbf{G}$. Again notice that the periodicity of $\mathbf{g}$ has to be used in defining $\mathbf{h}$, because the index on $\mathbf{g}$ will be negative for $m < k$. Also notice that $\mathbf{h}$ is periodic.

To summarize:

- **Convolution of discrete signals** Let $\mathbf{f}$ and $\mathbf{g}$ be periodic discrete signals. Define the *convolution* of $\mathbf{f}$ and $\mathbf{g}$ to be the periodic discrete signal $\mathbf{f} * \mathbf{g}$ where

$$(\mathbf{f} * \mathbf{g})[m] = \sum_{k=0}^{N-1}\mathbf{f}[k]\,\mathbf{g}[m-k]\,.$$

Then

$$\underline{\mathcal{F}}\,(\mathbf{f} * \mathbf{g}) = (\underline{\mathcal{F}}\,\mathbf{f})(\underline{\mathcal{F}}\,\mathbf{g})\,.$$

The product on the right hand side is the componentwise product of the DFT components.

You can verify (if you have the patience) the various algebraic properties of convolution, namely linearity, commutatively and associativity.

- It's also true that the DFT turns a product into a convolution:

$$\underline{\mathcal{F}}\,(\mathbf{f}\,\mathbf{g}) = \frac{1}{N}(\underline{\mathcal{F}}\,\mathbf{f} * \underline{\mathcal{F}}\,\mathbf{g})$$

(An extra factor of $1/N$. Agony.)

This equation can be derived from the first convolution property using duality. Let $\mathbf{F} = \underline{\mathcal{F}}^{-1}\mathbf{f}$ and $\mathbf{G} = uFT^{-1}\mathbf{g}$. Then $\mathbf{f} = \underline{\mathcal{F}}\,\mathbf{F},\ \mathbf{g} = \underline{\mathcal{F}}\,\mathbf{G}$ and

$$\mathbf{f}\,\mathbf{g} = (\underline{\mathcal{F}}\,\mathbf{F})(\underline{\mathcal{F}}\,\mathbf{G}) = \underline{\mathcal{F}}\,(\mathbf{F} * \mathbf{G})\,.$$

Hence

$$\underline{\mathcal{F}}\,(\mathbf{f}\,\mathbf{g}) = \underline{\mathcal{F}}\,\underline{\mathcal{F}}\,(\mathbf{F} * \mathbf{G}) = N(\mathbf{F} * \mathbf{G})^- = N(\frac{1}{N}(\underline{\mathcal{F}}\,\mathbf{f})^- * \frac{1}{N}(\underline{\mathcal{F}}\,\mathbf{g})^-)^- = \frac{1}{N}(\underline{\mathcal{F}}\,\mathbf{f} * \underline{\mathcal{F}}\,\mathbf{g})\,.$$

Shifts and convolution We note one general property combining convolution with delays, namely that the discrete shift works with discrete convolution just as it does in the continuous case:

$$\begin{aligned}((\tau_p\mathbf{f}) * \mathbf{g})[n] &= \sum_{k=0}^{N-1} \tau_p\mathbf{f}[n-k]\mathbf{g}[k] \\ &= \sum_{k=0}^{N-1} \mathbf{f}[n-k-p]\mathbf{g}[k] = (\mathbf{f} * \mathbf{g})[n-p] = \tau_p(\mathbf{f} * \mathbf{g})[n]\,.\end{aligned}$$

Thus, since convolution is commutative,

$$(\tau_p\mathbf{f}) * \mathbf{g} = \tau_p(\mathbf{f} * \mathbf{g}) = \mathbf{f} * (\tau_p\mathbf{g})\,.$$

6.8.2 More properties of $\boldsymbol{\delta}$

Two of the most useful properties of the continuous δ — if we can use the term "continuous" in connection with δ — are what it does when multiplied or convolved with a smooth function. For the discrete $\boldsymbol{\delta}$ we have similar results. For multiplication:

$$\mathbf{f}\boldsymbol{\delta}_0 = (\mathbf{f}[0]\cdot 1,\ \mathbf{f}[1]\cdot 0,\ \ldots,\ \mathbf{f}[N-1]\cdot 0) = (\mathbf{f}[0], 0, \ldots, 0) = \mathbf{f}[0]\boldsymbol{\delta}_0\,.$$

For convolution:

$$(\mathbf{f} * \boldsymbol{\delta}_0)[m] = \sum_{n=0}^{N-1} \mathbf{f}[m-n]\boldsymbol{\delta}_0[n]\,.$$

There are analogous properties for the shifted discrete $\boldsymbol{\delta}$.[3] For multiplication:

$$\mathbf{f}\boldsymbol{\delta}_k = \mathbf{f}[k]\boldsymbol{\delta}_k\,.$$

For convolution:

$$\mathbf{f} * \boldsymbol{\delta}_k = \mathbf{f} * \tau_k\boldsymbol{\delta}_0 = \tau_k(\mathbf{f} * \boldsymbol{\delta}_0) = \tau_k\mathbf{f}\,,$$

or in components

$$\mathbf{f} * \boldsymbol{\delta}_k[m] = \mathbf{f}[m-k]\,,$$

again in agreement with what we would expect from the continuous case.

There's more. Note that

$$\boldsymbol{\delta}_p\boldsymbol{\delta}_q = \begin{cases}\boldsymbol{\delta}_p & p = q\\ \mathbf{0} & p \neq q\end{cases}$$

and that

$$\boldsymbol{\delta}_p * \boldsymbol{\delta}_q = \boldsymbol{\delta}_{p+q}\,.$$

The former operation, multiplying δ's, is against the law in the continuous cases, but not in the discrete case. A cute observation making use of the convolution theorem is that

$$\boldsymbol{\omega}^p * \boldsymbol{\omega}^q = \begin{cases}N\boldsymbol{\omega}^p & p = q,\\ \mathbf{0} & p \neq q\end{cases}$$

Of course you can also see this directly, but it might not occur to you to look.

6.9 Different Definitions for the DFT

In this section I want to understand what is meant by the statement:

> "The DFT can be defined over any set of N consecutive indices."

For periodic functions $f(t)$ in the continuous setting, say of period 1, the n-th Fourier coefficient is

$$\hat{f}(n) = \int_0^1 e^{-2\pi int} f(t)\,dt\,.$$

The periodicity of f implies that $\hat{f}(n)$ can be obtained by integrating over any interval of length 1, however, and, morally, we're looking for the discrete version of that.

Take a good backward glance at what's been done, and "start over" now by giving a more general definition of the DFT:

> We consider discrete signals $\mathbf{f}\colon \mathbf{Z} \to \mathbf{C}$ that are periodic of period N. Let $\mathcal{P}$ and $\mathcal{Q}$ be "index sets" of N consecutive integers, say
>
> $$\mathcal{P} = [p : p+N-1], \quad \mathcal{Q} = [q : q+N-1]\,.$$

---

[3] Observe, incidentally, that

$$_k = \tau_k{}_0, \quad _k[m] = {}_0[m-k]$$

so a shifted discrete really does appear as delayed by k.

The DFT based on $\mathcal{P}$ and $\mathcal{Q}$ (the $(\mathcal{P},\mathcal{Q})$-DFT) is defined by

$$\underline{\mathcal{G}}\mathbf{f}[m] = \sum_{k\in\mathcal{P}} \mathbf{f}[k]\omega^{-km} = \sum_{k=p}^{p+N-1} \mathbf{f}[k]\omega^{-km}\,, \quad m\in\mathcal{Q}\,.$$

I've called the transform $\underline{\mathcal{G}}$ to distinguish it from $\underline{\mathcal{F}}$, which, in the present setup, is determined by the special choice of index sets $\mathcal{P}=\mathcal{Q}=[0:N-1]$.

The corresponding inverse transform is defined correspondingly — we'll work just with the forward transform.

Since $\mathbf{f}$ is periodic of period N, knowing $\mathbf{f}$ on any set of N consecutive numbers determines it everywhere, and the same should be true of a transform of $\mathbf{f}$. Thus one wants to establish that the definition of $\underline{\mathcal{G}}$ is independent of the choice of $\mathcal{P}$ and $\mathcal{Q}$. This is a sharper version of the informal statement in the first quote, above, but we have to say what "independent of the choice of $\mathcal{P}$" means.

For this, as a first step we extend $\underline{\mathcal{G}}\mathbf{f}$ to be periodic of period N. Since the exponentials in the definition of the $(\mathcal{P},\mathcal{Q})$-transform are periodic of period N, the extension of $\underline{\mathcal{G}}\mathbf{f}$ to be defined at any $m\in\mathbf{Z}$ is again given by the formula for $\underline{\mathcal{G}}\mathbf{f}$. To wit: Let $m\in\mathbf{Z}$ and write $m=n+kN$. Then

$$\begin{aligned}\sum_{\ell=p}^{p+N-1} \mathbf{f}[\ell]\omega^{-m\ell} &= \sum_{\ell=p}^{p+N-1} \mathbf{f}[\ell]\omega^{-(n+kN)\ell} \\ &= \sum_{\ell=p}^{p+N-1} \mathbf{f}[\ell]\omega^{-n\ell} = \underline{\mathcal{G}}\mathbf{f}[n] = \underline{\mathcal{G}}\mathbf{f}[n+kN] = \underline{\mathcal{G}}\mathbf{f}[m]\end{aligned}$$

The penultimate equality holds because we've extended $\underline{\mathcal{G}}\mathbf{f}$ to be periodic. Reading from bottom right to top left then shows that

$$\underline{\mathcal{G}}\mathbf{f}[m] = \sum_{\ell=p}^{p+N-1} \mathbf{f}[\ell]\omega^{-m\ell}\,.$$

for all $m\in\mathbf{Z}$. (All $m\in\mathbf{Z}$. Same formula. That's the point.) And *now* we want to show that

$$\underline{\mathcal{G}}\mathbf{f}[m] = \underline{\mathcal{F}}\,\mathbf{f}[m]$$

for all $m\in\mathbf{Z}$. (Observe that $\underline{\mathcal{F}}\,\mathbf{f}[m]$ on the right-hand-side is defined for any $m\in\mathbf{Z}$, as we've just seen for a general transform.) In other words, *any* DFT is *the* DFT, the one given by $\underline{\mathcal{F}}$ defined using the index set $[0:N-1]$. This, finally, is *the* precise meaning of the phrase "The DFT can be defined over any set of N consecutive indices." The point was to get to this statement — the claim that $\underline{\mathcal{G}}\mathbf{f}[m]=\underline{\mathcal{F}}\,\mathbf{f}[m]$ itself is pretty easy to establish.

Looking back on the development to this point, we haven't yet made any use of the assumption that $\mathbf{f}$ is periodic. That comes in now, in the final part of the argument. Write $p=rN+s$, where r and s are integers and $s\in[0:N-1]$. Then for any $m\in\mathbf{Z}$,

$$\begin{aligned}\underline{\mathcal{G}}\mathbf{f}[m] &= \sum_{k=p}^{p+N-1} \mathbf{f}[k]\omega^{-km} \\ &= \sum_{k=0}^{N-1} \mathbf{f}[k+p]\omega^{-(k+p)m} = \sum_{k=0}^{N-1} \mathbf{f}[k+s]\omega^{-(k+s)m} \quad \text{(using } p=rN+s \text{ and periodicity)} \\ &= \sum_{k=s}^{N-1} \mathbf{f}[k]\omega^{-km} + \sum_{k=N}^{N-1+s} \mathbf{f}[k]\omega^{-km} = \sum_{k=s}^{N-1} \mathbf{f}[k]\omega^{-km} + \sum_{k=0}^{s-1} \mathbf{f}[k]\omega^{-km} \quad \text{(periodicity again)} \\ &= \underline{\mathcal{F}}\,\mathbf{f}[m]\end{aligned}$$

Sorry for all of this, but it had to be said.

Of course, so too "the inverse DFT can be defined over any set of N consecutive indices." No argument there, from me or anyone else.

6.10 The FFT Algorithm

Before we look at further properties and applications of the DFT, I want to consider the practical problem of how it's computed. Then we'll go through the famous Cooley-Tukey Fast Fourier Transform algorithm.

You can't beat the matrix form

$$(\underline{\mathcal{F}})_{mn} = \omega^{-mn}, \quad m, n = 0, 1, \ldots, N-1$$

as a compact way of writing the DFT. It contains all you need to know. But you *can* beat it for minimizing the actual number of multiplications necessary to compute a given DFT. You can beat it by a lot, and that's the point of the fast Fourier transform. The FFT is an algorithm for computing the DFT with fewer than N^2 multiplications, the number of multiplications that seems required to find $\mathbf{F} = \underline{\mathcal{F}}_N \mathbf{f}$ by multiplying $\mathbf{f}$ by the $N \times N$ matrix. Here's how it works — from Gauss to Cooley and Tukey.

Reducing calculations: merge and sort To set the stage for a discussion of the FFT algorithm, I thought it first might be useful to see an example of a somewhat simpler but related idea, a way of reducing the total number of steps in a multistep calculation by a clever arranging and rearranging of the individual steps.

Consider the classic (and extremely important) problem of sorting N numbers from smallest to largest. Say the numbers are

5	2	3	1	6	4	8	7

From this list we want to create a new list with the numbers ordered from 1 to 8. The direct assault on this problem is to search the entire list for the smallest number, remove that number from the list and put it in the first slot of the new list, then search the remaining original list for the smallest number, remove that number and put it in the second slot of the new list, and so on:

Zeroth step	5	2	3	1	6	4	8	7			
First step	5	2	3	6	4	8	7		1		
Second step	5	3	6	4	8	7			1	2	
Third step	5	6	4	8	7				1	2	3
Fourth step	5	6	8	7					1	2	3	4

The successive steps each produce two lists, one that is unsorted and one that is sorted. The sorted list is created one number at a time and the final sorted list emerges in step 8.

In general, how many operations does such an algorithm require to do a complete sort? If we have N numbers then each step requires roughly N comparisons — true, the size of the list is shrinking, but the number of comparisons is *of the order* N — and we have to repeat this procedure N times. (There are N steps, not counting the zeroth step which just inputs the initial list.) Thus the number of operations used to sort N numbers by this procedure is of the order N^2, or, as it's usually written $O(N^2)$ (read "Big Oh").

The problem with this simple procedure is that the $(n+1)$-st step doesn't take into account the comparisons done in the n-th step. All that work is wasted, and it's wasted over and over. An alternate approach, one that makes use of intermediate comparisons, is to sort sublists of the original list, merge the results and sort again. Here's how it goes — we'll assess the efficiency after we see how the method works.

Start by going straight through the list and breaking it up into sublists that have just two elements; say that's step zero.

5	2		3	1		6	4		8	7

Step one is to sort each of these (four) 2-lists, producing two sets of 2-lists, called "top" lists and "bottom" lists just to keep then straight (and we'll also use top and bottom in later work on the FFT):

top

2	5

1	3

bottom

4	6

7	8

(This step requires four comparisons.)

Step two merges these 2-lists into two sorted 4-lists (again called top and bottom). Here's the algorithm, applied separately to the top and bottom 2-lists. The numbers in the first slots of each 2-list are the smaller of those two numbers. Compare these two numbers and promote the smaller of the two to the first slot of the top (respectively, bottom) 4-list. That leaves a 1-list and a 2-list. Compare the single element of the 1-list to the first element of the 2-list and promote the smaller to the second slot of the top (resp. bottom) 4-list. We're down to two numbers — compare them and put them in the three and four slots of the top (resp. bottom) 4-list. For the example we're working with this results in the two 4-lists:

top

1	2	3	5

bottom

4	6	7	8

(With this example this step requires five comparisons.)

Following this same sort procedure, step three is to merge the top and bottom sorted 4-lists into a single sorted 8-list:

1	2	3	4	5	6	7	8

(With this example this step requires five comparisons.)

In this process we haven't cut down (much) the number of comparisons we have to make at each step, but we have cut down the number of steps from 8 to 3.[4] In general how many operations are involved in getting to the final list of sorted numbers? It's not hard to see that the number of comparisons involved in merging two sublists is of the order of the total length of the sublists. Thus with N numbers total (at the start) the number of comparisons in any merge-sort is $O(N)$:

$$\text{number of comparisons in a merge-sort} = O(N)\,.$$

How many merge-sort steps are there? At each stage we halve the number of sublists, or, working the other way, from the final sorted list each step "up" doubles the number of sublists. Thus if there are n

[4] It wasn't an accident that I took eight numbers here. The procedure is most natural when we have a power of 2 numbers to sort, something we'll see again when we look at the FFT.

doublings (n steps) then $2^n = N$, or

$$\text{number of merge-sort steps} = \log_2 N\,.$$

We conclude that

$$\text{number of steps to sort } N \text{ numbers} = O(N \log N)\,.$$

(We can take the log in any base since this only changes the log by a constant factor, and that's thrown into the "big Oh".) If N is large this is a *huge* savings in steps from the $O(N^2)$ estimate for the simple sort that we did first. For example, if N is one million then $O(N^2)$ is a million million or 10^{12} steps while $N \log_{10} N = 10^6 \times 6$, a mere six million operations.

That's a correct accounting of the number of operations involved, but *why* is there a savings in using merge-sort rather than a straight comparison? By virtue of the sorting of sublists we only need to compare first elements of the sublists in the merge part of the algorithm. In this way the $(n+1)$-st step takes advantage of the comparisons made in the n-th step, the thing that is not done in the straight comparison method.

A sample calculation of the DFT As we consider how we might calculate the DFT more efficiently than by straight matrix multiplication, let's do a sample calculation with $N = 4$ so we have on record what the answer is and what we're supposed to come up with by other means. The DFT matrix is

$$\underline{\mathcal{F}}_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega_4^{-1} & \omega_4^{-2} & \omega_4^{-3} \\ 1 & \omega_4^{-2} & \omega_4^{-4} & \omega_4^{-6} \\ 1 & \omega_4^{-3} & \omega_4^{-6} & \omega_4^{-9} \end{pmatrix}.$$

We want to reduce this as much as possible, "reduction" being somewhat open to interpretation.

Using $\omega_4 = e^{2\pi i/4}$ and $\omega_4^{-4} = 1$ we have

$$\omega_4^{-6} = \omega_4^{-4}\omega_4^{-2} = \omega_4^{-2} \quad \text{and} \quad \omega_4^{-9} = \omega_4^{-8}\omega_4^{-1} = \omega_4^{-1}\,.$$

In general, to simplify ω_4 to a power we take the remainder of the exponent on dividing by 4, that is, we "reduce modulo 4". With these reductions $\underline{\mathcal{F}}_4$ becomes

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega_4^{-1} & \omega_4^{-2} & \omega_4^{-3} \\ 1 & \omega_4^{-2} & \omega_4^{-4} & \omega_4^{-6} \\ 1 & \omega_4^{-3} & \omega_4^{-6} & \omega_4^{-9} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega_4^{-1} & \omega_4^{-2} & \omega_4^{-3} \\ 1 & \omega_4^{-2} & 1 & \omega_4^{-2} \\ 1 & \omega_4^{-3} & \omega_4^{-2} & \omega_4^{-1} \end{pmatrix}.$$

But we don't stop there. Note that

$$\omega_4^{-2} = (e^{2\pi i/4})^{-2} = e^{-\pi i} = -1\,,$$

also a worthy simplification. (We have called attention to this $N/2$-th power of ω_N before.) So, finally,

$$\underline{\mathcal{F}}_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega_4^{-1} & \omega_4^{-2} & \omega_4^{-3} \\ 1 & \omega_4^{-2} & \omega_4^{-4} & \omega_4^{-6} \\ 1 & \omega_4^{-3} & \omega_4^{-6} & \omega_4^{-9} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega_4^{-1} & -1 & -\omega_4^{-1} \\ 1 & -1 & 1 & -1 \\ 1 & -\omega_4^{-1} & -1 & \omega_4^{-1} \end{pmatrix}.$$

Therefore we find

$$\underline{\mathcal{F}}_4\mathbf{f} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega_4^{-1} & -1 & -\omega_4^{-1} \\ 1 & -1 & 1 & -1 \\ 1 & -\omega_4^{-1} & -1 & \omega_4^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{f}[0] \\ \mathbf{f}[1] \\ \mathbf{f}[2] \\ \mathbf{f}[3] \end{pmatrix} = \begin{pmatrix} \mathbf{f}[0] + \mathbf{f}[1] + \mathbf{f}[2] + \mathbf{f}[3] \\ \mathbf{f}[0] + \mathbf{f}[1]\omega_4^{-1} - \mathbf{f}[2] - \mathbf{f}[3]\omega_4^{-1} \\ \mathbf{f}[0] - \mathbf{f}[1] + \mathbf{f}[2] - \mathbf{f}[3] \\ \mathbf{f}[0] - \mathbf{f}[1]\omega_4^{-1} - \mathbf{f}[2] + \mathbf{f}[3]\omega_4^{-1} \end{pmatrix}.$$

The matrix looks simpler, true, but it still took 16 multiplications to get the final answer. You can see those two special components that we called attention to earlier, the sum of the inputs in the zero slot and the alternating sum of the inputs in the $N/2 = 2$ slot.

This is about as far as we can go without being smart. Fortunately, there have been some smart people on the case.

6.10.1 Half the work is twice the fun: the Fast Fourier Transform

We agree that the DFT has a lot of structure. The trick to a faster computation of a DFT of order N is to use that structure to rearrange the products to bring in DFT's of order $N/2$. Here's where we use that N is even; in fact, to make the algorithm most efficient in being applied repeatedly we'll eventually want to assume that N is a power of 2.

We need a few elementary algebraic preliminaries on the ω_N, all of which we've used before. We also need to introduce some temporary(!) notation or we'll sink in a sea of subscripts and superscripts. Let's write powers of ω with two arguments:

$$\omega[p, q] = \omega_p^q \,.$$

I think this will help. It can't hurt. For our uses p will be N, $N/2$, etc.

First notice that

$$\omega[N/2, -1] = e^{-2\pi i/(N/2)} = e^{-4\pi i/N} = \omega[N, -2] \,.$$

Therefore powers of $\omega[N/2, -1]$ are *even* powers of $\omega[N, -1] = \omega_N^{-1}$:

$$\omega[N/2, -n] = \omega[N, -2n]$$

and in general,

$$\omega[N, -2nm] = \omega[N/2, -nm] \,.$$

What about odd powers of $\omega_N^{-1} = \omega[N, -1]$? An odd power is of the form $\omega[N, -(2n+1)]$ and so

$$\omega[N, -(2n+1)] = \omega[N, -1]\,\omega[N, -2n] = \omega[N, -1]\,\omega[N/2, -n] \,.$$

Thus also

$$\omega[N, -(2n+1)m] = \omega[N, -m]\,\omega[N/2, -nm]$$

Finally, recall that

$$\omega[N, -N/2] = e^{(-2\pi i/N)(N/2)} = e^{-\pi i} = -1 \,.$$

Splitting the sums Here's how we'll use this. For each m we want to split the single sum defining $\mathbf{F}[m]$ into two sums, one over the even indices and one over the odd indices:

$$\begin{aligned}\mathbf{F}[m] &= \sum_{n=0}^{N-1} \mathbf{f}[n]\,\omega[N, -nm] \\ &= \sum_{n=0}^{N/2-1} \mathbf{f}[2n]\,\omega[N, -(2n)m] \;+\; \sum_{n=0}^{N/2-1} \mathbf{f}[2n+1]\,\omega[N, -(2n+1)m] \,.\end{aligned}$$

Everything is accounted for here, all the terms are there — make sure you see that. Also, although both sums go from 0 to $N/2 - 1$, notice that for the first sum the first and last terms are $\mathbf{f}[0]$ and $\mathbf{f}[N-2]\,\omega[N, -(N-2)m]$, and for the second they are $\mathbf{f}[1]\omega[N, -m]$ and $\mathbf{f}[N-1]\,\omega[N, -(N-1)m]$.

Next, according to our observations on powers of ω we can also write $\mathbf{F}[m]$ as

$$\begin{aligned}\mathbf{F}[m] &= \sum_{n=0}^{N/2-1} \mathbf{f}[2n]\,\omega[N/2,-nm] \;+\; \sum_{n=0}^{N/2-1} \mathbf{f}[2n+1]\,\omega[N/2,-nm]\,\omega[N,-m] \\ &= \sum_{n=0}^{N/2-1} \mathbf{f}[2n]\,\omega[N/2,-nm] \;+\; \omega[N,-m] \sum_{n=0}^{N/2-1} \mathbf{f}[2n+1]\,\omega[N/2,-nm]\,.\end{aligned}$$

Let's study these sums more closely. There are $N/2$ even indices and $N/2$ odd indices and we appear, in each sum, *almost* to be taking a DFT of order $N/2$ of the $N/2$-tuples $\mathbf{f}[\text{even}]$ and $\mathbf{f}[\text{odd}]$. Why "almost"? The DFT of order $N/2$ accepts as input an $N/2$-tuple *and returns* an $N/2$-tuple. But the sums above give all N entries of the N-tuple $\mathbf{F}$ as m goes from 0 to $N-1$. We're going to do two things to bring in $\underline{\mathcal{F}}_{N/2}$.

- First, if we take m to go from 0 to $N/2-1$ then we get the first $N/2$ outputs $\mathbf{F}[m]$, and we write, informally,

$$\mathbf{F}[m] = (\underline{\mathcal{F}}_{N/2}\,\mathbf{f}_{\text{even}})[m] \;+\; \omega[N,-m]\,(\underline{\mathcal{F}}_{N/2}\,\mathbf{f}_{\text{odd}})[m]\,, \quad m = 0,1,\ldots,N/2-1\,.$$

That makes sense; $N/2$-tuples go in and $N/2$-tuples comes out.

- Second, what is the story for an index m in the second half of the range, from $N/2$ to $N-1$? Instead of letting m go from $N/2$ to $N-1$ we can write these indices in the form $m+N/2$, where m goes from 0 to $N/2-1$ and we ask what forms the sums take for $\mathbf{F}[m+N/2]$.

Look at the powers of $\omega_{N/2}$. In both the sums, over even and over odd indices, we have the powers $\omega[N/2,-mn]$, and with $m+N/2$ instead of m these are

$$\omega[N/2,-(m+N/2)n] = \omega[N/2,-mn]\,\omega[N/2,-n(N/2)] = \omega[N/2,-mn]\,,$$

since $\omega[N/2,-N/2] = 1$. So nothing's changed in that part of the sums from what we first wrote. However, for the sum over the odd indices there's also the factor $\omega[N,-m]$ out front, and this becomes

$$\omega[N,-(m+N/2)] = \omega[N,-m]\,\omega[N,-N/2] = -\omega[N,-m]\,.$$

Putting these observations together, the second half of the outputs $\mathbf{F}[m]$, from $\mathbf{F}[N/2]$ to $\mathbf{F}[N-1]$, are given by

$$\mathbf{F}[m+N/2] = \sum_{n=0}^{N/2-1} \mathbf{f}[2n]\,\omega[N/2,-nm] - \omega[N,-m] \sum_{n=0}^{N/2-1} \mathbf{f}[2n+1]\,\omega[N/2,-nm]$$

for $m=0$ to $m=N/2-1$, and just as we did for the first half of the outputs we write this as

$$\mathbf{F}[m+N/2] = (\underline{\mathcal{F}}_{N/2}\,\mathbf{f}_{\text{even}})[m] - \omega[N,-m]\,(\underline{\mathcal{F}}_{N/2}\,\mathbf{f}_{\text{odd}})[m]\,, \quad m = 0,1,\ldots,N/2-1\,.$$

The description of the FFT algorithm It's really very significant what we've done here. Let's summarize:

- We started with an input N-tuple $\mathbf{f}$ and want to compute its N-tuple output $\mathbf{F} = \underline{\mathcal{F}}_N\,\mathbf{f}$.

- The steps we took served to compute the component outputs $\mathbf{F}[m]$ for $m = 0, 1, \ldots, N-1$ by computing a DFT on two sequences, each of half the length, and arranging properly. That is:

  1. Separate $\mathbf{f}[n]$ into two sequences, the even and odd indices (0 is even), each of length $N/2$.
  2. Compute $\underline{\mathcal{F}}_{N/2}\,\mathbf{f}_{\text{even}}$ and $\underline{\mathcal{F}}_{N/2}\,\mathbf{f}_{\text{odd}}$.
  3. The outputs $\mathbf{F}[m]$ are obtained by arranging the results of this computation according to

$$\begin{aligned}\mathbf{F}[m] &= (\underline{\mathcal{F}}_{N/2}\,\mathbf{f}_{\text{even}})[m] + \omega[N,-m]\,(\underline{\mathcal{F}}_{N/2}\,\mathbf{f}_{\text{odd}})[m]\\ \mathbf{F}[m+N/2] &= (\underline{\mathcal{F}}_{N/2}\,\mathbf{f}_{\text{even}})[m] - \omega[N,-m]\,(\underline{\mathcal{F}}_{N/2}\,f_{\text{odd}})[m]\end{aligned}$$

  for $m = 0, 1, \ldots, N/2$.

Another look at $\underline{\mathcal{F}}_4$ Let's do the case $N = 4$ as an example, comparing it to our earlier calculation. The first step is to rearrange the inputs to group the even and odd indices. This is done by a *permutation matrix*

$$M = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}$$

whose effect is

$$\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{f}[0]\\ \mathbf{f}[1]\\ \mathbf{f}[2]\\ \mathbf{f}[3] \end{pmatrix} = \begin{pmatrix} \mathbf{f}[0]\\ \mathbf{f}[2]\\ \mathbf{f}[1]\\ \mathbf{f}[3] \end{pmatrix}.$$

M is defined by what it does to the natural basis $\mathbf{e}_0$, $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ of $\mathbf{R}^4$, namely $M\mathbf{e}_0 = \mathbf{e}_0$, $M\mathbf{e}_1 = \mathbf{e}_2$, $M\mathbf{e}_2 = \mathbf{e}_1$ and $M\mathbf{e}_3 = \mathbf{e}_3$.

Next, the even and odd indices are fed respectively to two DFT's of order $4/2 = 2$. This is the crucial reduction in the FFT algorithm.

$$\begin{pmatrix} 1 & 1 & 0 & 0\\ 1 & \omega_2^{-1} & 0 & 0\\ 0 & 0 & 1 & 1\\ 0 & 0 & 1 & \omega_2^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{f}[0]\\ \mathbf{f}[2]\\ \mathbf{f}[1]\\ \mathbf{f}[3] \end{pmatrix} = \begin{pmatrix} \mathbf{f}[0] + \mathbf{f}[2]\\ \mathbf{f}[0] + \mathbf{f}[2]\omega_2^{-1}\\ \mathbf{f}[1] + \mathbf{f}[3]\\ \mathbf{f}[1] + \mathbf{f}[3]\omega_2^{-1} \end{pmatrix}$$

On the left we have a "block diagonal" matrix. It's a 4×4 matrix with the 2×2 $\underline{\mathcal{F}}_2$ matrices down the diagonal and zeros everywhere else. We saw this step, but we didn't see the intermediate result written just above on the right because our formulas passed right away to the reassembly of the $\mathbf{F}[m]$'s. That reassembly is the final step.

So far we have

$$\underline{\mathcal{F}}_2\,\mathbf{f}_{\text{even}} = \begin{pmatrix} f_0 + \mathbf{f}[2]\\ f_0 + \mathbf{f}[2]\omega_2^{-1} \end{pmatrix}, \quad \underline{\mathcal{F}}_2\,\mathbf{f}_{\text{odd}} = \begin{pmatrix} \mathbf{f}[1] + \mathbf{f}[3]\\ \mathbf{f}[1] + \mathbf{f}[3]\omega_2^{-1} \end{pmatrix}$$

and in each case the indexing is $m = 0$ for the first entry and $m = 1$ for the second entry. The last stage, to get the $\mathbf{F}[m]$'s, is to recombine these half-size DFT's in accordance with the even and odd sums we wrote down earlier. In putting the pieces together we want to leave the even indices alone, put a $+\omega_4^{-m}$ in front of the m-th component of the first half of the $\underline{\mathcal{F}}_2$ of the odds and a $-\omega_4^{-m}$ in front of the m-th component of the the $\underline{\mathcal{F}}_2$ of the second half of the odds. This is done by the matrix

$$B_4 = \begin{pmatrix} 1 & 0 & 1 & 0\\ 0 & 1 & 0 & \omega_4^{-1}\\ 1 & 0 & -1 & 0\\ 0 & 1 & 0 & -\omega_4^{-1} \end{pmatrix}.$$

It works like this:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & \omega_4^{-1} \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -\omega_4^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{f}[0]+\mathbf{f}[2] \\ \mathbf{f}[0]+\mathbf{f}[2]\omega_2^{-1} \\ \mathbf{f}[1]+\mathbf{f}[3] \\ \mathbf{f}[1]+\mathbf{f}[3]\omega_2^{-1} \end{pmatrix} = \begin{pmatrix} \mathbf{f}[0]+\mathbf{f}[2]+\mathbf{f}[1]+\mathbf{f}[3] \\ \mathbf{f}[0]+\mathbf{f}[2]\omega_2^{-1}+\mathbf{f}[1]\omega_4^{-1}+\mathbf{f}[3]\omega_4^{-1}\omega_2^{-1} \\ \mathbf{f}[0]+\mathbf{f}[2]-\mathbf{f}[1]-\mathbf{f}[3] \\ \mathbf{f}[0]+\mathbf{f}[2]\omega_2^{-1}-\mathbf{f}[1]\omega_4^{-1}-\mathbf{f}[3]\omega_4^{-1}\omega_2^{-1} \end{pmatrix}$$
$$= \begin{pmatrix} \mathbf{f}[0]+\mathbf{f}[1]+\mathbf{f}[2]+\mathbf{f}[3] \\ \mathbf{f}[0]+\mathbf{f}[1]\omega_4^{-1}-\mathbf{f}[2]-\mathbf{f}[3]\omega_4^{-1} \\ \mathbf{f}[0]-\mathbf{f}[1]+\mathbf{f}[2]-\mathbf{f}[3] \\ \mathbf{f}[0]-\mathbf{f}[1]\omega_4^{-1}-\mathbf{f}[2]+\mathbf{f}[3]\omega_4^{-1} \end{pmatrix},$$

where we (finally) used $\omega_2^{-1} = e^{-2\pi i/2} = -1$. This checks with what we got before.

One way to view this procedure is as a factorization of $\underline{\mathcal{F}}_4$ into simpler matrices. It looks like

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega_4^{-1} & \omega_4^{-2} & \omega_4^{-3} \\ 1 & \omega_4^{-2} & \omega_4^{-4} & \omega_4^{-6} \\ 1 & \omega_4^{-3} & \omega_4^{-6} & \omega_4^{-9} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega_4^{-1} & -1 & -\omega_4^{-1} \\ 1 & -1 & 1 & -1 \\ 1 & -\omega_4^{-1} & -1 & \omega_4^{-1} \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & \omega_4^{-1} \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -\omega_4^{-1} \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & \omega_2^{-1} & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & \omega_2^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & \omega_4^{-1} \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -\omega_4^{-1} \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Look at all the zeros! There are 48 entries total in the three matrices that multiply together to give $\underline{\mathcal{F}}_4$, but only 20 entries are nonzero.

In the same way, the general shape of the factorization to get DFT_N via $DFT_{N/2}$ is

$$\underline{\mathcal{F}}_N = \begin{pmatrix} I_{N/2} & \Omega_{N/2} \\ I_{N/2} & -\Omega_{N/2} \end{pmatrix} \begin{pmatrix} \underline{\mathcal{F}}_{N/2} & 0 \\ 0 & \underline{\mathcal{F}}_{N/2} \end{pmatrix} \begin{pmatrix} \text{Sort the even} \\ \text{and odd indices} \end{pmatrix}$$

$I_{N/2}$ is the $N/2 \times N/2$ identity matrix. 0 is the zero matrix (of size $N/2 \times N/2$ in this case). $\Omega_{N/2}$ is the diagonal matrix with entries 1, ω_N^{-1}, ω_N^{-2}, $\ldots \omega_N^{-(N/2-1)}$ down the diagonal.[5] $\underline{\mathcal{F}}_{N/2}$ is the DFT of half the order, and the permutation matrix puts the $N/2$ even indices first and the $N/2$ odd indices second.

Thus the way this factorization works is:

- The inputs are $\mathbf{f}[0]$, $\mathbf{f}[1]$, $\ldots$, $\mathbf{f}[N-1]$.
- The matrix on the right is a permutation matrix that puts the even indices in the first $N/2$ slots and the odd indices in the second $N/2$ slots.

[5] The notation $\Omega_{N/2}$ isn't the greatest — it's written with $N/2$ because of the dimensions of the matrix, though the entries are powers of ω_N. Still, it will prove useful to us later on, and it appears in the literature.

  - Alternatively, think of the operation as first starting with $\mathbf{f}[0]$ and taking every other $\mathbf{f}[n]$ — this collects $\mathbf{f}[0]$, $\mathbf{f}[2]$, $\mathbf{f}[4]$, and so on — and then starting back with $\mathbf{f}[1]$ and taking every other $\mathbf{f}[n]$ — this collects $\mathbf{f}[1]$, $\mathbf{f}[3]$, $\mathbf{f}[5]$, and so on. As we iterate the process, this will be a more natural way of thinking about the way the first matrix chooses how to send the inputs on to the second matrix.

- The outputs of the first matrix operation are a pair of $N/2$-vectors. The matrix in the middle accepts these as inputs. It computes half-size DFT's on these half-size inputs and outputs two $N/2$-vectors, which are then passed along as inputs to the third matrix.
- The third matrix, on the left, reassembles the outputs from the half-size DFT's and outputs the $\mathbf{F}[0]$, $\mathbf{F}[1]$, $\ldots \mathbf{F}[N-1]$.
  - This is similar in spirit to a step in the "merge-sort" algorithm for sorting numbers. Operations (comparisons in that case, DFT's in this case) are performed on smaller lists which are then merged to longer lists.
- The important feature, as far as counting the multiplications go, is that suddenly there are a lot of zeros in the matrices.

As to this last point, we can already assess some savings in the number of operations when the even/odd splitting is used versus the straight evaluation of the DFT from its original definition. If we compute $\mathbf{F} = \underline{\mathcal{F}}_N \mathbf{f}$ just as a matrix product there are N^2 multiplications and N^2 additions for a total of $2N^2$ operations. On the other hand, with the splitting the computations in the inner block matrix of two DFT's of order $N/2$ require $2(N/2)^2 = N^2/2$ multiplications and $2(N/2)^2 = N^2/2$ additions. The sorting and recombining by the third matrix require another $N/2$ multiplications and N additions — and this is *linear* in N. Thus the splitting method needs on the order of N^2 operations while the straight DFT needs $2N^2$. We've cut the work in half, pretty much, though it's still of the same order. We'll get back to this analysis later.

Divide and conquer At this point it's clear what we'd like to do — repeat the algorithm, each time halving the size of the DFT. The factorization from N to $N/2$ is the top level:

$$\underline{\mathcal{F}}_N = \begin{pmatrix} I_{N/2} & \Omega_{N/2} \\ I_{N/2} & -\Omega_{N/2} \end{pmatrix} \begin{pmatrix} \underline{\mathcal{F}}_{N/2} & 0 \\ 0 & \underline{\mathcal{F}}_{N/2} \end{pmatrix} \begin{pmatrix} \text{Sort the even} \\ \text{and odd indices} \end{pmatrix}$$

At the next level "down" we don't do anything to the matrices on the ends, but we factor each of the two $\underline{\mathcal{F}}_{N/2}$'s the same way, into a permutation matrix on the right, a block matrix of $\underline{\mathcal{F}}_{N/4}$'s in the middle and a reassembly matrix on the left. (I'll come back to the sorting — it's the most interesting part.). That is,

$$\underline{\mathcal{F}}_{N/2} = \begin{pmatrix} I_{N/4} & \Omega_{N/4} \\ I_{N/4} & -\Omega_{N/4} \end{pmatrix} \begin{pmatrix} \underline{\mathcal{F}}_{N/4} & 0 \\ 0 & \underline{\mathcal{F}}_{N/4} \end{pmatrix} \begin{pmatrix} \text{Sort } N/2\text{-lists to} \\ \text{two } N/4\text{-lists} \end{pmatrix}$$

and putting this into the top level picture the operations become "nested" (or recursive):

$$\underline{\mathcal{F}}_N = \begin{pmatrix} I_{N/2} & \Omega_{N/2} \\ I_{N/2} & -\Omega_{N/2} \end{pmatrix} \cdot \begin{pmatrix} \begin{pmatrix} I_{N/4} & \Omega_{N/4} \\ I_{N/4} & -\Omega_{N/4} \end{pmatrix} \begin{pmatrix} \underline{\mathcal{F}}_{N/4} & 0 \\ 0 & \underline{\mathcal{F}}_{N/4} \end{pmatrix} \begin{pmatrix} N/2 \text{ to} \\ N/4 \text{ sort} \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} I_{N/4} & \Omega_{N/4} \\ I_{N/4} & -\Omega_{N/4} \end{pmatrix} \begin{pmatrix} \underline{\mathcal{F}}_{N/4} & 0 \\ 0 & \underline{\mathcal{F}}_{N/4} \end{pmatrix} \begin{pmatrix} N/2 \text{ to} \\ N/4 \text{ sort} \end{pmatrix} \end{pmatrix} \cdot \begin{pmatrix} \text{Sort the } N/2\text{-even} \\ \text{and } N/2\text{-odd indices} \end{pmatrix}.$$

To be able to repeat this, to keep halving the size of the DFT, we now see that we need to take N to be a power of 2. The construction then continues, "going down" levels till we get from $\underline{\mathcal{F}}_N$ to $\underline{\mathcal{F}}_1$. Note that the DFT of order 1 takes a single input and returns it unchanged, i.e., it is the identity transform.

When the halving is all over here's what happens. The work is in the initial sorting and in the reassembling, since the final DFT in the factorization is $\underline{\mathcal{F}}_1$ which leaves alone whatever it gets. Thus, reading from right to left, the initial inputs $(\mathbf{f}[0], \mathbf{f}[1], \ldots, \mathbf{f}[N-1])$ are first sorted and then passed back up through a number of reassembly matrices, ultimately winding up as the outputs $(\mathbf{F}[0], \mathbf{F}[1], \ldots, \mathbf{F}[N-1])$.

It's clear, with the abundance of zeros in the matrices, that there should be a savings in the total number of operations, though it's not clear how much. The *entire trip*, from $\mathbf{f}$'s to $\mathbf{F}$'s is called the Fast Fourier Transform (FFT). It's fast because of the reduction in the number of operations. Remember, the FFT is not a new transform, *it is just computing the DFT of the initial inputs.*

6.10.2 Factoring the DFT matrix

Rather than trying now to describe the general process in more detail, let's look at an example more thoroughly, and from the matrix point of view. One comment about this approach, a matrix factorization description versus other sorts of descriptions of the algorithm. Since the initial ideas of Cooley and Tukey there have been many other styles of FFT algorithms proposed and implemented, similar in some respects to Cooley & Tukey's formulation and different in others. It became a mess. In a 1992 book, *Computational Frameworks for the Fast Fourier Transform*, Charles Van Loan showed how many of the ideas could be unified via a study of different matrix factorizations of the DFT. This is not the only way to organize the material, but it has been very influential.

Let's take the case $N = 16$, just to live it up. Once again, the initial input is a 16-tuple (or vector) $\mathbf{f}$ and the final output is another 16-tuple, $\mathbf{F} = \underline{\mathcal{F}}_{16}\mathbf{f}$. At the top level, we can write this as

$$\mathbf{F} = \underline{\mathcal{F}}_{16}\mathbf{f} = \begin{pmatrix} I_8 & \Omega_8 \\ I_8 & -\Omega_8 \end{pmatrix} \begin{pmatrix} \underline{\mathcal{F}}_8 & 0 \\ 0 & \underline{\mathcal{F}}_8 \end{pmatrix} \begin{pmatrix} \mathbf{f}_{\text{even}} \\ \mathbf{f}_{\text{odd}} \end{pmatrix} = \begin{pmatrix} I_8 & \Omega_8 \\ I_8 & -\Omega_8 \end{pmatrix} \begin{pmatrix} \mathbf{G} \\ \mathbf{H} \end{pmatrix}$$

where $\mathbf{G}$ and $\mathbf{H}$ are the results of computing $\underline{\mathcal{F}}_8$ on $\mathbf{f}_{\text{even}}$ and $\mathbf{f}_{\text{odd}}$, respectively. Write this as

$$\mathbf{F} = B_{16} \begin{pmatrix} \mathbf{G} \\ \mathbf{H} \end{pmatrix}, \quad B_{16} = \begin{pmatrix} I_8 & \Omega_8 \\ I_8 & -\Omega_8 \end{pmatrix}$$

where B is supposed to stand for "butterfly" — more on this, later.

But how, in the nesting of operations, did we get to $\mathbf{G}$ and $\mathbf{H}$? The next level down (or in) has

$$\mathbf{G} = \underline{\mathcal{F}}_8\mathbf{f}_{\text{even}} = \begin{pmatrix} I_4 & \Omega_4 \\ I_4 & -\Omega_4 \end{pmatrix} \begin{pmatrix} \mathbf{G}' \\ \mathbf{H}' \end{pmatrix} = B_8 \begin{pmatrix} \mathbf{G}' \\ \mathbf{H}' \end{pmatrix}$$

where $\mathbf{G}'$ and $\mathbf{H}'$ are the result of computing $\underline{\mathcal{F}}_4$'s on the even and odd subsets of $\mathbf{f}_{\text{even}}$. Got it? Likewise we write

$$\mathbf{H} = \underline{\mathcal{F}}_8\mathbf{f}_{\text{odd}} = \begin{pmatrix} I_4 & \Omega_4 \\ I_4 & -\Omega_4 \end{pmatrix} \begin{pmatrix} \mathbf{G}'' \\ \mathbf{H}'' \end{pmatrix} = B_8 \begin{pmatrix} \mathbf{G}'' \\ \mathbf{H}'' \end{pmatrix}$$

where $\mathbf{G}''$ and $\mathbf{H}''$ are the result of computing $\underline{\mathcal{F}}_4$'s on the even and odd subsets of $\mathbf{f}_{\text{odd}}$.

Combining this with what we did first, we have

$$\mathbf{F} = \underline{\mathcal{F}}_{16}\mathbf{f} = B_{16}\begin{pmatrix} B_8 & 0 \\ 0 & B_8 \end{pmatrix}\begin{pmatrix} \mathbf{G}' \\ \mathbf{H}' \\ \mathbf{G}'' \\ \mathbf{H}'' \end{pmatrix}$$

Continue this for two more steps — it remains to find DFT's of order 4 and 2. The result then looks like

$$\begin{aligned}\mathbf{F} &= \underline{\mathcal{F}}_{16}\mathbf{f} \\ &= B_{16}\begin{pmatrix} B_8 & 0 \\ 0 & B_8 \end{pmatrix}\begin{pmatrix} B_4 & 0 & 0 & 0 \\ 0 & B_4 & 0 & 0 \\ 0 & 0 & B_4 & 0 \\ 0 & 0 & 0 & B_4 \end{pmatrix}\begin{pmatrix} B_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & B_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & B_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & B_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & B_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_2 \end{pmatrix}\begin{pmatrix} 16 \times 16 \text{ permutation} \\ \text{matrix that sorts} \\ \text{the inputs} \end{pmatrix}\mathbf{f}\,.\end{aligned}$$

Note that

$$B_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Each B_2 receives a pair of inputs coming from a pair of DFT_1's, and since the DFT_1's don't do anything, each B_2 receives a pair of the original inputs $\mathbf{f}[m]$, but shuffled from the ordering $\mathbf{f}[0], \mathbf{f}[1], \ldots, f_{15}$. We'll get back to the question of sorting the indices, but first let's be sure that it's worth it.

And the point of this is, again? There are lots of zeros in the factorization of the DFT. After the initial sorting of the indices (also lots of zeros in that matrix) there are 4 reassembly stages. In general, for $N = 2^n$ there are $n = \log_2 N$ reassembly stages after the initial sorting. The count $\log_2 N$ for the number of reassembly stages follows in the same way as the count for the number of merge-sort steps in the sorting algorithm, but I want to be a little more precise this time.

We now consider the "computational complexity" of the FFT algorithm in general. Let $C(N)$ denote the number of elementary operations involved in finding the DFT via the FFT algorithm; these include additions and multiplications. We reassemble $\underline{\mathcal{F}}_N$ from two $\underline{\mathcal{F}}_{N/2}$'s by another set of elementary operations. From our earlier considerations. or from the factorization, that number of operations can easily be shown to be proportional to N. Thus the basic recursion relationship is

$$C(N) = 2C(N/2) + KN\,.$$

We can solve this recurrence equation as follows. Let

$$n = \log_2 N$$

and let

$$T(n) = \frac{C(N)}{N}$$

so that

$$C(N) = NT(n)\,.$$

Then $n - 1 = \log_2(N/2)$ and thus

$$T(n-1) = \frac{C(N/2)}{N/2} = 2\frac{C(N/2)}{N}, \quad \text{or} \quad NT(n-1) = 2C(N/2)\,.$$

Substituting into the recurrence relationship for C then gives

$$NT(n) = NT(n-1) + KN$$

or simply

$$T(n) = T(n-1) + K\,.$$

This already implies that $T(n)$ is linear. But $C(1)$ is obviously 0, because there aren't any operations needed to compute the DFT of a single point. Hence $T(0) = C(1) = 0,\ T(1) = K$ and in general

$$T(n) = Kn\,.$$

In terms of C this says

$$C(N) = KN\log_2 N\,.$$

Various implementations of the FFT try to make the constant K as small as possible. The best one around now, I think, brings the number of multiplications down to $N\log_2 N$ and the number of additions down to $3\log_2 N$. Remember that this is for complex inputs. Restricting to real inputs cuts the number of operations in half.

As we pointed out when talking about the problem of sorting, when N is large the reduction in computation from N^2 to $N\log_2 N$ is an enormous savings. For example, take $N = 1024 = 2^{10}$. Then $N^2 = 2^{20}$ is about a million while $2^{10}\log_2 2^{10}$, about ten thousand, or even cut down to five thousand for real signals. That's a substantial reduction, and for larger N it's even more dramatic.

6.10.3 Sorting the indices

If we think of recursively factoring the inner DFT matrix, then in implementing the whole FFT the first thing that's done is to sort and shuffle the inputs.[6] It's common to display a flow diagram for the FFT, and much of the pictorial splendor in many treatments of the FFT is in showing how the $\mathbf{f}[m]$'s are shuffled and passed on from stage to stage. The flow chart of the complete FFT algorithm is called a *butterfly diagram* — hence the naming of the matrices B. You can find butterfly diagrams in any of the standard works.

The principle of sorting the inputs is as stated earlier. Start with the first input, $\mathbf{f}[0]$, and take every other one. Then start with the second input, $\mathbf{f}[1]$, (which was skipped over in the first pass) and take every other one. This produces $\mathbf{f}_{\text{even}}$ and $\mathbf{f}_{\text{odd}}$. The next sorting repeats the process *for the subsequences* $\mathbf{f}_{\text{even}}$ and $\mathbf{f}_{\text{odd}}$, and so on.

For $N = 8$ (I don't have the stamina for $N = 16$ again) this looks like

[6] Shuffling is actually an apt description: See, D. Knuth, *The Art of Computer Programming*, Vol. 3, p. 237.

$\mathbf{f}[0]$	$\mathbf{f}[0]$	$\mathbf{f}[0]$	$\mathbf{f}[0]$
$\mathbf{f}[1]$	$\mathbf{f}[2]$	$\mathbf{f}[4]$	$\mathbf{f}[4]$
$\mathbf{f}[2]$	$\mathbf{f}[4]$	$\mathbf{f}[2]$	$\mathbf{f}[2]$
$\mathbf{f}[3]$	$\mathbf{f}[6]$	$\mathbf{f}[6]$	$\mathbf{f}[6]$
$\mathbf{f}[4]$	$\mathbf{f}[1]$	$\mathbf{f}[1]$	$\mathbf{f}[1]$
$\mathbf{f}[5]$	$\mathbf{f}[3]$	$\mathbf{f}[5]$	$\mathbf{f}[5]$
$\mathbf{f}[6]$	$\mathbf{f}[5]$	$\mathbf{f}[3]$	$\mathbf{f}[3]$
$\mathbf{f}[7]$	$\mathbf{f}[7]$	$\mathbf{f}[7]$	$\mathbf{f}[7]$

Though there's no more shuffling from the third to the fourth column we've written the last column to indicate that the inputs go in, one at a time, in that order to the waiting B's.

The FFT algorithm with $N = 8$ is thus

$$\mathbf{F} = B_8 \begin{pmatrix} B_4 & 0 \\ 0 & B_4 \end{pmatrix} \begin{pmatrix} B_2 & 0 & 0 & 0 \\ 0 & B_2 & 0 & 0 \\ 0 & 0 & B_2 & 0 \\ 0 & 0 & 0 & B_2 \end{pmatrix} \begin{pmatrix} \mathbf{f}[0] \\ \mathbf{f}[4] \\ \mathbf{f}[2] \\ \mathbf{f}[6] \\ \mathbf{f}[1] \\ \mathbf{f}[5] \\ \mathbf{f}[3] \\ \mathbf{f}[7] \end{pmatrix}.$$

The sorting can be described in a neat way via binary numbers. Each sort puts a collection of inputs into a "top" bin or a "bottom" bin. Let's write 0 for top and 1 for bottom (as in 0 for even and 1 for odd). Assigning digits from right to left, the least significant bit is the first sort, the next most significant bit is the second sort and the most significant bit (for the three sorts needed when $N = 8$) is the final sort. We thus augment the table, above, to (read the top/bottom descriptions right to left):

$\mathbf{f}[0]$	$\mathbf{f}[0]$	0	top	$\mathbf{f}[0]$	00	top-top	$\mathbf{f}[0]$	000	top-top-top
$\mathbf{f}[1]$	$\mathbf{f}[2]$	0	top	$\mathbf{f}[4]$	00	top-top	$\mathbf{f}[4]$	100	bottom-top-top
$\mathbf{f}[2]$	$\mathbf{f}[4]$	0	top	$\mathbf{f}[2]$	10	bottom-top	$\mathbf{f}[2]$	010	top-bottom-top
$\mathbf{f}[3]$	$\mathbf{f}[6]$	0	top	$\mathbf{f}[6]$	10	bottom-top	$\mathbf{f}[6]$	110	bottom-bottom-top
$\mathbf{f}[4]$	$\mathbf{f}[1]$	1	bottom	$\mathbf{f}[1]$	01	top-bottom	$\mathbf{f}[1]$	001	top-top-bottom
$\mathbf{f}[5]$	$\mathbf{f}[3]$	1	bottom	$\mathbf{f}[5]$	01	top-bottom	$\mathbf{f}[5]$	101	bottom-top-bottom
$\mathbf{f}[6]$	$\mathbf{f}[5]$	1	bottom	$\mathbf{f}[3]$	11	bottom-bottom	$\mathbf{f}[3]$	011	top-bottom-bottom
$\mathbf{f}[7]$	$\mathbf{f}[7]$	1	bottom	$\mathbf{f}[7]$	11	bottom-bottom	$\mathbf{f}[7]$	111	bottom-bottom-bottom

The numbers in the final column are exactly the binary representations for $0, 4, 2, 6, 1, 5, 3, 7$.

Now notice that we get from the initial natural ordering

$\mathbf{f}[0]$	000
$\mathbf{f}[1]$	001
$\mathbf{f}[2]$	010
$\mathbf{f}[3]$	011
$\mathbf{f}[4]$	100
$\mathbf{f}[5]$	101
$\mathbf{f}[6]$	110
$\mathbf{f}[7]$	111

to the ordering that we feed in,

$\mathbf{f}[0]$	000
$\mathbf{f}[4]$	100
$\mathbf{f}[2]$	010
$\mathbf{f}[6]$	110
$\mathbf{f}[1]$	001
$\mathbf{f}[5]$	101
$\mathbf{f}[3]$	011
$\mathbf{f}[7]$	111

by *reversing the binary representation* of the numbers in the first table.

What happens to $\mathbf{f}[6]$ in the sort, for example? 6 in binary is 110. That's bottom-bottom-top: the first sort puts $\mathbf{f}[6]$ in the top 4-list, the second sort generates 4 "2-lists", two top "2-lists" and two bottom "2-lists", and puts $\mathbf{f}[6]$ (along with $\mathbf{f}[2]$) in the bottom of the two top 2-lists. The final sort puts $\mathbf{f}[6]$ just below $\mathbf{f}[2]$. The slot for $\mathbf{f}[6]$ in the final sort, corresponding to "bottom-bottom-top", is the fourth one down — that's 110, the reverse of binary 011 (the fourth slot in the original ordering).

This same procedure for sorting works for all N. That is:

1. Write the numbers 0 to $N-1$ in binary. (With leading 0's so all numbers have the same length.) That enumerates the slots from 0 to $N-1$.
2. Reverse the binary digits of each slot number. For a binary number m call this reversed number $\overleftarrow{m}$
3. The input $f_{\overleftarrow{m}}$ goes in slot m.

This step in the FFT algorithm is called *bit reversal*, for obvious reasons. In fact, people spend plenty of time coming up with efficient bit reversal algorithms[7]. In running an FFT routine, like in Matlab, you don't do the sorting, of course. The program takes care of that. If, in the likely case, you don't happen to have 2^n samples in whatever data you've collected, then a common dodge is to add zeros to get up to the closest power of 2. This is referred to as "zero padding", and some FFT routines will automatically do it for you. But, like anything else, it can be dangerous if used improperly. We discuss zero padding later, and you've had a homework problem on it.

Bit reversal via permutation matrices To write down the permutation matrix that does this sorting you perform the "every other one" algorithm to the rows (or columns) of the $N \times N$ identity matrix,

[7] See P. Rosel, "Timing of some bit reversal algorithms", *Signal Processing*, 18 (1989) 425–433 for a survey of 12 (!) different bit-reversal algorithms.

reorder the rows according to that, then repeat. Thus for $N = 8$ there are two steps

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \text{ sort and rearrange the rows } \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\text{sort and rearrange top and bottom halves } \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

And sure enough

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{f}[0] \\ \mathbf{f}[1] \\ \mathbf{f}[2] \\ \mathbf{f}[3] \\ \mathbf{f}[4] \\ \mathbf{f}[5] \\ \mathbf{f}[6] \\ \mathbf{f}[7] \end{pmatrix} = \begin{pmatrix} \mathbf{f}[0] \\ \mathbf{f}[4] \\ \mathbf{f}[2] \\ \mathbf{f}[6] \\ \mathbf{f}[1] \\ \mathbf{f}[5] \\ \mathbf{f}[3] \\ \mathbf{f}[7] \end{pmatrix}.$$

6.11 Zero Padding

As we have seen, the FFT algorithm for computing the DFT is set up to work with an input length that is a power of 2. While not all implementations of the FFT require input to be of that length, many programs only accept inputs of certain lengths, and when this is not met it's common to add enough zeros to the end of the signal to bring the input up to the length required. This procedure is called *zero padding* and many programs will do it automatically (if needed) when you call an FFT.

Let $\mathbf{f} = (\mathbf{f}[0], \mathbf{f}[1], \ldots, \mathbf{f}[N-1])$ be the original input. For an integer $M > N$, define

$$\mathbf{g}[n] = \begin{cases} \mathbf{f}[n] & 0 \le n \le N-1 \\ 0 & N \le n \le M-1 \end{cases}$$

Then

$$\mathbf{G}[m] = \underline{\mathcal{F}}_M \mathbf{g}[m] = \sum_{n=0}^{M-1} \omega_M^{-mn} \mathbf{g}[n] = \sum_{n=0}^{N-1} \omega_M^{-mn} \mathbf{f}[n]$$

Work a little bit with ω_M^{-mn}:

$$\omega_M^{-mn} = e^{-2\pi imn/M} = e^{-2\pi imnN/MN} = e^{-2\pi in(mN/M)/N} = \omega_N^{n(mN/M)}.$$

Thus whenever mN/M is an integer we have

$$\mathbf{G}[m] = \sum_{n=0}^{N-1} \omega_N^{-n(mN/M)} \mathbf{f}[n] = \mathbf{F}[mN/M]\,.$$

We could also write this equation for $\mathbf{F}$ in terms of the $\mathbf{G}$ as

$$\mathbf{F}[m] = \mathbf{G}[mM/N]$$

whenever mM/N is an integer. This is what we're more interested in: the program computes the zero padded transform $\mathbf{G} = \underline{\mathcal{F}}_M \mathbf{g}$ and we'd like to know what the outputs $\mathbf{F}[m]$ of our original signal are in terms of the $\mathbf{G}$'s. The answer is the m-th component of $\mathbf{F}$ is the mM/N-th component of $\mathbf{G}$ whenever mM/N is an integer.

Let's pursue this a little, starting with getting rid of the stupid proviso that mM/N is an integer. We can *choose* M so let's choose

$$M = kN$$

for some integer k; so M is twice as large as N, or 3 times as large as N, or whatever. Then $mM/N = km$, always an integer, and

$$\mathbf{F}[m] = \mathbf{G}[km]$$

which is much easier to say in words:

- If $\mathbf{f}$ is zero padded to a signal $\mathbf{g}$ of length M, where $M = kN$, then the m-th component of $\mathbf{F} = \underline{\mathcal{F}}\,\mathbf{f}$ is the km-th component of $\mathbf{G} = \underline{\mathcal{F}}\,\mathbf{g}$.

Zero padding the inputs has an important consequence for the spacing of the grid points in the frequency domain. Suppose that the discrete signal $\mathbf{f} = (\mathbf{f}[0], \mathbf{f}[1], \ldots, \mathbf{f}[N-1])$ comes from sampling a continuous signal at points t_n, so that $\mathbf{f}[n] = f(t_n)$. Suppose also that the N sample points in the time domain of $f(t)$ are spaced Δt apart. Then the length of the interval on which $f(t)$ is defined is $N\Delta t$ and the spectrum $\mathcal{F}f(s)$ is spread out over an interval of length $1/\Delta t$. Remember, knowing N and Δt determines everything. Going from N inputs to $M = kN$ inputs by padding with zeros lengthens the interval in the time domain to $M\Delta t$ but it doesn't change the spacing of the sample points, i.e., it doesn't change the sampling rate in the time domain. What is the effect in the frequency domain? For the sample points associated with the discrete signals $\mathbf{f}$ and $\mathbf{F} = \underline{\mathcal{F}}_N \mathbf{f}$ we have

$$\Delta t \Delta\nu_{\text{unpadded}} = \frac{1}{N}$$

by the reciprocity relations (see Section 6.3), and for $\mathbf{g}$ and $\mathbf{G} = \underline{\mathcal{F}}_M \mathbf{g}$ we have

$$\Delta t \Delta\nu_{\text{padded}} = \frac{1}{M} = \frac{1}{kN}\,.$$

The Δt in both equations is the same, so

$$\frac{\Delta\nu_{\text{padded}}}{\Delta\nu_{\text{unpadded}}} = \frac{1}{k} \quad \text{or} \quad \Delta\nu_{\text{padded}} = \frac{1}{k}\Delta\nu_{\text{unpadded}}\,,$$

that is, the spacing of the sample points *in the frequency domain* for the padded sequence has decreased by the factor $1/k$. At the same time, the total extent of the grid in the frequency domain has not changed because it is $1/\Delta t$ and Δt has not changed. What this means is:

- Zero padding in the time domain *refines* the grid in the frequency domain.

There's a warning that goes along with this. Using zero padding to refine the grid in the frequency domain is only a valid thing to do if the original continuous signal f is already known to be zero outside of the original interval. If not then you're killing off real data by filling $\mathbf{f}$ out with zeros.

Chapter 7

Linear Time-Invariant Systems

7.1 Linear Systems

A former Dean of Stanford's School of Engineering used to joke that we didn't really have departments of Electrical Engineering, Mechanical Engineering, Chemical Engineering, and so on, we had departments of large systems, small systems, mechanical systems, chemical systems, and so on. If "system" is a catch-all phrase describing the process of going from inputs to outputs then that's probably as good a description of our organization as any, maybe especially as a way of contrasting engineering with fields that seem to be stuck on "input".

For us, a *system* is a mapping from input *signals* to output signals, and we'll typically write this as

$$w(t) = L(v(t))$$

or, without the variable, as

$$w = L(v)\,.$$

We often think of the signals as functions of time or of a spatial variable. The system operates on that signal in some way to produce another signal. This is the "continuous case", where input and outputs are function of a continuous variable. The discrete case will often arise for us by sampling continuous functions.

To be precise we would have to define the domain of L, i.e., the space of signals that we can feed into the system as inputs. For example, maybe it's only appropriate to consider L as operating on finite energy signals (a natural restriction), or on band-limited signals, whatever. We might also have to spell out what kind of continuity properties L has. These are genuine issues, but they aren't so important for us in setting up the basic properties, and the mathematical difficulties can detract from the key ideas — it keeps us stuck at the inputs.[1]

With such an extreme degree of generality, one shouldn't expect to be able to say anything terribly interesting — a system is some kind of operation that relates an incoming signal to an outgoing signal. Someway. Somehow. Great. Imposing more structure can make system a more interesting notion, and the simplest nontrivial extra assumption is that the system is linear.

[1] Generally speaking, the problems come from working with infinite dimensional spaces of signals and settling on appropriate definitions of continuity etc.. However, just as we did for the infrastructure that supports the theory of distributions, we're setting this aside for our work here. The area of mathematics that comes into play is called *functional analysis*.

The system is *linear* if L is linear. This means exactly that for all signals v, v_1, v_2 and all scalars α

$$\begin{aligned} L(v_1(t) + v_2(t)) &= L(v_1(t)) + L(v_2(t)) && (L \text{ is } \textit{additive}) \\ L(\alpha v(t)) &= \alpha L(v(t)) && (L \text{ is } \textit{homogeneous}) \end{aligned}$$

Note that to define this notion we have to be able to add and scale inputs and outputs. Not all systems can be linear because, depending on the application, it just might not make sense to add inputs or to scale them. Ditto for the outputs. A system where you *can* add and scale inputs and outputs but where one or both of the properties, above, do not hold is generically referred to as *nonlinear*. By the way, a common notational convention when dealing with linear operators is to drop the parentheses when L is acting on a single signal, that is, we write $Lv(t)$ instead of $L(v(t))$. This convention comes from the analogy of general linear systems to those given by multiplication by a matrix — more on that connection later.

One immediate comment. If the zero signal is the input to a linear system, then the output is also the zero signal, since

$$L(0) = L(0 \cdot 0) = 0 \cdot L(0) = 0\,.$$

If a system is nonlinear, it may not be that $L(0) = 0$; take, for example $L(v(t)) = v(t) + 1$.[2]

An expression of the form $\alpha_1 v_1(t) + \alpha_2 v_2(t)$ is called a *linear combination* or a *superposition* of v_1 and v_2. Thus a linear systems is often said to satisfy the *principle of superposition* — adding the inputs results in adding the outputs and scaling the input scales the output by the same amount. One can extend these properties directly to finite sums and, with proper assumptions of continuity of L and convergence of the sums, to infinite sums. That is,

$$L\Big(\sum_{n=0}^{N} v_n(t)\Big) = \sum_{n=0}^{N} Lv_n(t) \quad \text{and} \quad L\Big(\sum_{n=0}^{\infty} v_n(t)\Big) = \sum_{n=0}^{\infty} Lv_n(t).$$

We won't make an issue of convergence and continuity for the kinds of things we want to do. However, don't minimize the importance of these properties; if a signal can be written as a sum of its components (think Fourier series) and if we know the action of L on the components (think complex exponentials) then we can find the action of L on the composite signal.

7.2 Examples

Working with and thinking in terms of systems — and linear systems in particular — is as much an adopted attitude as it is an application of a collection of results. I don't want to go so far as saying that any problem should be viewed through the lens of linear systems from start to finish, but it often provides, at the very least, a powerful organizing principle. As we develop some general properties of linear systems it will be helpful to have in mind some examples that led to those general properties, so you too can develop an attitude.

I'll divide the examples into broad categories, but they really aren't so neatly separated. There will be a fair amount of "this example has aspects of that example"; this is an important aspect of the subject and something you should look for.

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[2] This may be the quickest mental checkoff to see if a system is *not* linear. Does the zero signal go to the zero signal? If not then the system *cannot* be linear.

7.2.1 Multiplying in the time domain

The most basic example of a linear system is the relationship of "A is directly proportional to B". Suitably interpreted (a slippery phrase), you see this in some form or function in almost all linear constructions.

Usually one thinks of "direct proportion" in terms of a "constant of proportionality", as in "the voltage is proportional to the current, $V = RI$", or "the acceleration is proportional to the force, $a = (1/m)F$". The conclusions are of the type, "doubling the current corresponds to doubling the voltage". But the "constant" can be a function, and the key property of linearity is still present: Fix a function $h(t)$ and define

$$Lv(t) = h(t)v(t)\,.$$

Then, to be formal about it, L is linear because

$$L(av_1(t) + bv_2(t)) = h(t)(av_1(t) + bv_2(t)) = ah(t)v_1(t) + bh(t)v_2(t) = aLv_1(t) + bLv_2(t)\,.$$

This is such a common construction that it's already come up many times, though not in the context of "linearity" per se. Here are two examples.

Switching on and off is a linear system Suppose we have a system consisting of a "switch". When the switch is closed a signal goes through unchanged and when the switch is open the signal doesn't go through at all (so by convention what comes out the other end is the zero signal). Suppose that the switch is closed for $-\frac{1}{2} \le x \le \frac{1}{2}$. Is this a linear system? Sure; it's described precisely by

$$Lv(t) = \Pi(t)v(t)\,,$$

i.e., multiplication by Π.

We could modify this any number of ways:

- Switching on and off at various time intervals.
  - This is modeled by multiplication by a sum of shifted and scaled Π's.
- Switching on and staying on, or switching off and staying off.
  - This is modeled by multiplication by the unit step $H(t)$ or by $1 - H(t)$.

All of these are linear systems, and you can come up with many other systems built on the same principle.

Sampling is a linear system Sampling is multiplication by Ш. To sample a signal $v(t)$ with sample points spaced p apart is to form

$$Lv(t) = v(t)\text{Ш}_p(t) = \sum_{k=-\infty}^{\infty} v(kp)\delta(t - kp)\,.$$

It doesn't hurt to say in words what the consequences of linearity are, namely: "Sampling the sum of two signals is the sum of the sampled signals." It's better if you say that fast.

7.2.2 Matrices and integrals

Similar to simple direct proportion, but one level up in sophistication, are linear systems defined by matrix multiplication and by integration.

Matrix multiplication The most basic example of a discrete linear system is multiplication by a matrix, i.e.,

$$\mathbf{w} = A\mathbf{v}\,,$$

where $\mathbf{v} \in \mathbf{R}^n$, $\mathbf{w} \in \mathbf{R}^m$, and A is an $m \times n$ matrix. Written out,

$$\mathbf{w}[i] = \sum_{j=1}^{n} a_{ij}\mathbf{v}[j]\,, \quad i = 1, \ldots, m\,.$$

The inputs are n-vectors and the outputs are m-vectors. The linear system might have special properties according to whether A has special properties — symmetric, Hermitian, unitary. The DFT, for example, can thus be thought of as a linear system.

Linear dynamical systems Speaking of matrix multiplication, to mix the continuous and discrete, and for those taking EE 263, the linear system

$$L\mathbf{v}(t) = e^{At}\mathbf{v}$$

where A is an $n \times n$ matrix and $\mathbf{v} \in \mathbf{R}^n$, is the linear system associated with the initial value problem

$$\dot{\mathbf{x}}(t) = A\mathbf{x}\,, \quad \mathbf{x}(0) = \mathbf{v}\,.$$

Here the matrix e^{At} varies in time, and the system describes how the initial value $\mathbf{v}$ evolves over time.

Linear systems via integrals The case of a linear system defined via matrix multiplication is a model for more complicated situations. The continuous, infinite-dimensional version of matrix multiplication is the linear system given by an operation of the form

$$w(x) = Lv(x) = \int_a^b k(x,y)v(y)\,dy.$$

Here, $k(x,y)$ is called the *kernel* and one speaks of "integrating $v(y)$ against a kernel". This is certainly a linear system, since

$$\begin{aligned} L(\alpha_1 v_1(x) + \alpha_2 v_2(x)) &= \int_a^b k(x,y)(\alpha_1 v_1(y) + \alpha_2 v_2(y))\,dy \\ &= \alpha_1 \int_a^b k(x,y)v_1(y)\,dy + \alpha_2 \int_a^b k(x,y)v_2(y)\,dy \\ &= \alpha_1 L v_1(x) + \alpha_2 L v_2(x). \end{aligned}$$

To imagine how this generalizes the (finite) matrix linear systems, think, if you dare, of the values of v as being listed in an infinite column, $k(x,y)$ as an infinite (square) matrix, $k(x,y)v(y)$ as the product of the (x,y)-entry of k with the y-th entry of v, and the integral

$$w(x) = \int_a^b k(x,y)v(y)\,dy$$

as summing the products $k(x,y)$ across the x-th row of k with the entries of the column v, resulting in the x-th value of the output, $w(x)$. You really won't be misleading yourself thinking this way, though, of course, there are questions of convergence in pursuing the analogy.

The system may have special properties according to whether the kernel $k(x, y)$ has special properties; symmetric ($k(x, y) = k(y, x)$) or Hermitian ($k(x, y) = \overline{k(y, x)}$). To push the analogy further, the adjoint (or transpose) of L is usually defined to be $L^T v(y) = \int_a^b k(x, y)v(x)\,dx$; note that we're integrating with respect to the *first* variable in $k(x, y)$ here. Thus if $k(x, y) = k(y, x)$ then $L^T = L$. Nice.

To take one quick example, as a linear system the Fourier transform is of this form:

$$\mathcal{F}f(s) = \int_{-\infty}^{\infty} e^{-2\pi ist} f(t)\,dt = \int_{-\infty}^{\infty} k(s, t)f(t)\,dt$$

In this case the kernel is $k(s, t) = e^{-2\pi ist}$.

7.2.3 Convolution: continuous and discrete

One example of a linear system defined by integrating against a kernel is convolution, but convolution is so important in applications that it rises above "special case" status in the list of examples. In the continuous realm, fix a signal $g(x)$ and define

$$w(x) = Lv(x) = (g * v)(x)\,.$$

In the category "linear systems via integration" this is, explicitly,

$$w(x) = (g * v)(x) = \int_{-\infty}^{\infty} g(x - y)v(y)\,dy = \int_{-\infty}^{\infty} k(x, y)v(y)\,dy, \quad \text{where } k(x, y) = g(x - y)\,.$$

The discrete case Convolution in the finite discrete case naturally involves periodic discrete signals. Fix a periodic sequence, say $\mathbf{h}$, and define $\mathbf{w} = L\mathbf{v}$ by convolution with $\mathbf{h}$:

$$\mathbf{w}[m] = (\mathbf{h} * \mathbf{v})[m] = \sum_{n=0}^{N-1} \mathbf{h}[m - n]\mathbf{v}[n]\,,$$

Just as continuous convolution fits into the framework of linear systems via integration, discrete convolution is an example of linear systems via matrix multiplication. Since the operation $\mathbf{w} = \mathbf{h} * \mathbf{v}$ from the input $\mathbf{v}$ to the output $\mathbf{w}$ is a linear transformation of $\mathbf{C}^N$ to itself, it must be given by multiplication by some $N \times N$ matrix:

$$\mathbf{w} = \mathbf{h} * \mathbf{v} = L\mathbf{v}\,.$$

What is the matrix L?

First, to be precise, it's not really "the" matrix, because the matrix form of a linear transformation depends on the choice of bases for the inputs and outputs. But if we use the natural basis for $\mathbf{C}^n$ we quickly get an answer. Borrowing from our work with the DFT, we write the basis as the discrete $\boldsymbol{\delta}$'s, namely $\boldsymbol{\delta}_0$, $\boldsymbol{\delta}_1$, ..., $\boldsymbol{\delta}_{N-1}$. The n-th column of the matrix L is the vector $L\boldsymbol{\delta}_n$ (indexed from $n = 0$ to $n = N - 1$). We know what happens when we convolve with $\boldsymbol{\delta}$'s, we shift the index — the m-th entry in the n-th column of L is

$$L\boldsymbol{\delta}_n[m] = (\mathbf{h} * \boldsymbol{\delta}_n)[m] = \mathbf{h}[m - n]\,,$$

written simply as

$$(L)_{mn} = \mathbf{h}[m - n]\,.$$

The matrix L is constant along the diagonals and is filled out, column by column, by the shifted versions of $\mathbf{h}$. Note again the crucial role played by periodicity.

To take an example, if

$$\mathbf{h} = (\mathbf{h}[0], \mathbf{h}[1], \mathbf{h}[2], \mathbf{h}[3]) = (1, 2, 3, 4)$$

then the matrix L for which

$$\mathbf{w} = \mathbf{h} * \mathbf{v} = L\mathbf{v}$$

has columns

$$\begin{pmatrix} \mathbf{h}[0] \\ \mathbf{h}[1] \\ \mathbf{h}[2] \\ \mathbf{h}[3] \end{pmatrix}, \quad \begin{pmatrix} \mathbf{h}[-1] \\ \mathbf{h}[0] \\ \mathbf{h}[1] \\ \mathbf{h}[2] \end{pmatrix} = \begin{pmatrix} \mathbf{h}[3] \\ \mathbf{h}[0] \\ \mathbf{h}[1] \\ \mathbf{h}[2] \end{pmatrix}, \quad \begin{pmatrix} \mathbf{h}[-2] \\ \mathbf{h}[-1] \\ \mathbf{h}[0] \\ \mathbf{h}[1] \end{pmatrix} = \begin{pmatrix} \mathbf{h}[2] \\ \mathbf{h}[3] \\ \mathbf{h}[0] \\ \mathbf{h}[1] \end{pmatrix}, \quad \begin{pmatrix} \mathbf{h}[-3] \\ \mathbf{h}[-2] \\ \mathbf{h}[-1] \\ \mathbf{h}[0] \end{pmatrix} = \begin{pmatrix} \mathbf{h}[1] \\ \mathbf{h}[2] \\ \mathbf{h}[3] \\ \mathbf{h}[0] \end{pmatrix}$$

which is

$$L = \begin{pmatrix} 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}.$$

In general, square matrices that are constant along the diagonals (different constants for different diagonals allowed) are called *Toeplitz* matrices, after the mathematician Otto Toeplitz who singled them out for special study. They have all sorts of interesting properties. There's even more structure to the Toeplitz matrices that correspond to convolution because of the (assumed) periodicity of the columns. This special class of Toeplitz matrices are called *circulant matrices*. We won't pursue their features any farther, but see Chapter 4.7 of the book *Matrix Computation* by G. Golub and C. van Loan, and the references there.

7.2.4 Translation or shifting

Signals, whether functions of a continuous or discrete variable, can be delayed or advanced to produce new signals. The operation is this:

$$Lv(t) = v(t - \tau)$$

This is very familiar to us, to say the least.

Think of t as time "now", where now starts at $t = 0$, and τ as "time ago". Think of $t - \tau$ as a delay in time by an amount τ — "delay" if $\tau > 0$, "advance" if $\tau < 0$. To delay a signal 24 hours from current time ("now") is to consider the difference between current time and time 24 hours ago, i.e., $t - 24$. The signal v delayed 24 hours is $v(t - 24)$ because it's not until $t = 24$ that the signal "starts".

Convolving with δ We could show directly that translation in time (or in space, if that's the physical variable) is a linear system. But we can also observe that translation in time is nothing other than convolving with a translated δ. That is,

$$v(t - \tau) = (\delta_\tau * v)(t)\,,$$

and the same for a discrete signal:

$$\mathbf{v}[m - n] = (\boldsymbol{\delta}_n * \mathbf{v})[m]\,.$$

Periodizing is a linear system By the same token, we see that periodizing a signal is a linear system, since this amounts to convolution with a Ш_p. Thus

$$w(t) = Lv(t) = (\text{Ш}_p * v)(t)$$

is a linear system. In words, "the periodization of the sum of two signals is the sum of the periodizations", with a similar statement for scaling by a constant.

7.3 Cascading Linear Systems

An important operation is to *compose* or *cascade* two (or more) linear systems. That is, if L and M are linear systems, then — as long as the operations make sense — ML is also a linear system:

$$\begin{aligned}(ML)(\alpha_1 v_1 + \alpha_2 v_2) &= M(L(\alpha_1 v_1 + \alpha_2 v_2)) \\ &= M(\alpha_1 L v_1 + \alpha_2 L v_2) = \alpha_1 ML v_1 + \alpha_2 ML v_2\end{aligned}$$

In general we do *not* have $ML = LM$.

The phrase "as long as the operations make sense" means that we do have to pay attention to the domains and ranges of the individual systems. For example, if we start out with an integrable function $f(t)$ then $\mathcal{F}f$ makes sense but $\mathcal{F}\mathcal{F}f$ may not.[3]

Cascading linear systems defined by matrix multiplication amounts to multiplying the matrices. There's a version of this in the important case of cascading two linear systems when one is given as an integral. If L is the linear system given by

$$Lv(x) = \int_a^b k(x,y)v(y)\,dy,$$

and M is another linear system (not necessarily given by an integral), then the composition ML is the linear system

$$MLv(x) = \int_a^b M(k(x,y))v(y)\,dy.$$

What does this mean, and when is it true? First, $k(x,y)$ has to be a signal upon which M can operate, and in writing $M(k(x,y))$ (and then integrating with respect to y) we intend that its operation on $k(x,y)$ is in its x-dependence. To bring M inside the integral requires some continuity assumptions, but I won't spell all this out. The restrictions are mild, and we can be safe in assuming that we can perform such operations for the applications we're interested in.

To take an example, what does this look like if M is also given by integration against a kernel? Say

$$Mv(x) = \int_a^b \ell(x,y)v(y)\,dy.$$

Then

$$\begin{aligned}MLv(y) &= \int_a^b \ell(x,y)Lv(y)\,dy \\ &= \int_a^b \ell(x,y)\left(\int_a^b k(y,z)v(z)\,dz\right)dy \\ &\qquad \text{(we introduced a new variable of integration)} \\ &= \int_a^b \int_a^b \ell(x,y)k(y,z)v(z)\,dz\,dy.\end{aligned}$$

Now if all the necessary hypotheses are satisfied, which is always the tasteful assumption, we can further write

$$\int_a^b \int_a^b \ell(x,y)k(y,z)v(z)\,dz\,dy = \int_a^b \left(\int_a^b \ell(x,y)k(y,z)\,dy\right)v(z)\,dz.$$

[3] If $f(t)$ is a Schwartz function, however, then we can keep applying $\mathcal{F}$. Duality results imply, however, that $\mathcal{F}\mathcal{F}\mathcal{F}\mathcal{F} =$ identity, so cascading $\mathcal{F}$ doesn't go on producing new signals forever.

Thus the cascaded system MLv is also given by integration against a kernel:

$$MLv(x) = \int_a^b K(x,z)v(z)\,dz\,,$$

where

$$K(x,z) = \int_a^b \ell(x,y)k(y,z)\,dy\,.$$

The formula for the kernel $K(x,z)$ should call to mind an analogy to a matrix product.

7.4 The Impulse Response, or, The deepest fact in the theory of distributions is well known to all electrical engineers

It's not just that defining a (continuous) linear system by an integral is a nice example and a nice thing to do. It's the *only* thing to do. Under very minimal assumptions, *all* linear systems are of this form. Here's what I mean, and here's how to think about such a statement.

We're used to recovering the values of a signal by convolving with δ. That is,

$$v(x) = (\delta * v)(x) = \int_{-\infty}^{\infty} \delta(x-y)v(y)\,dy\,.$$

Now suppose that L is a linear system. Applying L to $v(x)$ then gives

$$w(x) = Lv(x) = \int_{-\infty}^{\infty} L\delta(x-y)v(y)\,dy\,.$$

In the integrand only $\delta(x-y)$ depends on x, so that's what L operates on.

What we need to know is what L does to δ, or, as it's usually put, how the system *responds* to the impulse $\delta(x-y)$. We're ready for an important definition.

- Let L be a linear system. The *impulse response* is

$$h(x,y) = L\delta(x-y)\,.$$

What this means in practice is that we see how the system responds to a very short, very peaked signal. The limit of such responses is the impulse response. You will note the usual mathematical modus operandi — we answer the question of how a system responds to an impulse via a definition. I think credit for introducing the impulse response belongs to engineers, however.

Putting this into the earlier integral, we have what is sometimes called the

Superposition Theorem If L is a linear system with impulse response $h(x,y)$, then

$$w(x) = Lv(x) = \int_{-\infty}^{\infty} h(x,y)v(y)\,dy\,.$$

In this way we have realized the linear system L as integrating the input signal against a kernel. The kernel is the impulse response.

Can this be made more precise? What does it mean for L to operate on $\delta(x-y)$? Is that a function or a distribution? And so on. The answer is yes, all of this can be made precise and it has a natural home in the context of distributions. The superposition theorem of electrical engineering is known as the Schwartz kernel theorem in mathematics, and the impulse response (which *is* a distribution) is known as the Schwartz kernel. Moreover, there is a uniqueness statement. It says, in the present context, that for each linear system L there is a *unique* kernel $h(x,y)$ such that

$$Lv(x) = \int_{-\infty}^{\infty} h(x,y)v(y)\,dy.$$

The uniqueness is good to know — if you've somehow expressed a linear system as an integral with a kernel then you have *found* the impulse response. Thus, for example, the impulse response of the Fourier transform is $h(s,t) = e^{-2\pi ist}$ since

$$\mathcal{F}f(s) = \int_{-\infty}^{\infty} e^{-2\pi ist} f(t)\,dt = \int_{-\infty}^{\infty} h(s,t)f(t)\,dt\,.$$

We conclude from this that

$$\mathcal{F}\delta(t-s) = e^{-2\pi ist}\,,$$

which checks with what we know from earlier work.[4]

The Schwartz kernel theorem is considered probably the hardest result in the theory of distributions. So, by popular demand, we won't take this any farther. We can, however, push the analogy with matrices a little more, and this might be helpful to you.

If you know how an $m \times n$ matrix A acts on a basis for $\mathbf{C}^n$ (meaning you know the products of the matrix with the basis vectors), then you can figure out what it does to any vector by expressing that vector in terms of the basis and using linearity. To review this: Suppose $\mathbf{v}_1, \dots \mathbf{v}_n$ is a basis of $\mathbf{C}^n$ and A is a linear transformation. And suppose you know $\mathbf{w}_1 = A\mathbf{v}_1, \dots, \mathbf{w}_n = A\mathbf{v}_n$. Every vector $\mathbf{v}$ can be written as a linear combination

$$\mathbf{v} = \sum_{k=1}^{n} \alpha_k \mathbf{v}_k$$

and therefore, by linearity,

$$A\mathbf{v} = A\Big(\sum_{k=1}^{n} \alpha_k \mathbf{v}_k\Big) = \sum_{k=1}^{n} \alpha_k A\mathbf{v}_k = \sum_{k=1}^{n} \alpha_k \mathbf{w}_k\,.$$

The continuous case is analogous. Think of

$$v(x) = (\delta * v)(x) = \int_{-\infty}^{\infty} \delta(x-y)v(y)\,dy$$

[4] This isn't circular reasoning, but neither is it a good way to find the Fourier transform of a shifted δ-function; it's hard to prove the Schwartz kernel theorem, but it's easy (with distributions) to find the Fourier transform of δ.

as a continuous version of expressing the signal v as a "sum" (integral in this case) of its "components" (its values at all points y) times "basis signals" (the shifted delta functions $\delta(x-y)$). That is, the fact that we can write $v(x)$ in this way is some way of saying that the $\delta(x-y)$ are a "basis"; they're analogous to the natural basis of $\mathbf{C}^n$, which, as we've seen, are exactly the shifted discrete $\boldsymbol{\delta}$'s.

Applying a linear system L to $v(x)$ as expressed above then gives

$$w(x) = Lv(x) = \int_{-\infty}^{\infty} L\delta(x-y)v(y)\,dy = \int_{-\infty}^{\infty} h(x,y)v(y)\,dy,$$

where $h(x,y)$ is the impulse response. Thus $h(x,y)$ is the "matrix representation" for the linear system L in terms of the basis signals $\delta(x-y)$.

The discrete case The discrete case is the same, but easier: no integrals, no distributions. A linear system in the discrete case is exactly matrix multiplication, and we did part of the analysis of this case just above. Here's how to finish it. Suppose the linear system is

$$\mathbf{w} = A\mathbf{v}\,,$$

where the linear transformation A is written as a matrix using the natural basis. This means that the columns of A are exactly the products of A and the basis $\boldsymbol{\delta}_0$, $\boldsymbol{\delta}_1, \ldots, \boldsymbol{\delta}_{N-1}$. In this basis the matrix A *is* the impulse response.

7.5 Linear Time-Invariant (LTI) Systems

If I run a program tomorrow I expect to get the same answers as I do when I run it today. Except I'll get them tomorrow. The circuits that carry the currents and compute the 0's and 1's will behave (ideally) today just as they did yesterday and into the past, and just as they should tomorrow and into the future. We know that's not true indefinitely, of course — components fail — but as an approximation this kind of time invariance is a natural assumption for many systems. When it holds it has important consequences for the mathematical (and engineering) analysis of the system.

The time-invariance property is that a shift in time of the inputs should result in an identical shift in time of the outputs. Notice that this kind of time invariance is spoken in terms of "shifts in time", or "differences in time", as in "it's the same tomorrow as it is today", implying that we're looking at whether behavior changes over a time interval, or between two instants of time. "Absolute time" doesn't make sense, but differences between two times does.

What is the mathematical expression of this? If $w(t) = Lv(t)$ is the output of the system at current time then to say that the system is *time invariant*, or is an LTI system, is to say that a delay of the input signal by an amount τ produces a delay of the output signal by the same amount, but no other changes. As a formula, this is

$$Lv(t-\tau) = w(t-\tau)\,.$$

Delaying the input signal by 24 hours produces a delay in the output by 24 hours, but that's the only thing that happens. (Sometimes LTI is translated to read "Linear Translation Invariant" system, recognizing that the variable isn't always time, but the operation is always translation. You also see LSI in use, meaning Linear Shift Invariant.)

What about the impulse response for an LTI system? For a general linear system the impulse response $h(t,\tau) = L\delta(t-\tau)$ depends independently on t and τ, i.e. the response can have different forms for impulses at different times. But this *isn't the case* for an LTI system. Let's say that

$$L\delta(t) = h(t)\,,$$

so this is the impulse response at $\tau = 0$. Then by the time invariance

$$L\delta(t-\tau) = h(t-\tau)\,.$$

That is, the impulse response does not depend independently on t and τ but rather only on their difference, $t-\tau$. The character of the impulse response means that the superposition integral assumes the form of a convolution:

$$w(t) = Lv(t) = \int_{-\infty}^{\infty} h(t-\tau)v(\tau)\,d\tau = (h*v)(t)\,.$$

Conversely, let's show that a linear system given by a convolution integral is time invariant. Suppose

$$w(t) = Lv(t) = (g*v)(t) = \int_{-\infty}^{\infty} g(t-\tau)v(\tau)\,d\tau$$

Then

$$\begin{aligned} L(v(t-t_0)) &= \int_{-\infty}^{\infty} g(t-\tau)v(\tau-t_0)\,d\tau \\ &\quad \text{(make sure you understand how we substituted the shifted } v \\ &= \int_{-\infty}^{\infty} g(t-t_0-s)v(s)\,ds \quad (\text{substituting } s=\tau-t_0) \\ &= (g*v)(t-t_0) = w(t-t_0)\,. \end{aligned}$$

Thus L is time invariant, as we wanted to show. Furthermore, we see that when L is defined this way, by a convolution, $g(t-\tau)$ *must* be the impulse response, because for a time invariant system the impulse response is determined by

$$L\delta(t) = (g*\delta)(t) = g(t)\,,$$

that is,

$$L\delta(t-\tau) = g(t-\tau)\,.$$

This is a very satisfactory state of affairs. Let's summarize what we have learned:

If L is a linear system, then

$$Lv(x) = \int_{-\infty}^{\infty} h(x,y)v(y)\,dy,$$

where $h(x,y)$ is the impulse response

$$h(x,y) = L\delta(x-y)\,.$$

The system is time invariant *if and only if* it is a convolution. The impulse response is a function of the difference $x-y$, and the convolution is with the impulse response,

$$Lv(x) = \int_{-\infty}^{\infty} h(x-y)v(y)\,dy = (h*v)(x)\,.$$

This last result is another indication of how fundamental, and natural, the operation of convolution is.

How to spot a discrete LTI system A discrete system given by a convolution

$$\mathbf{w} = \mathbf{h} * \mathbf{v}$$

is time invariant, as you can check, and if a system is given to you in this form there's not much to spot. But a general discrete linear system is given in terms of matrix multiplication, say,

$$\mathbf{w} = L\mathbf{v}\,,$$

Can you spot when this is time invariant?

We observed that the matrix L associated to the system $\mathbf{w} = \mathbf{h} * \mathbf{v}$ is a circulant matrix, filled out column by column by the shifted versions of $\mathbf{h}$ (periodicity!). As an exercise you can show: If L is a circulant matrix then $\mathbf{w} = L\mathbf{v}$ is an LTI system. In terms of convolution it is given by $\mathbf{w} = \mathbf{h} * \mathbf{v}$ where $\mathbf{h}$ is the first column of L.

How to get a raise, if handled politely Finally, we had a list of examples of linear systems, starting with the most basic example of direct proportion. How do those examples fare with regard to time invariance? Does "direct proportion" pass the test? Afraid not, except in the simplest case.

Suppose that

$$Lv(t) = h(t)v(t) \quad \text{(multiplication, not convolution)}$$

is time invariant. Then for any τ, on the one hand,

$$L(v(t-\tau)) = h(t)v(t-\tau)\,,$$

and on the other hand

$$L(v(t-\tau)) = (Lv)(t-\tau) = h(t-\tau)v(t-\tau)\,.$$

Thus,

$$h(t-\tau)v(t-\tau) = h(t)v(t-\tau)\,.$$

This is to hold for every input v and every τ, and that can only happen if $h(t)$ is *constant*. Hence the relationship of direct proportion will only define a time-invariant linear system when the proportionality factor is constant ("genuine" direct proportion).

So, if your boss comes to you and says: "I want to build a set of switches and I want you to model that for me by convolution, because although I don't know what convolution means I know it's an important idea." you will have to say: "That cannot be done because while switches can be modeled as a linear system, the simple (even for you, boss) relation of direct proportion that we would use does not define a time-invariant system, as convolution must." You win.

Later, however, your boss comes back and says; "OK, no convolution, but find the impulse response of my switch system and find it fast." This you can do. To be definite, take the case of a single switch system modeled by

$$Lv(t) = \Pi(t)v(t)\,.$$

Then

$$h(t,\tau) = L\delta(t-\tau) = \Pi(t)\delta(t-\tau) = \Pi(\tau)\delta(t-\tau)\,.$$

Sure enough, the impulse response is *not* a function of $t-\tau$ only.

For extra credit, you also offer your boss the superposition integral, and you show him that it works:

$$\begin{aligned} Lv(t) &= \int_{-\infty}^{\infty} h(t,\tau)v(\tau)\,d\tau \\ &= \int_{-\infty}^{\infty} \Pi(\tau)\delta(t-\tau)v(\tau)\,d\tau \quad \text{(this is the superposition integral)} \\ &= \Pi(t)v(t). \end{aligned}$$

It works. You take the rest of the day off.

7.6 Appendix: The Linear Millennium

It's been quite a 1000 years, and I feel some obligation to contribute to the millennial reminiscences before we get too far into the new century. I propose *linearity* as one of the most important themes of mathematics and its applications in times past and times present.

Why has linearity been so successful? I offer three overall reasons.

1. On a small scale, smooth functions are approximately linear.

   This is the basis of calculus, of course, but it's a very general idea. Whenever one quantity changes smoothly (differentiably) with another, small changes in one quantity produce, approximately, directly proportional changes in the other.

2. There's a highly developed, highly successful assortment of existence and uniqueness results for linear problems, and existence and uniqueness are related for linear problems.

   When does an equation have a solution? If it has one, does it have more than one? These are fundamental questions. Think of solving systems of linear equations as the model here. Understanding the structure of the space of solutions to linear systems in the finite dimensional, discrete case (i.e., matrices) is important in itself, and it has served as the model of what to look for in the infinite dimensional case, discrete and continuous alike.

   When people studied the "classical differential equations of mathematical physics" like the heat equation, the wave equation, and Laplace's equation, they all knew that they satisfied the "principle of superposition" and they all knew that this was important. The change in point of view was to take this "linear structure" of the space of solutions as the starting point; one could add solutions and get another solution *because* the differential equations themselves defined linear systems.

3. There's generally some group structure associated with linear problems.

   This mixes the analysis with algebra and geometry and contributes to all of those perspectives. It brings out symmetries in the problem and symmetries in the solutions. This has turned out to be very important in sorting out many phenomena in Fourier analysis.

Finally, I'm willing to bet (but not large sums) that "linearity" won't hold sway too far into the new millennium, and that nonlinear phenomena will become increasingly more central. Nonlinear problems have always been important, but it's the *computational* power now available that is making them more accessible to analysis.

7.7 Appendix: Translating in Time and Plugging into L

From my own bitter experience, I can report that knowing when and how to plug the time shifts into a formula is not always easy. So I'd like to return briefly to the definition of time invariance and offer a streamlined, mathematical way of writing this, and also a way to think of it in terms of cascading systems (and block diagrams). For me, at least, because I'm used to thinking in these terms, this helps to settle the issue of what gets "plugged in" to L.

The first approach is to write the act of shifting by an amount b as an operation on a signal.[5] That is, bring back the "translate by b" operator and define

$$(\tau_b v)(x) = v(x-b)\,.$$

(Now I must write my variables as x and y instead of t and τ — it's always something.) If $w(x) = Lv(x)$ then

$$w(x-b) = (\tau_b w)(x)\,,$$

and the time invariance property then says that

$$L(\tau_b v) = \tau_b(Lv), \quad \text{without writing the variable } x,$$

or

$$L(\tau_b v)(x) = \tau_b(Lv)(x) = (Lv)(x-b), \quad \text{writing the variable } x.$$

It's the placement of parentheses here that means everything — it says that translating by x and then applying L (the left hand side) has the same effect as applying L and then translating by τ (the right hand side). One says that an LTI system L "commutes" with translation. Most succinctly

$$L\,\tau_b = \tau_b\,L\,.$$

In fact, "commuting" is just the way to look at time invariance from a second point of view. We already observed that "translation by b" is itself a linear system. Then the combination of τ_b and L, in that order, produces the output $L(v(x-b))$. To say that L is an LTI system is to say that the system τ_b followed by L produces the same result as the system L followed by τ_b.

Now go back to that plugging in we did earlier in the convolution:

$$Lv(x) = (g*v)(x) = \int_{-\infty}^{\infty} g(x-y)v(y)\,dy$$

Then

$$L(v(x-x_0)) = \int_{-\infty}^{\infty} g(x-y)v(y-x_0)\,dy\,.$$

We can show this carefully by writing

$$\begin{aligned}
L(\tau_{x_0}v)(x) &= \int_{-\infty}^{\infty} g(x-y)\tau_{x_0}v(y)\,dy \\
&\qquad \text{(it's the } \textit{translated} \text{ signal } \tau_{x_0}v \text{ that's getting convolved)} \\
&= \int_{-\infty}^{\infty} g(x-y)v(y-x_0)dy \\
&= \int_{-\infty}^{\infty} g(x-x_0-s)v(s)\,ds \quad \text{(substituting } s = y-x_0) \\
&= (g*v)(x-x_0) \;=\; \tau_{x_0}(g*v)(x) \;=\; \tau_{x_0}(Lv)(x)\,.
\end{aligned}$$

[5] We did this when we talked about the shift theorem for distributions — we really had to in that case.

7.8 The Fourier Transform and LTI Systems

The fact that LTI systems are identical with linear systems defined by convolution should trip the Fourier transform switch in your head. Given the LTI system

$$w(t) = (h * v)(t)$$

we take Fourier transforms and write

$$W(s) = H(s)V(s)\,,$$

turning convolution in the time domain to multiplication in the frequency domain. Recall that $H(s)$ is called the *transfer* function of the system. We introduced the transfer function earlier, in Chapter 3, and I refer you there for a quick review. The extra terminology is that the system $w = h * v$ is called a *filter* (or an LTI filter) and sometimes the impulse response h is called the filter function.

One catch phrase used to describe LTI filters is that they "add no new frequencies". Rather than say what that means, here's an example of a system that does add new frequencies. Consider

$$Lv(t) = v(t)^2\,.$$

This is nonlinear. If for example we feed in $v(t) = \cos 2\pi t$ we get out

$$w(t) = Lv(t) = \cos^2 2\pi t = \tfrac{1}{2} + \tfrac{1}{2}\cos 4\pi t\,.$$

Although the input has a single frequency at 1 Hz, the output has a DC component of 1/2 and a frequency component at 2 Hz.

7.8.1 Complex exponentials are eigenfunctions

We want to pursue further properties of the transfer function. Consider an LTI system's response to a complex exponential of frequency ν, called the *frequency response* of the system. To find $L(e^{2\pi i\nu t})$ we work in the frequency domain.

$$\begin{aligned} W(s) &= H(s)\mathcal{F}(e^{2\pi i\nu t}) \\ &= H(s)\delta(s-\nu) \quad \text{(using the Fourier transform pairing } \delta(s-\nu) \rightleftharpoons e^{2\pi i\nu t}\text{)} \\ &= H(\nu)\delta(s-\nu) \quad \text{(using the property of a function times } \delta\text{)} \end{aligned}$$

Now take the inverse transform. Because $H(\nu)$ is a *constant* we find that

$$L(e^{2\pi i\nu t}) = H(\nu)e^{2\pi i\nu t}\,.$$

This is quite an important discovery. We already know that L is a linear operator. This equation says that

- The exponentials $e^{2\pi i\nu t}$ are *eigenfunctions* of L; that is, the output is a scalar mulitple of these inputs.
- The corresponding eigenvalues are the values of the transfer function $H(\nu)$.

This is *the* reason that complex exponentials are fundamental for studying LTI systems.

Contrast this fact to what happens to a sine or cosine signal under an LTI system L with a real-valued impulse response. For example, feed in a cosine signal $\cos(2\pi\nu t)$. What is the response? We have

$$v(t) = \cos(2\pi\nu t) = \tfrac{1}{2}e^{2\pi i\nu t} + \tfrac{1}{2}e^{-2\pi i\nu t}\,.$$

Hence

$$\begin{aligned} Lv(t) &= \tfrac{1}{2}H(\nu)e^{2\pi i\nu t} + \tfrac{1}{2}H(-\nu)e^{-2\pi i\nu t} \\ &= \tfrac{1}{2}H(\nu)e^{2\pi i\nu t} + \tfrac{1}{2}\overline{H(\nu)}e^{-2\pi i\nu t} \quad (H(-\nu) = \overline{H(\nu)} \text{ because } h(t) \text{ is real-valued}) \\ &= \tfrac{1}{2}(H(\nu)e^{2\pi i\nu t} + \overline{H(\nu)e^{2\pi i\nu t}}) \\ &= \operatorname{Re} H(\nu)e^{2\pi i\nu t} \\ &= |H(\nu)|\cos(2\pi\nu t + \phi_H(\nu)), \end{aligned}$$

where

$$H(\nu) = |H(\nu)|e^{i\phi_H(\nu)}\,.$$

The response is a cosine of the same frequency, but with a changed amplitude and phase. We would find a similar result for the response to a sine signal.

This shows that *neither the cosine nor the sine are themselves eigenfunctions of L*. It is only the *complex exponential* that is an eigenfunction. We are (sort of) back to where we started in the course — with complex exponentials as a basis for decomposing a signal, and now for decomposing an operator.

Let's take this one step further. Suppose again that L has a real-valued impulse response $h(t)$. Suppose also that we input a real periodic signal, which we represent as a Fourier series

$$v(t) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi int}\,.$$

Recall that because $v(t)$ is real the Fourier coefficients $c_n = \hat{v}(n)$ satisfy

$$c_{-n} = \overline{c_n}\,.$$

If we apply L to $v(t)$ we find

$$Lv(t) = \sum_{n=-\infty}^{\infty} c_n L e^{2\pi int} = \sum_{n=-\infty}^{\infty} c_n H(n) e^{2\pi int}\,.$$

But the fact that $h(t)$ is real valued implies that

$$H(-n) = \overline{H(n)}\,,$$

the same symmetry as the Fourier coefficients. That is, if

$$C_n = c_n H(n)$$

then $C_{-n} = \overline{C_n}$ and the output $w(t)$ is also a real, periodic function with Fourier series

$$w(t) = \sum_{n=-\infty}^{\infty} C_n e^{2\pi int}\,.$$

The discrete case The situation for discrete LTI filters is entirely analogous. Suppose the system is given by

$$\mathbf{w} = L\mathbf{v} = \mathbf{h} * \mathbf{v}\,.$$

Take $\mathbf{v} = \boldsymbol{\omega}^k$ as input and take the discrete Fourier transform of both sides of $\mathbf{w} = \mathbf{h} * \boldsymbol{\omega}^k$:

$$\begin{aligned}\underline{\mathcal{F}}\,\mathbf{w}[m] &= \underline{\mathcal{F}}\,\mathbf{h}[m]\,\underline{\mathcal{F}}\boldsymbol{\omega}^k[m] \\ &= \underline{\mathcal{F}}\,\mathbf{h}[m]\,N\boldsymbol{\delta}[m-k] \quad \text{(remember the extra factor } N\text{)} \\ &= \mathbf{H}[m]\,N\boldsymbol{\delta}[m-k] \\ &= \mathbf{H}[k]\,N\boldsymbol{\delta}[m-k]\end{aligned}$$

Now take the inverse DFT of both sides (remembering that $\underline{\mathcal{F}}\,\boldsymbol{\delta}_k = (1/N)\boldsymbol{\omega}^k$):

$$\mathbf{w} = \mathbf{H}[k]\,\boldsymbol{\omega}^k\,.$$

Hence

$$L\boldsymbol{\omega}^k = \mathbf{H}[k]\,\boldsymbol{\omega}^k$$

and we see that $\boldsymbol{\omega}^k$ is an eigenvector of L for $k = 0, 1, \ldots, N-1$ with eigenvalue $\mathbf{H}[k]$.

Remark We already know that $\mathbf{1}$, $\boldsymbol{\omega}$, $\boldsymbol{\omega}^2$, $\ldots$, $\boldsymbol{\omega}^{N-1}$ are an orthogonal basis for $\mathbf{C}^N$. This is the orthogonality of the vector complex exponentials — the basis for much of the theory of the DFT. This new result says that if L is a discrete LTI system then the complex exponentials are an orthogonal basis for $\mathbf{C}^N$ consisting of eigenvectors of L. They "diagonalize" L, meaning that if the matrix of L is expressed in this basis it is a diagonal with diagonal entries $\mathbf{H}[k]$.

7.9 Matched Filters

LTI systems are used extensively in the study of communications systems, where a fundamental concern is to distinguish the signal from noise and to design a filter that will do the job. That is, the filter should "respond strongly" to one particular signal and only to that signal. This is *not* a question of recovering or extracting a particular signal from the noise, it's a question of *detecting* whether the signal is present — think radar. If the filtered signal rises above a certain threshold an alarm goes off, so to speak, and we believe that the signal we want is there.

Here's a *highly* condensed discussion of this central problem, but even a condensed version fits naturally with what we're doing. With $w(t) = (h * v)(t)$, and $W(s) = H(s)V(s)$, we'll try to design the transfer function $H(s)$ so that the system responds strongly (a term still to be defined) to a particular signal $v_0(t)$.

Let's begin with some general observations. Suppose an incoming signal is of the form $v(t) + p(t)$ where $p(t)$ is "noise". Then the output is $h(t) * (v(t) + p(t)) = w(t) + q(t)$, where $q(t)$, the contribution of the noise to the output, has total energy

$$\int_{-\infty}^{\infty} |q(t)|^2\,dt\,,$$

which, using Parseval's theorem and the transfer function, we can write as

$$\int_{-\infty}^{\infty} |q(t)|^2\,dt = \int_{-\infty}^{\infty} |Q(s)|^2\,ds = \int_{-\infty}^{\infty} |H(s)|^2|P(s)|^2\,ds\,.$$

Now we make an assumption about the nature of the noise. Take the special case of "white noise". A reasonable definition of that term — one that translates into a workable condition — is that $p(t)$ should

have *equal power in all frequencies.*[6] This means simply that $|P(s)|$ is *constant*, say $|P(s)| = C$, and so the output energy of the noise is

$$E_{\text{noise}} = C^2 \int_{-\infty}^{\infty} |H(s)|^2 \, ds \, .$$

We can compare the energy of the noise output to the strength of the output signal $w(t) = (h * v)(t)$. Using Fourier inversion we write

$$w(t) = \int_{-\infty}^{\infty} W(s) e^{2\pi i s t} \, ds = \int_{-\infty}^{\infty} H(s) V(s) e^{2\pi i s t} \, ds \, .$$

Long dormant since our discussion of Fourier series, used briefly in a problem on autocorrelation, just waiting for this moment, the Cauchy-Schwarz inequality now makes its triumphant reappearance.[7] According to Cauchy-Schwarz,

$$\begin{aligned} |w(t)|^2 &= \left| \int_{-\infty}^{\infty} H(s) V(s) e^{2\pi i s t} \, ds \right|^2 \\ &\leq \int_{-\infty}^{\infty} |H(s)|^2 \, ds \int_{-\infty}^{\infty} |V(s)|^2 \, ds \quad \text{(we also used } |e^{2\pi i s t}| = 1) \end{aligned}$$

That is,

$$\frac{|w(t)|^2}{E_{\text{noise}}} \leq \frac{1}{C^2} \int_{-\infty}^{\infty} |V(s)|^2 \, ds \, .$$

By definition, the fraction $|w(t)|^2 / E_{\text{noise}}$ is the *signal-to-noise* ratio, abbreviated SNR. The biggest the SNR can be is when there is equality in the Cauchy-Schwarz inequality. Thus the filter that gives the strongest response, meaning largest SNR, when a given signal $v_0(t)$ is part of a combined noisy signal $v_0(t) + p(t)$ is the one with transfer function proportional to $\overline{V_0(s) e^{2\pi i s t}}$, where $V_0(s) = \mathcal{F} v(t)$.

This result is sometimes referred to as the *matched filter theorem*:

- To design a filter that has the strongest response to a particular signal $v_0(t)$, in the sense of having the largest signal-to-noise ratio, design it so the transfer function $H(s)$ "has the same shape" as $\overline{V_0(s)}$.

To recapitulate, when the filter is designed this way, then

$$\begin{aligned} |w_0(t)|^2 &= \left| \int_{-\infty}^{\infty} H(s) V_0(s) e^{2\pi i s t} \, ds \right|^2 \\ &= \frac{1}{C^2} E_{\text{noise}} \int_{-\infty}^{\infty} |V_0(s)|^2 \, ds = \frac{1}{C^2} E_{\text{noise}} \int_{-\infty}^{\infty} |v_0(t)|^2 \, dt \quad \text{(by Parseval)} \end{aligned}$$

Thus the SNR is

$$\frac{|w_0(t)|^2}{E_{\text{noise}}} = \frac{1}{C^2} \int_{-\infty}^{\infty} |v_0(t)|^2 \, dt = \frac{1}{C^2} (\text{Energy of } v(t)) \, .$$

[6] There are other ways to get to this definition, e.g., the autocorrelation of $p(t)$ should be zero.

[7] It says, in the form we're going to use it, $|\int f \overline{g} \, dt| \leq (\int |f|^2 \, dt)^{1/2} (\int |g|^2 \, dt)^{1/2}$. With equality if and only if f is a constant multiple of $\overline{g}$

7.10 Causality

In the words of our own Ron Bracewell: "It is well known that effects never precede their causes." This commonsense sentiment, put into action for systems, and for inputs and outputs, is referred to as *causality*. In words, we might describe causality as saying that the past determines the present but not vice versa. More precisely, if L is a system and $Lv(t) = w(t)$ then the value of $w(t)$ for $t = t_0$ depends only on the values of $v(t)$ for $t < t_0$. More precisely still:

If $v_1(t) = v_2(t)$ for $t < t_0$ then $Lv_1(t) = Lv_2(t)$ for $t < t_0$, and this holds for all t_0.

At first glance you may be puzzled by the statement, or wonder if there's any statement at all: If two signals are the same then mustn't their outputs be the same? But the signals $v_1(t)$ and $v_2(t)$ are assumed to exist for all time, and the system L might produce outputs based not only on the values of the inputs up to a certain time, t_0, but on times "into the future" as well. Thus it is a nontrivial requirement that a system be causal — that the values of the output depend only on the values of the input "up to the present".

The definition given above is the general definition for a system L to be causal.[8] One often sees other versions of the definition, depending on additional assumptions on the system, and it's worthwhile sorting these out.

For example, we can formulate the causality condition a little more compactly, and a little more conveniently, when L is linear (which, if you notice, I didn't assume above). First, if L is linear *and causal* then $v(t) = 0$ for $t < t_0$ implies that $w(t) = 0$ for $t < t_0$. Why? Watch the logic here. Let $u(t) \equiv 0$ be the zero signal. Then $Lu(t) = 0$ because L is linear. Causality means that $v(t) = 0 = u(t)$ for $t < t_0$ implies $w(t) = Lv(t) = Lu(t) = 0$ for $t < 0$.

Conversely, suppose that L is linear and that $v(t) = 0$ for $t < t_0$ implies $w(t) = 0$ for $t < t_0$, where $w(t) = Lv(t)$, as usual. We claim that L is causal. For this, if $v_1(t) = v_2(t)$ for $t < t_0$ then $v(t) = v_1(t) - v_2(t) = 0$ for $t < t_0$, so, by the hypothesis on L, if $t < t_0$ then $0 = Lv(t) = L(v_1(t) - v_2(t)) = Lv_1(t) - Lv_2(t)$, i.e. $Lv_1(t) = Lv_2(t)$ for $t < t_0$. These arguments together show that:

- A linear system is causal if and only if $v(t) = 0$ for $t < t_0$ implies $w(t) = 0$ for $t < t_0$.

Finally, for an LTI system one t_0 is as good as the next, so to speak, and we can simplify the definition of causality even further. If L is linear and causal then $v(t) = 0$ for $t < 0$ implies $w(t) = 0$ for $t < 0$. (Note: $t < 0$ here, not $t < t_0$.) Conversely, suppose L is an LTI system such that $v(t) = 0$ for $t < 0$ implies $Lv(t) = 0$ for $t < 0$. We claim that L is causal. For this, suppose that $v(t)$ is zero for $t < t_0$ and let $w(t) = Lv(t)$. The signal $u(t) = v(t + t_0)$ is zero for $t < 0$ and hence $Lu(t) = 0$ for $t < 0$. But by time invariance $Lu(t) = Lv(t + t_0) = w(t + t_0)$. Thus $w(t + t_0) = 0$ for $t < 0$, i.e. $w(t) = 0$ for $t < t_0$. The conclusion is:

- An LTI system is causal if and only if $v(t) = 0$ for $t < 0$ implies $w(t) = 0$ for $t < 0$.

In many treatments of causal systems it is only this last definition that is presented, with the assumptions, sometimes taken tacitly, that linearity and time invariance are in force. In fact, one often runs directly into the following two definitions, usually given without the preceding motivation:

[8] One also sees the definition stated with "$<$" replaced by "$\leq$", i.e., to include "the present". I gave the definition with "$<$" because it's more standard, especially in connection with the characterization of causality via the impulse response, which we'll soon see.

- A function $v(t)$ is *causal* if it is zero for $t < 0$.
- An LTI system is *causal* if causal inputs go to causal outputs.

That's a fine definition, but it seems a little stark.

Still another definition of causality for LTI systems can be given in terms of the impulse response. Since $\delta(t) = 0$ for $t < 0$ the impulse response $h(t) = L\delta(t)$ of a causal LTI system satisfies $h(t) = 0$ for $t < 0$. Conversely, if the impulse response for an LTI system satisfies this condition, and if $v(t)$ is a signal with $v(t) = 0$ for $t < 0$ then the output $w(t) = (v * h)(t)$ is zero for $t < 0$, too. Thus L is causal. This is the definition one finds in Bracewell's book, to cite one example:

- An LTI system is causal if and only if its impulse response $h(t)$ is zero for $t < 0$.

Many systems are causal and many aren't, and as an exercise you should look over the examples we've already had and decide which is which. Causality is sometimes called the condition of "physical realizability" because of its "past determines the present" interpretation. Many systems governed by differential equations are causal (current in RC circuits, for example). Indeed, the idea that a differential equation plus initial conditions determines the solution uniquely is another way causality is often put to use.[9]

If causality seems naturally to apply to a system where "time is running" and "past" and "present" make sense, it's important to realize that causality may not apply where "everything is there already". For example, optical systems may not be causal. The variable is typically not time but rather may represent some kind of spatial relation. The input may be an object and the output an image. There is no past and present, all the information that there is and ever will be is present. Or to take an example when time is still the relevant variable, one may want to *design* a noncausal system to filter *prerecorded* music. In the case of prerecorded music the full signal is known, and one may *want* to take into account past, present and future values of the input to decide on what the output should sound like.

7.11 The Hilbert Transform

Let's now consider a causal LTI system given to us as a convolution

$$w(t) = (h * v)(t)\,.$$

Taking the Fourier transform,

$$W(s) = H(s)V(s)\,.$$

Owing to the fact that $h(t)$ is causal, the transfer function $H(s)$ has some special properties that are important in many applications.

To analyze $H(s)$ it is convenient to use the unit step function

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

We've modified this slightly from how we first defined it. For one thing we are calling it u instead of H, since we're using H for the transfer function, and for another it's more natural to have $u(t) = 0$ for $t < 0$ instead of for $t \leq 0$ for use in connection with causality as we've defined it. No big deal.

[9] I'm not even going to *touch* what happens to causality in quantum mechanics.

Recall that the Fourier transform of $u(t)$ is

$$U(s) = \tfrac{1}{2}\left(\delta(s) + \frac{1}{\pi i s}\right).$$

Since $h(t) = 0$ for $t < 0$, we can write

$$h(t) = u(t)h(t).$$

Therefore

$$\begin{aligned} H(s) &= U(s) * H(s) \\ &= \tfrac{1}{2}\left(\delta(s) + \frac{1}{\pi i s}\right) * H(s) \\ &= \tfrac{1}{2}(\delta * H)(s) + \frac{1}{2\pi i s} * H(s) = \tfrac{1}{2}H(s) + \frac{1}{2\pi i s} * H(s). \end{aligned}$$

Thus we have the weird looking result

$$\tfrac{1}{2}H(s) = \frac{1}{2\pi i s} * H(s) \quad \text{or} \quad H(s) = \frac{1}{\pi i s} * H(s).$$

Let's work with this a little more.

Split $H(s)$ into its real and imaginary parts, writing, for the time being,

$$H(s) = R(s) + iI(s),$$

where

$$R(s) = \operatorname{Re} H(s), \quad I(s) = \operatorname{Im} H(s).$$

Then

$$\begin{aligned} R(s) + iI(s) &= \frac{1}{\pi i s} * \big(R(s) + iI(s)\big) \\ &= \frac{1}{\pi i s} * R(s) + \frac{1}{\pi s} * I(s) = \frac{1}{\pi s} * I(s) - i\frac{1}{\pi s} * R(s) \quad \text{(using } 1/i = -i\text{)}. \end{aligned}$$

Now equate real and imaginary parts on the two sides of the equation. We see that

$$\begin{aligned} \operatorname{Re} H(s) &= \frac{1}{\pi s} * \operatorname{Im} H(s) \\ \operatorname{Im} H(s) &= -\frac{1}{\pi s} * \operatorname{Re} H(s) \end{aligned}$$

The real and imaginary parts of the transfer function $H(s)$ are "paired" with each other; we get one from the other via a convolution with $1/\pi s$. Ordinarily, the real and imaginary parts of a complex-valued function have nothing to do with each other, but in this case they are very strongly related — if we know one then we know the other. It's as if we get all of $H(s)$ by knowing only half of it. This merits some additional discussion.

The Hilbert transform as an operator Way back in Chaper 4 I threatened to look at a convolution with $1/x$. The time has come. The *Hilbert transform*[10] of a function $v(x)$ is defined by

$$\mathcal{H}v(x) = -\frac{1}{\pi x} * v(x) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{v(y)}{x-y}\,dy = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{v(y)}{y-x}\,dy.$$

[10] Named after David Hilbert, (1862–1943). Hilbert made fundamental contributions to a great many areas of mathematics, including number theory, geometry, logic, and differential and integral equations. The Hilbert transform comes up in several areas of harmonic analysis, but I think it may well be true that more engineers know more about its properties than many mathematicians.

A few comments. The integral is improper since the denominator in the integrand is zero when $y = x$.[11] As we discussed in Chapter 4 (and we'll do no more now), for functions $v(x)$ that we typically work with, the integral exists as a *Cauchy principal value*. That is, it exists, and is defined, via taking a *symmetric* limit:

$$\mathcal{H}v(x) = \frac{1}{\pi}\text{pr.v.}\int_{-\infty}^{\infty}\frac{v(y)}{y-x}\,dy = \frac{1}{\pi}\lim_{\epsilon\to 0}\left(\int_{-\infty}^{x-\epsilon}\frac{v(y)}{y-x}\,dy + \int_{x+\epsilon}^{\infty}\frac{v(y)}{y-x}\,dy\right)$$

Having said this once I won't say it again, and I won't write "pr.v." in front of the integral anymore.

In Chapter 4 we found the Fourier transform of $1/x$ to be

$$\mathcal{F}\left(\frac{1}{x}\right) = -\pi i \operatorname{sgn} s\,,$$

where

$$\operatorname{sgn}(x) = \begin{cases} +1 & x > 0 \\ -1 & x < 0 \end{cases}$$

Thus

$$\mathcal{F}\left(-\frac{1}{\pi x}\right) = i \operatorname{sgn} s\,,$$

and

$$\mathcal{F}(\mathcal{H}v)(s) = \mathcal{F}(-\frac{1}{\pi x} * v(x)) = i \operatorname{sgn} s V(s)\,.$$

For right now we want to draw just one conclusion from the calculation of the Fourier transform. If we apply $\mathcal{H}$ *twice* to a function then, on taking the Fourier transform,

$$\mathcal{F}(\mathcal{H}(\mathcal{H}v))(s) = \mathcal{F}\left((-\frac{1}{\pi x}) * (-\frac{1}{\pi x}) * v(x)\right) = (i \operatorname{sgn} s)(i \operatorname{sgn} s)V(s) = -\operatorname{sgn}^2 s\, V(s) = -V(s)\,.$$

Taking the inverse Fourier transform we see that

$$\mathcal{H}(\mathcal{H}v(x)) = -v(x)\,.$$

Hence the Hilbert transform is *invertible* and its inverse is just its negative:

$$\mathcal{H}^{-1}v(x) = -\mathcal{H}v(x) = \frac{1}{\pi x} * v(x)\,,$$

a result that's far from obvious.

Let's now use the Hilbert transform and its inverse to recast the results on the real and imaginary parts of the transfer function for a causal LTI system:

If L is a causal LTI system with transfer function $H(s)$, then

$$\operatorname{Im} H(s) = \mathcal{H}(\operatorname{Re} H(s))$$
$$\operatorname{Re} H(s) = \mathcal{H}^{-1}(\operatorname{Im} H(s))$$

Sorry if I can't make this statement more thrilling at this point, but it turns out to be quite significant. The Hilbert transform comes up in many applications, especially in communications. We'll examine a few more of its properties.

[11] It's also improper because it has limits $\pm\infty$, but that's the kind of improper integral we're used to working with.

The Hilbert transform as an LTI system Since it's a convolution, we can view the Hilbert transform as an LTI system:

$$w(t) = (\mathcal{H}v)(t) = -\frac{1}{\pi t} * v(t)\,.$$

However, though $\mathcal{H}$ is naturally associated with causal systems, $w = \mathcal{H}v$ is *not* a causal system since its impulse response is $-1/\pi t$.

As a filter, this convolution is a rather strange operation. We can write it in terms of Fourier transforms as

$$W(s) = i\,\mathrm{sgn}\,s\,V(s)\,,$$

or, in terms of amplitude and phase,

$$|W(s)|e^{i\phi_W(s)} = e^{i\pi/2}\mathrm{sgn}\,s\,|V(s)|\,e^{i\phi_V(s)}\,.$$

The amplitudes of the spectrum are unchanged but the phases are rotated. The phases corresponding to positive frequencies are rotated by $+\pi/2$, since $\mathrm{sgn}\,s = 1$ for $s > 0$. The phases corresponding to negative frequencies are rotated by $-\pi/2$, since $\mathrm{sgn}\,s = -1$ for $s < 0$, and

$$-1 \cdot e^{i\pi/2} = e^{-i\pi}e^{i\pi/2} = e^{-i\pi/2}\,.$$

In summary

$$W(s) = \begin{cases} |V(s)|e^{i(\phi_V(s)+\pi/2)} & s > 0 \\ |V(s)|e^{i(\phi_V(s)-\pi/2)} & s < 0 \end{cases}$$

For this reason $\mathcal{H}v(t)$ is referred to as the *quadrature function* of $v(t)$ in some literature. Among the various definitions, the term "quadrature" means the process of making something square, and in astronomy it refers to "any configuration in which the angular separation of two celestial bodies, as measured from a third, is 90°."

Suppose we pass a cosine through the Hilbert LTI filter. Take $v(t) = \cos 2\pi at$, $a > 0$, for example. Then $V(s) = (1/2)(\delta(s-a) + \delta(s+a))$ and the Fourier transform of the output $w(t)$ is

$$\begin{aligned} W(s) &= \frac{i}{2}\mathrm{sgn}\,s(\delta(s-a) + \delta(s+a)) \\ &= \frac{i}{2}(\mathrm{sgn}\,a\,\delta(s-a) + \mathrm{sgn}\,(-a)\delta(s+a)) \\ &= \frac{i}{2}(\delta(s-a) - \delta(s+a)) = -\frac{1}{2i}(\delta(s-a) - \delta(s+a)) \end{aligned}$$

We recognize this last expression as the Fourier transform of $-\sin 2\pi at$. Thus

$$\mathcal{H}\cos 2\pi at = -\sin 2\pi at\,.$$

Using $\mathcal{H}^{-1} = -\mathcal{H}$ we then also see that

$$\mathcal{H}\sin 2\pi at = \cos 2\pi at\,.$$

Personally, I can't think of a more complicated way to turn a sine into a cosine than convolving with $1/\pi t$, but there you are.

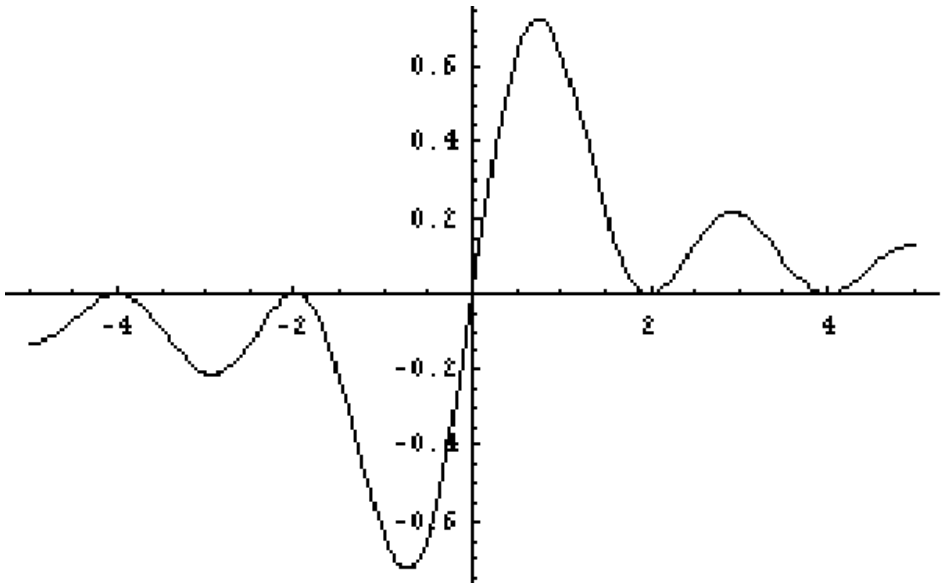
For another example, direct integration gives the Hilbert transform of the rectangle function:

$$\mathcal{H}\Pi(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Pi(y)}{y-x}\,dy = \frac{1}{\pi}\int_{-1/2}^{1/2}\frac{1}{y-x}\,dy = \frac{1}{\pi}\ln\left|\frac{x-\frac{1}{2}}{x+\frac{1}{2}}\right|$$

And in case you're wondering, in the Section 7.13 I'll give the calculations that show

$$\mathcal{H}\,\mathrm{sinc}\,t = -\frac{\pi t}{2}\mathrm{sinc}^2\frac{t}{2}\,.$$

The graph is:



The Hilbert transform and analytic signals Those of you who have had courses in circuits are well aware of how much using "phasors" (also known as complex exponentials) can simplify the analysis. In this course we have likewise seen many examples of the benefits of using complex exponentials to represent real signals, but the one drawback of introducing complex exponentials is that they seem inevitably to require that we consider negative and positive frequencies in representing real signals. This manifests itself in the spectrum of a real signal having two sidebands due to the relationship $\mathcal{F}f(-s) = \overline{\mathcal{F}f(s)}$. However, the use of the Hilbert transform allows one to complete a real signal to an "analytic signal" — a complex-valued form of the signal that involves only *positive* frequencies. This representation can simplify results and methods in some cases, especially in applications to communications systems. In the present notes we'll just go through the basic construction — take a course, any course, on communications to see this in daily use.

To take an example, the real signal $\cos 2\pi at$ can be written directly, as usual, as

$$\cos 2\pi at = \tfrac{1}{2}(e^{2\pi iat} + e^{-2\pi iat})\,.$$

The right hand side includes both the frequencies a and $-a$. But it's also natural to recover the real signal as the real part of the complex or "analytic signal"

$$\cos 2\pi at = \mathrm{Re}(\cos 2\pi at + i \sin 2\pi at) = \mathrm{Re}\, e^{2\pi iat}\,,$$

which involves only positive frequencies. In this example the real and imaginary parts are a Hilbert transform pair, $\sin 2\pi at = -\mathcal{H}\cos 2\pi at$. The pairing of real and imaginary parts via the Hilbert transform is what's done in general. Here's how.

Take a real-valued signal $v(t)$ and denote its Fourier transform by $V(s)$, as usual. Since $v(t)$ is real valued, $V(s)$ has the symmetry property

$$V(-s) = \overline{V(s)}\,,$$

so, in particular, v has positive and negative frequencies in its spectrum. Cut off the negative frequencies by multiplying $V(s)$ by the unit step function $u(s)$ (*not* the Fourier transform of u, just u). Actually, as we'll see, it's best to consider $2u(s)V(s)$.

Now form

$$\mathcal{Z}(t) = \mathcal{F}^{-1}(2u(s)V(s))\,.$$

By construction $\mathcal{Z}(t)$ has no negative frequencies in its spectrum. Watch how it's related to the original signal $v(t)$. We can write $2u(s) = 1 + \operatorname{sgn} s$, hence

$$\begin{aligned}\mathcal{F}^{-1}(2u(s)V(s)) &= \mathcal{F}^{-1}((1+\operatorname{sgn} s)V(s)) \\ &= \mathcal{F}^{-1}(V(s) + \operatorname{sgn} sV(s)) \\ &= \mathcal{F}^{-1}(V(s)) + \mathcal{F}^{-1}(\operatorname{sgn} sV(s)) \;=\; v(t) - i(\mathcal{H}v)(t)\end{aligned}$$

$\mathcal{Z}(t)$ is called the *analytic signal* associated with $v(t)$.

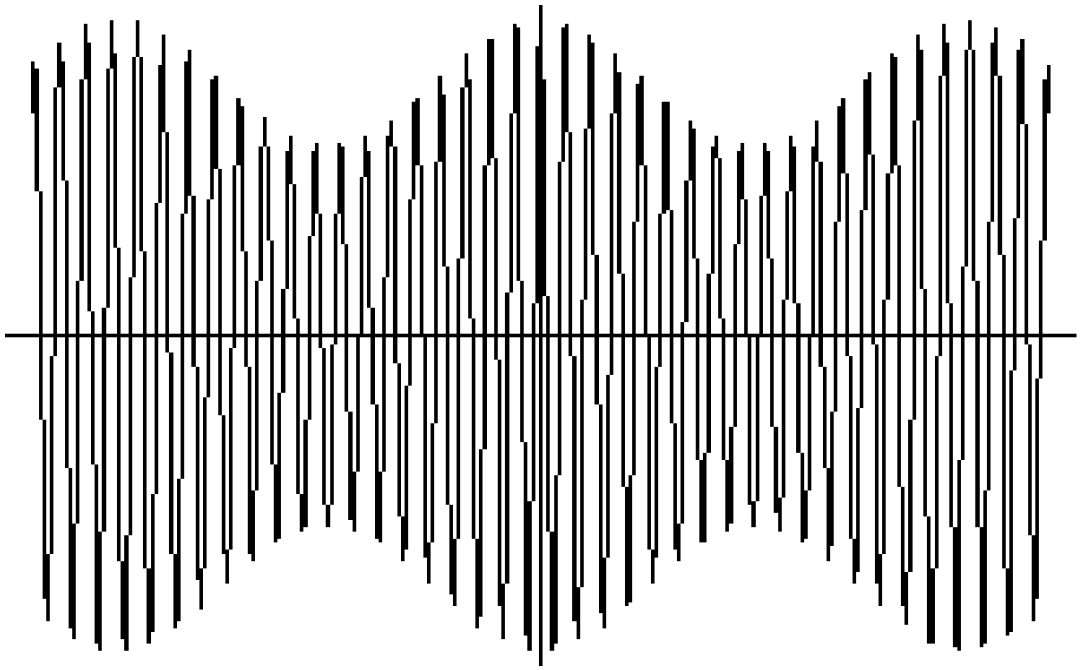
To summarize, let $v(t)$ be a real-valued signal with Fourier transform $V(s)$. Let $\mathcal{Z}(t) = \mathcal{F}^{-1}(2u(s)V(s))$. Then

1. $\mathcal{Z}(t) = v(t) - i(\mathcal{H}v)(t)$, so that $v(t) = \operatorname{Re}\mathcal{Z}(t)$.
2. $\mathcal{Z}(t)$ has no negative frequencies in its spectrum.
3. Briefly, to get the spectrum of $\mathcal{Z}(t)$ cut off the negative frequencies of $v(t)$ and double the amplitudes of the positive frequencies.

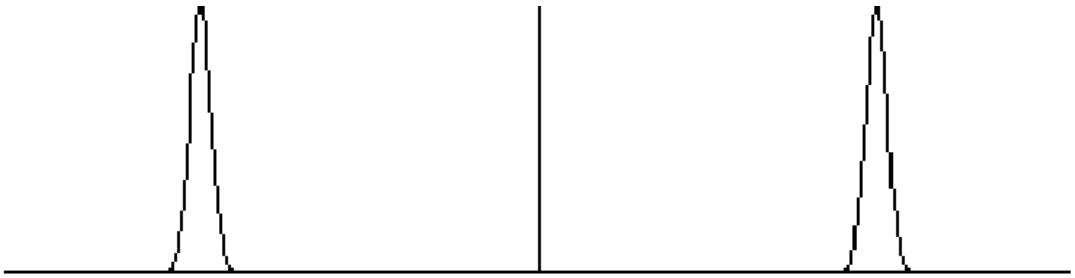
Two examples of this where we've already computed the Hilbert transform are:

$$\begin{aligned}v(t) = \cos 2\pi at \;&\Rightarrow\; \mathcal{Z}(t) = \cos 2\pi at + i\sin 2\pi at \\ v(t) = \operatorname{sinc} t \;&\Rightarrow\; \mathcal{Z}(t) = \operatorname{sinc} t + \frac{i\pi t}{2}\operatorname{sinc}^2\frac{t}{2}\end{aligned}$$

An example: Narrowband signals Let's look at just one application of the analytic signal, to the representation of *narrowband signals*. Without giving a precise definition, a narrowband signal might look something like this:



And its spectrum might look something like this:



The idea is that the spectrum is concentrated in two islands, roughly centered about some frequencies $\pm s_0$, and that the spread about $\pm s_0$ is much less than s_0. Often such a signal is of the form

$$v(t) = A(t)\cos(2\pi s_0 t + \phi(t))\,,$$

where the amplitude $A(t)$ and the phase $\phi(t)$ are *slowly varying*. Write $v(t)$ as

$$v(t) = \tfrac{1}{2}A(t)\left(e^{2\pi i s_0 t + i\phi(t)} + e^{-(2\pi i s_0 t + i\phi(t))}\right) = \tfrac{1}{2}A(t)e^{i\phi(t)}e^{2\pi i s_0 t} + \tfrac{1}{2}A(t)e^{-i\phi(t)}e^{-2\pi i s_0 t}\,.$$

Remember that we get the analytic signal corresponding to $v(t)$ by cutting off the negative frequencies and doubling the amplitudes of the positive frequencies. So to the extent that the spectral islands of $v(t)$ are really separated, and the amplitude and phase are slowly varying functions, we are justified in saying that, approximately,

$$\mathcal{Z}(t) = A(t)e^{i\phi(t)}e^{2\pi i s_0 t}\,.$$

In terms of $\mathcal{Z}(t)$ the envelope of the signal $v(t)$ is

$$A(t) = |\mathcal{Z}(t)|$$

and the phase is

$$\psi(t) = \arg \mathcal{Z}(t) = 2\pi s_0 t + \phi(t)\,.$$

In this context it is common to introduce the *instantaneous frequency*

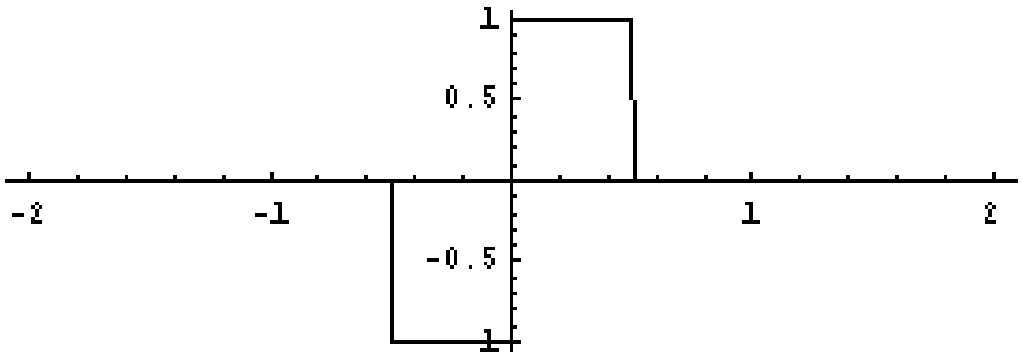
$$s_{\text{inst}} = \frac{1}{2\pi}\frac{d\psi}{dt} = s_0 + \frac{1}{2\pi}\frac{d\phi}{dt}\,.$$

7.12 Appendix: The Hilbert Transform of sinc

To find the Hilbert transform of the sinc function, we start in the frequency domain. That is,

$$\mathcal{F}(\mathcal{H}\,\text{sinc})(s) = i\,\text{sgn}\,s\,\mathcal{F}\,\text{sinc}(s) = i\,\text{sgn}\,s\,\Pi(s)\,.$$

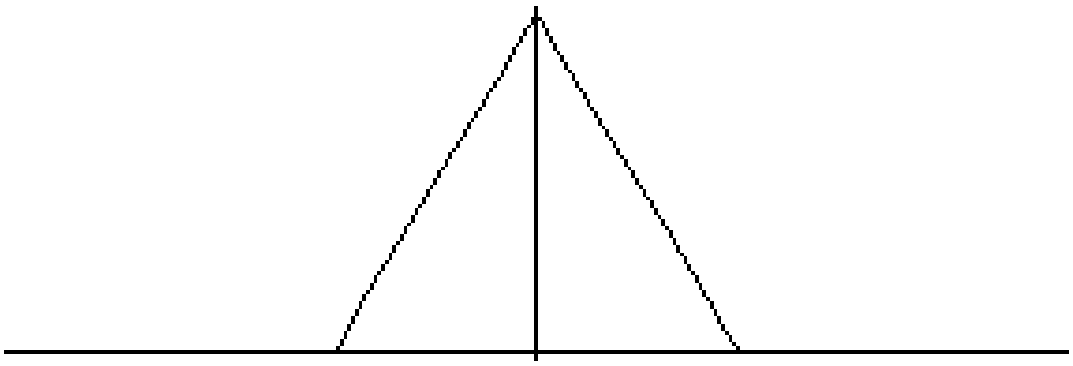
The graph of $\text{sgn}\,s\,\Pi(s)$ looks like this:



How do we find the inverse Fourier transform of something like that? Here's one way to do it. Remember the triangle function, defined by

$$\Lambda(x) = \begin{cases} 1 - |x| & |x| \le 1 \\ 0 & \text{otherwise} \end{cases}$$

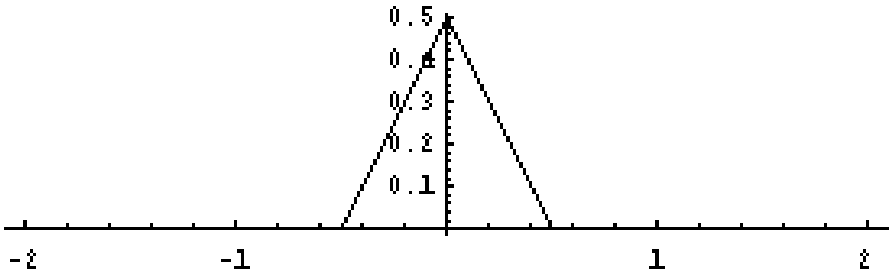
The graph is



Its Fourier transform is

$$\mathcal{F}\Lambda(s) = \operatorname{sinc}^2 s\,.$$

If we scale this to $\frac{1}{2}\Lambda(2x)$, which has the following graph,



then it's clear that

$$\operatorname{sgn} x\,\Pi(x) = -\frac{d}{dx}\frac{1}{2}\Lambda(2x)\,.$$

Thus

$$\mathcal{F}^{-1}(i\operatorname{sgn} s\,\Pi(s)) = \mathcal{F}^{-1}(-i\frac{d}{ds}\frac{1}{2}\Lambda(2s))$$

and we can find the right hand side using the derivative theorem (and the stretch theorem). This gives

$$\mathcal{F}^{-1}(-i\frac{d}{ds}\frac{1}{2}\Lambda(2s)) = -i(-2\pi it)\frac{1}{4}\operatorname{sinc}^2\frac{t}{2} = -\frac{\pi t}{2}\operatorname{sinc}^2\frac{t}{2}$$

and the final result

$$\mathcal{H}\operatorname{sinc} t = -\frac{\pi t}{2}\operatorname{sinc}^2\frac{t}{2}\,.$$

7.13 Filters Finis

We now want to live and work in the discrete world and discuss some typical digital filters and their uses. There are entire books and courses on digital filters (e.g., EE 264 and EE 265), so we'll touch on only a

few aspects, enough, I hope to give you a sense of how digital filters work and to encourage you to find out more.

The set up is a discrete LTI system, given by

$$\mathbf{w} = \mathbf{h} * \mathbf{v}\,,$$

where $\mathbf{v}$, $\mathbf{h}$, $\mathbf{w}$ are N-vectors, periodized, and $\mathbf{h}$ is the impulse response. Taking the DFT we write

$$\mathbf{W} = \mathbf{H}\,\mathbf{V}\,.$$

7.13.1 Averaging in the time domain

A common use of filters is to "smooth" data. The procedure starts out as a least squares fit, which turns into a running average, which is then realized to be a digital filter, which is finally refined by analysis in the frequency domain. We've spoken many times of convolution as a smoothing operation, and here it is quite explicitly.

Suppose we have a list of data, typically arising as samples of some function $f(x)$, say $(x_0, f(x_0))$, $(x_1, f(x_1))$, $\ldots$, $(x_n, f(x_n))$. We suppose that we've sampled at evenly spaced points. We plot the data and join successive points to make a continuous curve. It's continuous but it looks too jagged, perhaps, and we suspect too much noise in the measurements, whatever. We set out to smooth the data.

Averaging and least squares Let $g(x)$ be the smoothed version of $f(x)$. A first attempt to define $g(x)$ is this:

- Instead of taking the points two at a time and joining successively to make a graph, start by taking the first three data points $(x_0, f(x_0))$, $(x_1, f(x_1))$, $(x_2, f(x_2))$ and find the line that best fits these three points according to the method of least squares.
- As the first new data point, take the midpoint of the least squares fit. Since the x's are evenly spaced the midpoint has x-coordinate x_1, and $g(x_1)$ is then defined to be the corresponding y-coordinate on the line, i.e., the midpoint of the least squares fit is $(x_1, g(x_1))$.
- Repeat this procedure using the triple of original data points $(x_1, f(x_1))$, $(x_2, f(x_2))$, $(x_3, f(x_3))$. This gives the second new data point $(x_2, g(x_2))$.
- Then work with $(x_2, f(x_2))$, $(x_3, f(x_3))$, $(x_4, f(x_4))$ to get the third new data point, $(x_3, g(x_3))$ and so on through the list up to the triple $(x_{n-2}, f(x_{n-2})$, $(x_{n-1}, f(x_{n-1})$, $(x_n, f(x_n))$ producing the final new data point $x_{n-1}, g(x_{n-1}))$.

Note that this procedure "drops" the original left endpoint and the right endpoint, i.e., the new data points run from $(x_1, g(x_1))$ up to $(x_{n-1}, g(x_{n-1}))$.

Let's find a formula for $g(x_i)$, and let's switch to writing things in terms of discrete signals. Say we have taken N data points, which we write as a vector $\mathbf{f}$. For reasons of symmetry, especially later when we switch to the frequency domain, it's a little more convenient to write

$$\mathbf{f} = (\mathbf{f}[-\frac{N}{2}+1], \ldots, \mathbf{f}[0], \ldots, \mathbf{f}[\frac{N}{2}])\,,$$

letting the index range from $m = -N/2+1$ to $m = N/2$. We consider the samples $\mathbf{f}[m]$ to be tagged by the index m and the data points to be $(m, \mathbf{f}[m])$ (rather than $(x_m, f(x_m))$) and we want to find the new

points $(m, \mathbf{g}[m])$ determined by the least squares method just described. The formula for least squares (which I won't derive here — see EE 263) gives a very simple answer to this:

$$\mathbf{g}[m] = \tfrac{1}{3}(\mathbf{f}[m-1] + \mathbf{f}[m] + \mathbf{f}[m+1])$$

In words, the new value $\mathbf{g}[m]$ is the average of the old value $\mathbf{f}[m]$ with its adjacent values on either side, $\mathbf{f}[m-1]$ and $\mathbf{f}[m+1]$.

The formula for $\mathbf{g}$ is often called a *running average* because the point $\mathbf{f}[m]$ is replaced by the average of itself and its neighbors, and the neighboring points overlap. Note, however, that it's always the original data that's getting averaged, not a combination of old and new data[12].

Using $\boldsymbol{\delta}$'s it's very easy to express $\mathbf{g}$ as a convolution,

$$\mathbf{g} = \tfrac{1}{3}(\boldsymbol{\delta}_1 + \boldsymbol{\delta}_0 + \boldsymbol{\delta}_{-1}) * \mathbf{f}\,,$$

for we have

$$\begin{aligned}\mathbf{g}[m] &= \tfrac{1}{3}((\boldsymbol{\delta}_1 * \mathbf{f})[m] + (\boldsymbol{\delta}_0 * \mathbf{f})[m] + (\boldsymbol{\delta}_{-1} * \mathbf{f})[m]) \\ &= \tfrac{1}{3}(\mathbf{f}[m-1] + \mathbf{f}[m] + \mathbf{f}[m+1])\,,\end{aligned}$$

just as advertised. This is an appealing and useful way to think of the operation, and very suitable to computation, but convolution comes at a cost — there are a few consequences of the formula that we must sort out.

Modified averaging If we let m run from $-N/2+2$ to $N/2-1$, dropping the left and right endpoints, then we have described algebraically what we originally described geometrically, no more, no less. But if we let convolution be convolution, then it's natural (essential if the DFT comes into the picture) that the inputs be periodic, and with m running from $-N/2-1$ to $N/2$ the endpoints do come in and something funny happens there.

At the left endpoint $m = -N/2+1$ we have, using periodicity,

$$\begin{aligned}\mathbf{g}[-\frac{N}{2}+1] &= \tfrac{1}{3}\Big(\mathbf{f}[-\frac{N}{2}] + \mathbf{f}[-\frac{N}{2}+1] + \mathbf{f}[-\frac{N}{2}+2]\Big) \\ &= \tfrac{1}{3}\Big(\mathbf{f}[\frac{N}{2}] + \mathbf{f}[-\frac{N}{2}+1] + \mathbf{f}[-\frac{N}{2}+2]\Big) \\ &\qquad \text{(the left index changed from } -\frac{N}{2} \text{ to } -\frac{N}{2}+N = \frac{N}{2})\,.\end{aligned}$$

And at the right endpoint $m = N/2$, again using periodicity, we have,

$$\begin{aligned}\mathbf{g}[\frac{N}{2}] &= \tfrac{1}{3}\Big(\mathbf{f}[\frac{N}{2}-1] + \mathbf{f}[\frac{N}{2}] + \mathbf{f}[\frac{N}{2}+1]\Big) \\ &= \tfrac{1}{3}\Big(\mathbf{f}[\frac{N}{2}-1] + \mathbf{f}[\frac{N}{2}] + \mathbf{f}[-\frac{N}{2}+1]\Big) \\ &\qquad \text{(the right index changed from } \frac{N}{2}+1 \text{ to } \frac{N}{2}+1-N = -\frac{N}{2}+1)\,.\end{aligned}$$

We see that in computing the value of the new left endpoint the operation of convolution has averaged in the old value at the *right* endpoint, a value at the opposite end of the original data! The corresponding thing happens in computing the new right endpoint. This is certainly not what was called for in the original description of smoothing as the least squares line fit of three adjacent points, or put differently, it doesn't seem much like "smoothing" to average together data from opposite ends of the sampled values.

There are several things we could do.

[12] This is the difference between a *nonrecursive filter*, such as we have here, and a *recursive filter*, when the new values involve the old values together with previously computed new values.

- As mentioned first, we could run the convolution as is, and then just drop the computed values at the left and right endpoints. That would be in keeping with the original description as a moving least squares fit of three adjacent data points.

- Or before convolving we could replace both the original left endpoint and original right endpoint values by the average of the two, that is, the first *and* last values of the input $\mathbf{f}$ are modified to be

$$A = \tfrac{1}{2}\Big(\mathbf{f}[-\frac{N}{2}+1] + \mathbf{f}[\frac{N}{2}]\Big)\,.$$

  This sort of thing is done quite often in DFT calculations, and although we won't pursue it here the more you work with the DFT the more you will run into the dictum: replace the values at a jump by the average. The idea derives from the phenomenon that the Fourier series of a discontinuous function converges to the average value of the endpoints at a jump discontinuity.

What's most often done is to modify the filter function h rather than the data, and it's done for more than one reason. In the original running average filter the coefficients of the filter function all have weight 1/3. Rather then doing that, we take the neighbors, $\mathbf{f}[m-1]$ and $\mathbf{f}[m+1]$, to have half the weight of the midpoint $\mathbf{f}[m]$. This "modified running average filter" (or modified least squares filter) is thus defined by

$$\mathbf{g} = \big(\tfrac{1}{4}\boldsymbol{\delta}_1 + \tfrac{1}{2}\boldsymbol{\delta}_0 + \tfrac{1}{4}\boldsymbol{\delta}_{-1}\big) * \mathbf{f}\,,$$

or

$$\mathbf{g}[m] = \tfrac{1}{4}\mathbf{f}[m-1] + \tfrac{1}{2}\mathbf{f}[m] + \tfrac{1}{4}\mathbf{f}[m+1]\,.$$

We'll analyze the effects of this filter in the frequency domain, but first a confession. We did the three-point running average because it was easy to describe and easy to write out. It's more common to take a *five-point* running average. The least squares interpretation is as before: take the least squares fit to five consecutive data points and take the new data point to be the midpoint of this fit. With that description, the filter is

$$\mathbf{g} = \tfrac{1}{5}\big(\boldsymbol{\delta}_2 + \boldsymbol{\delta}_1 + \boldsymbol{\delta}_0 + \boldsymbol{\delta}_{-1} + \boldsymbol{\delta}_{-2}\big) * \mathbf{f}\,,$$

or

$$\mathbf{g}[m] = \tfrac{1}{5}(\mathbf{f}[m-2] + \mathbf{f}[m-1] + \mathbf{f}[m] + \mathbf{f}[m+1] + \mathbf{f}[m+2])\,.$$

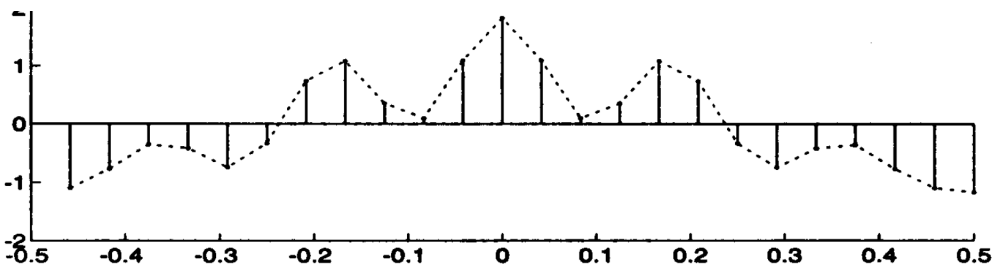
Once again, to address the endpoint problem, this is usually modified so that the two end values are given half the weight of the interior values. This results in the modified 5-point running average filter:

$$\mathbf{g} = \big(\tfrac{1}{8}\boldsymbol{\delta}_2 + \tfrac{1}{4}\boldsymbol{\delta}_1 + \tfrac{1}{4}\boldsymbol{\delta}_0 + \tfrac{1}{4}\boldsymbol{\delta}_{-1} + \tfrac{1}{8}\boldsymbol{\delta}_{-2}\big) * \mathbf{f} = \mathbf{h} * \mathbf{f}$$

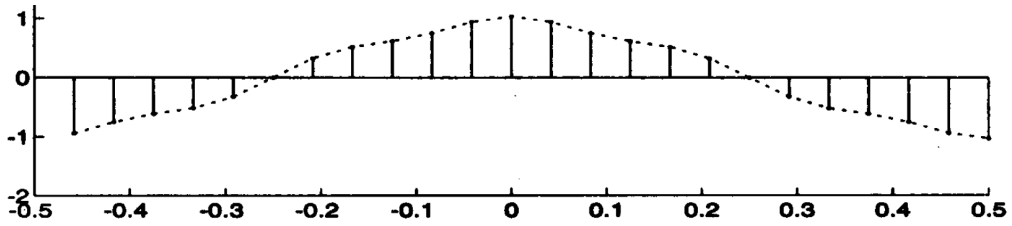
or

$$\mathbf{g}[m] = \tfrac{1}{8}\mathbf{f}[m-2] + \tfrac{1}{4}\mathbf{f}[m-1] + \tfrac{1}{4}\mathbf{f}[m] + \tfrac{1}{4}\mathbf{f}[m+1] + \tfrac{1}{8}\mathbf{f}[m+2]\,.$$

Here's an example of the five-point filter in action. The old data, $\mathbf{f}$, is a list of 24 samples of the sum of three sinusoids. The picture is this:



The new data, the output $\mathbf{g}$, is also sequence of 24 data points, and here's a picture of that.



It's clear that the old data has been smoothed, and it appears as though the higher frequencies have been eliminated. Whether this smoothing has lost something essential in the data is up to you and your scientific conscience to sort out.

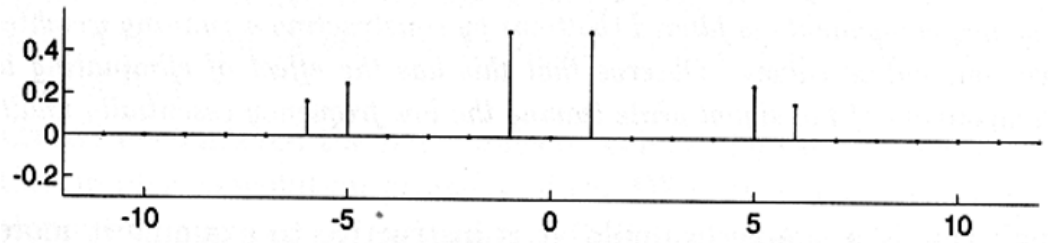
7.13.2 Analysis in the frequency domain

How have the higher frequencies been eliminated? That's where the picture in the frequency domain comes in.

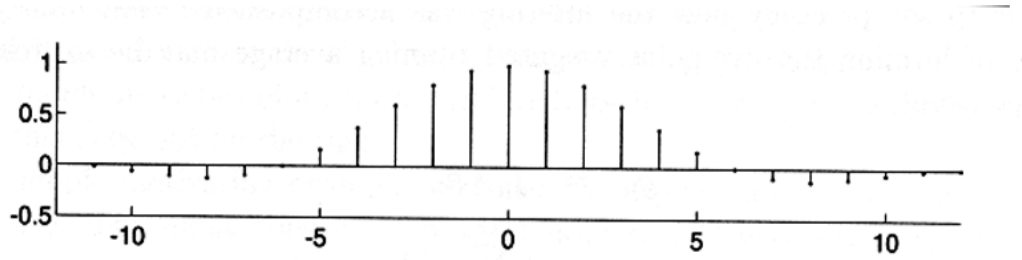
Take the DFT, of order 24, of $\mathbf{g} = \mathbf{h} * \mathbf{f}$ and write it as

$$\mathbf{G} = \mathbf{H}\,\mathbf{F}\,.$$

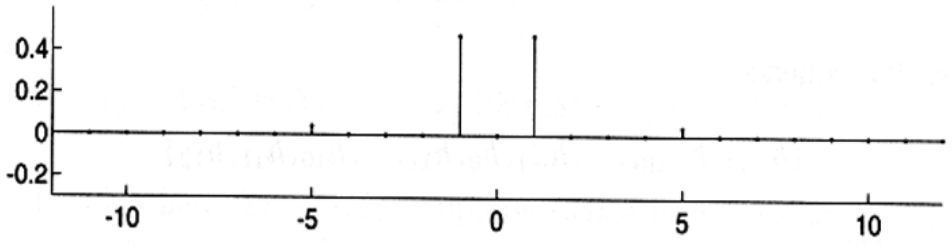
Here's the plot of $\mathbf{F}$ showing spikes at the three frequencies (positive and negative) in the signal.



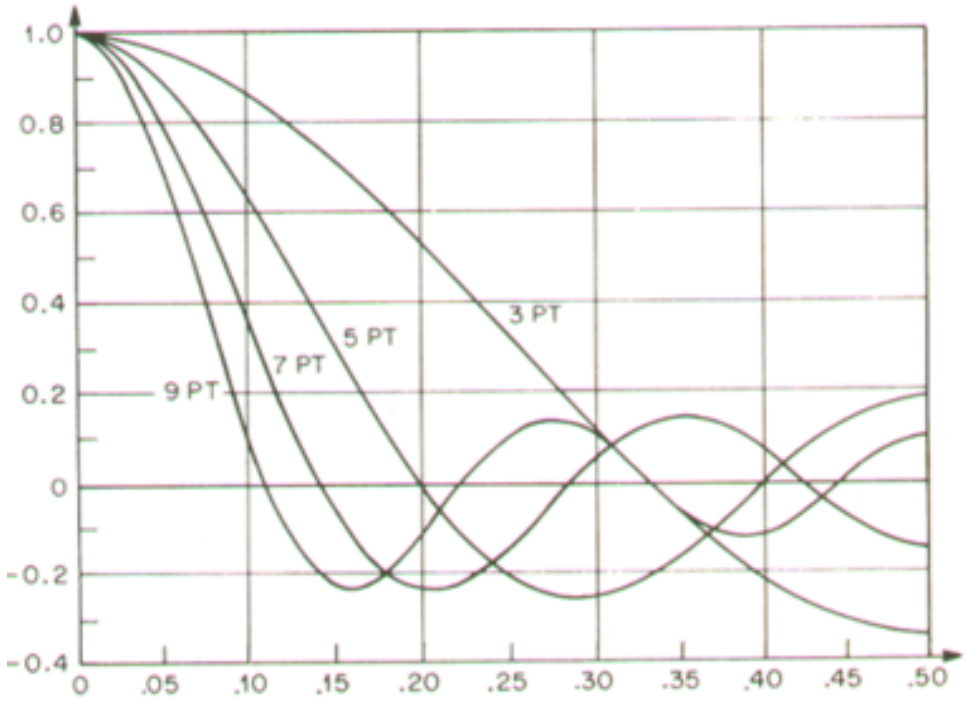
Here's the plot of the transfer function $\mathbf{H}$.



Taking the product, $\mathbf{H}\,\mathbf{F}$, of these two sequences to produce $\mathbf{G}$ has the effect of diminishing the higher frequencies in the original signal $\mathbf{f}$ while leaving the magnitude of the lower frequency components largely unchanged. Here's a plot of $\mathbf{G}$, showing exactly this phenomenon.



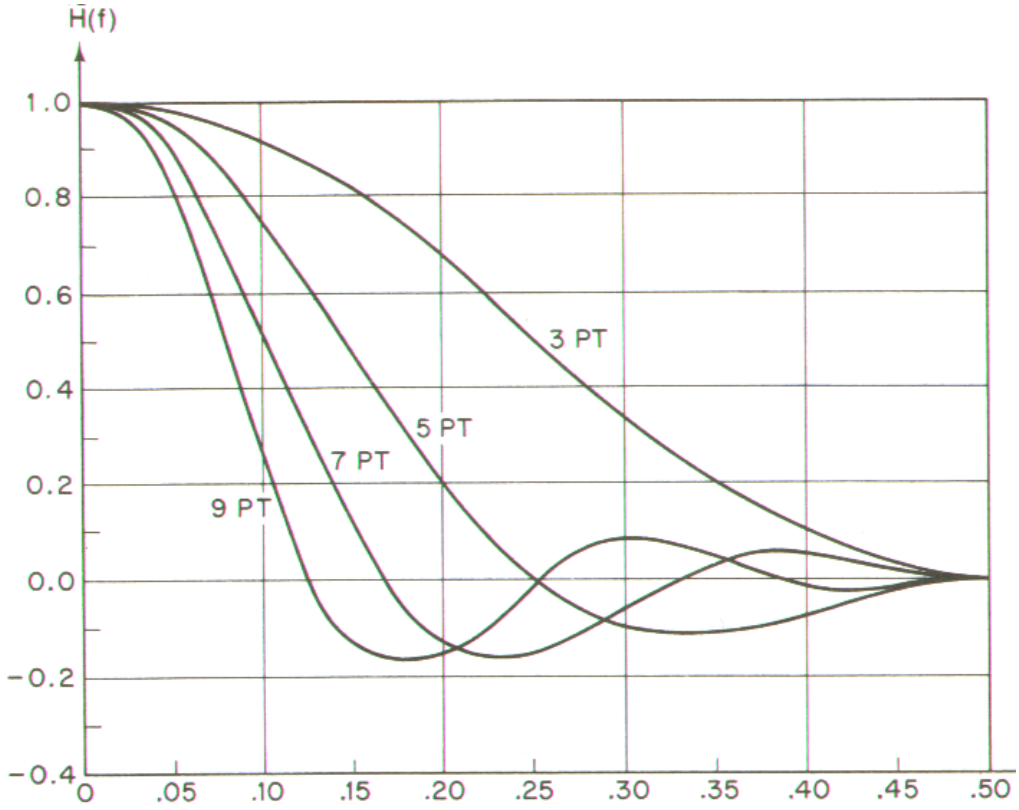
The pictures of the transfer functions in the frequency domain shed more light on the difference between the straight running average, where all values are weighted equally, and the modified running average, where the end values are given half the weight of the interior values. Here's a plot of the transfer function for running average filters of 3 points, 5 points, 7 points, and 9 points. (The transfer functions are even, so only the plots for positive frequencies are shown. These plots come from the book *Digital Filters* by R. Hamming.) Remember, the effect of the filter is to modify the input by *multiplying* its spectrum by the transfer function.



For each plot there are two features to note. In each case the transfer function has a "center lobe" from its maximum down to its first minimum (the plot for the 3 point filter doesn't go out far enough to show this), and then there are "sidelobes" where the transfer function oscillates. The frequencies close to the center are passed through pretty much unchanged, and the ones that are in the side lobe regions are decreased. The sidelobes die out, so the farther out we go the more the frequencies in those regions are eliminated, but it's a question of how fast the sidelobes are dying out.

Now here's a plot of the transfer functions for the modified running average.

The center lobes are wider here than they are for the unmodified running average, so more frequencies in the center are passed through, but more important is that for the modified running average the sidelobes damp down much more quickly. Thus frequencies outside the center lobe are decreased much more dramatically. The result is "smoother" data (fewer high frequencies) back in the time domain.



One word of caution. Although the modified running average as we've defined it is in widespread use, it's not the only way to smooth data. I hope one consequence of the discussion so far is that you can imagine designing and analyzing your own filters.

It's faster in frequency It's worth pausing to remember the gains in computational efficiency provided by the FFT algorithm for calculating the DFT, and realizing what this means for filtering, as above. Calculating the convolution of two (periodic) sequences of length N requires on the order of N^2 operations. On the other hand, using the FFT on the inputs ($O(N \log_2 N$ operations), multiplying *componentwise* the results (N operations), and then inverting (another $O(N \log_2 N)$ operations) requires a total of $O(N \log_2 N)$ operations. Filtering by passing to the frequency domain gives a considerable savings.

7.13.3 Find that filter

Let's finish our very brief discussion of digital filters with the discrete form of lowpass and bandpass filters. You can get highpass and notch filters directly from these.

Lowpass filters The example of finding a running average had us convolving in the time domain and multiplying in the frequency domain, and the effect was to eliminate, or at least attenuate, the higher frequencies. If we start in the frequency domain instead, and just cut off higher frequencies, we wind up with a lowpass filter. We discussed this in Chapter 3, strictly in the continuous case, and it might help to review that briefly here.

An ideal low-pass filter sets all frequencies above a certain amount ν_c to zero and lets all frequencies below ν_c pass through unchanged. If we write, as usual,

$$w(t) = (h * v)(t), \quad W(s) = H(s)V(s)\,,$$

then we want the transfer function

$$H(s) = \Pi_{2\nu_c}(s) = \begin{cases} 1 & |s| \le \nu_c \\ 0 & |s| > \nu_c \end{cases}$$

That is, the transfer function is just a scaled rect function. In the time domain the impulse response is

$$h(t) = 2\nu_c \operatorname{sinc}(2\nu_c t)\,.$$

In the discrete case with N-points, going, say, from $-N/2+1$ to $N/2$, the transfer function is defined to be

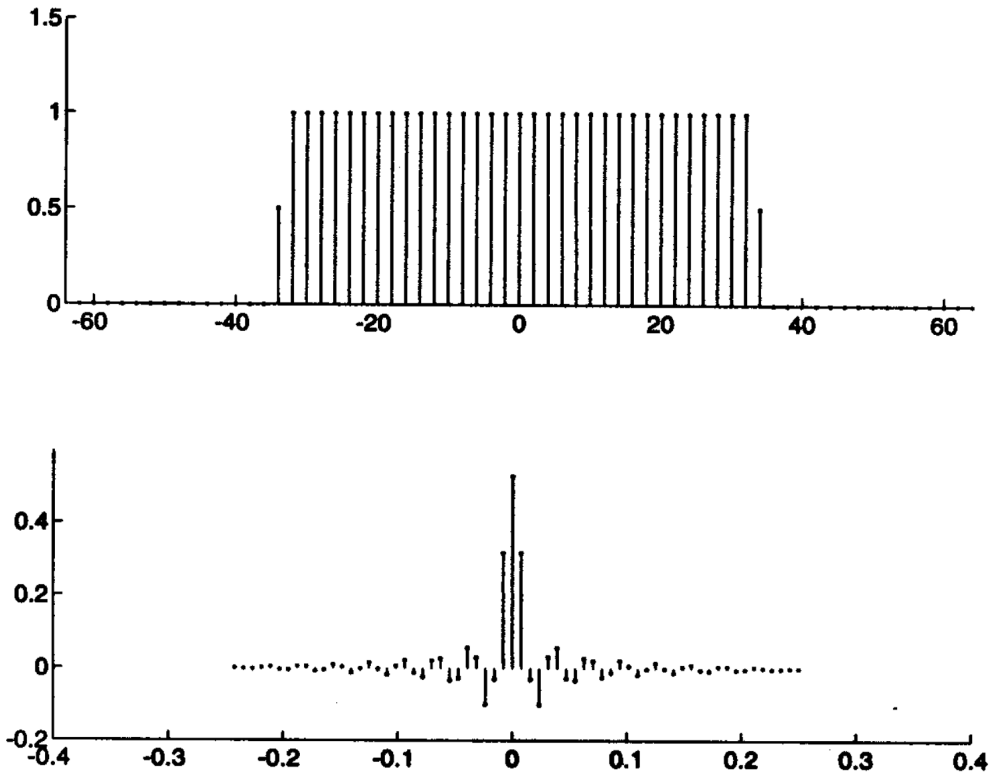
$$\mathbf{H}[m] = \begin{cases} 1 & |m| < m_c \\ \frac{1}{2} & |m| = m_c \\ 0 & m_c < |m| \le \frac{N}{2} \end{cases}$$

Here m_c is the index associated with the frequency where we want to cut off. We take the value to be $1/2$ at the endpoints. This choice comes from the "take the average at a jump discontinuity" principle.

In Section 7.15 I'll derive the explicit formula for the (discrete) impulse response:

$$\mathbf{h}[m] = \frac{\cos(\pi m/N)\sin(2\pi m m_c/N)}{\sin(\pi m/N)}$$

Here are plots of $\mathbf{H}[m]$ and $\mathbf{h}[m]$ with $N = 64$ and $m_c = 32$:

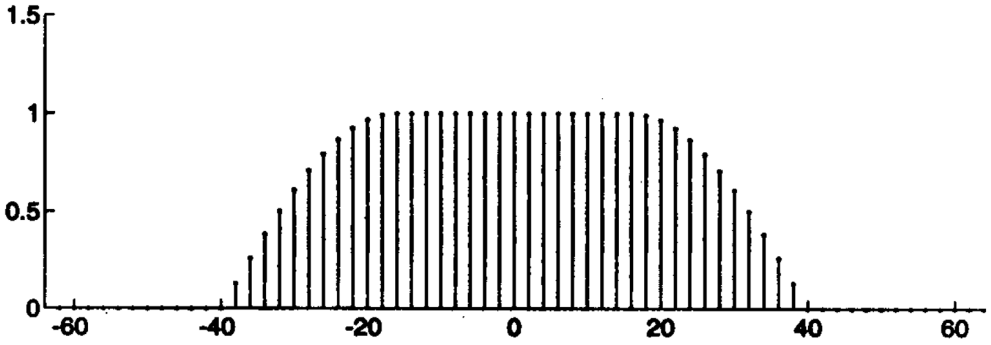


Notice again the sidelobes in the time domain, i.e. the many, small oscillations. These are less pronounced for this definition of **h**, i.e., with **H** defined to be 1/2 at the endpoints, than they would be if **H** jumped straight from 0 to 1, but even so, using this filter with a signal that itself has "edges" can cause some unwanted effects, called *ringing*. To counteract such effects one sometimes brings the transfer function down to zero more gradually. One example of how to do this is

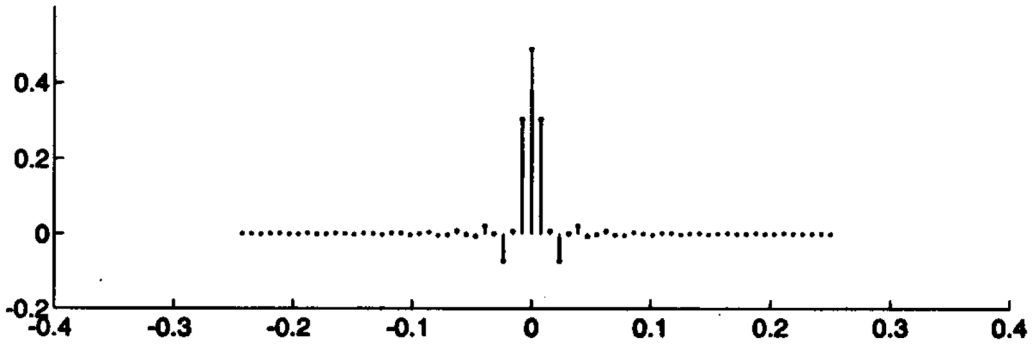
$$\mathbf{H}[m] = \begin{cases} 1 & |m| \le m_c - m_0 \\ \sin\left(\frac{\pi(m_c - m)}{2m_0}\right) & m_c - m_0 < |m| \le m_c \\ 0 & m_c < |m| \end{cases}$$

where, again, m_c is where you want the frequencies cut off, and m_0 is where you start bringing the transfer function down.

Here is the picture in frequency — the transfer function, **H**:



And here is the picture in time — the impulse response, **h**:



The sidelobes are definitely less pronounced.

Bandpass filters We also looked earlier at *bandpass filters*, filters that pass a particular band of frequencies through unchanged and eliminate all others. The transfer function $B(s)$ for a bandpass filter can be constructed by shifting and combining the transfer function $H(s)$ for the lowpass filter.

We center our bandpass filter at $\pm\nu_0$ and cut off frequencies more than ν_c above and below ν_0. That is,

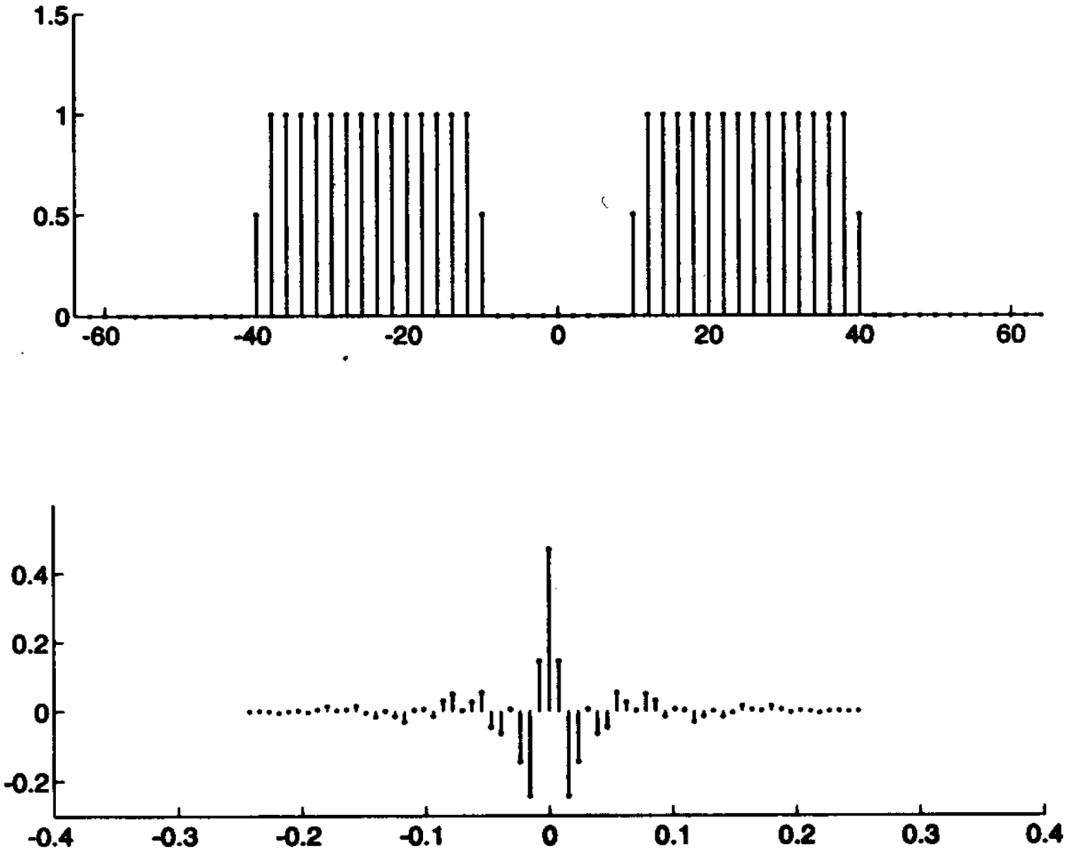
we define

$$B(s) = \begin{cases} 1 & \nu_0 - \nu_c \le |s| \le \nu_0 + \nu_c \\ 0 & |s| < \nu_0 - \nu_c \text{ or } |s| > \nu_0 + \nu_c \end{cases}$$
$$= H(s - \nu_0) + H(s + \nu_0)\,.$$

From the representation of $B(s)$ in terms of $H(s)$ it's easy to find the impulse response, $b(t)$:

$$b(t) = h(t)e^{2\pi i\nu_0 t} + h(t)e^{-2\pi i\nu_0 t} \quad \text{(using the shift theorem)}$$
$$= 2h(t)\cos(2\pi\nu_0 t).$$

Here's a picture of the discrete version of a bandpass filter in the frequency and time domains. Once again you notice the sidelobes. As before, it's possible to mollify this.



7.14 Appendix: Geometric Series of the Vector Complex Exponentials

There are times when explicit, closed form expressions for discrete filters are helpful, say if one wants to do a careful analysis of "endpoint effects" and related phenomena, or try out modifications in particular situations. The formulas also work out nicely, by and large, and further advance the notion that the discrete case can be made to look like the continuous case (and thus allow us to use what we know from the latter).

Such undertakings always seem to depend on calculations with the vector complex exponentials. One particular calculation that comes up a lot is a formula for a geometric sum of the $\boldsymbol{\omega}$, a sum of the form

$$\mathbf{1} + \boldsymbol{\omega} + \boldsymbol{\omega}^2 + \cdots + \boldsymbol{\omega}^{q-1}\,,$$

or more generally

$$\mathbf{1} + \boldsymbol{\omega}^p + \boldsymbol{\omega}^{2p} + \cdots + \boldsymbol{\omega}^{(q-1)p}\,.$$

We take $1 \le q \le N$, and for the second sum the case that will be of interest is when $pq = N$. It's easy enough to work out these sums componentwise, but it's also interesting to proceed as in the scalar case. Thus if we write

$$\mathbf{S} = \mathbf{1} + \boldsymbol{\omega} + \boldsymbol{\omega}^2 + \cdots + \boldsymbol{\omega}^{q-1}$$

then

$$\boldsymbol{\omega}\,\mathbf{S} = \boldsymbol{\omega} + \boldsymbol{\omega}^2 + \boldsymbol{\omega}^3 + \cdots + \boldsymbol{\omega}^{q}\,,$$

and

$$(\mathbf{1} - \boldsymbol{\omega})\mathbf{S} = \mathbf{1} - \boldsymbol{\omega}^q\,.$$

On the left hand side, $\mathbf{1} - \boldsymbol{\omega}$ has a 0 in the slot 0 and nowhere else. On the right hand side, $\mathbf{1} - \boldsymbol{\omega}^q$ is also zero in the slot 0, and possibly in other slots. We thus have to determine the zeroth slot of $\mathbf{S}$ directly, and it's just the sum of q 1's. We can write

$$\mathbf{S} = q\boldsymbol{\delta}_0 + \mathbf{T}\,,$$

where $\mathbf{T}$ is 0 in the zeroth slot. The remaining components of $\mathbf{T}$ are the components from 1 to $N-1$ in $(\mathbf{1} - \boldsymbol{\omega}^q)/(\mathbf{1} - \boldsymbol{\omega})$ understood as the componentwise quotient. All of the quotients are defined, and some of them may be zero. If $q = N$ then $\mathbf{1} - \boldsymbol{\omega}^N$ is identically zero. A rather precise way of writing the final answer is

$$\mathbf{1} + \boldsymbol{\omega} + \boldsymbol{\omega}^2 + \cdots + \boldsymbol{\omega}^{q-1} = \mathbf{S} = q\delta_0 + (\mathbf{1} - \boldsymbol{\delta}_0)\frac{\mathbf{1} - \boldsymbol{\omega}^q}{\mathbf{1} - \boldsymbol{\omega}}\,,$$

but I'm tempted to write the formula for the sum as

$$\mathbf{1} + \boldsymbol{\omega} + \boldsymbol{\omega}^2 + \cdots + \boldsymbol{\omega}^{q-1} = \frac{\mathbf{1} - \boldsymbol{\omega}^q}{\mathbf{1} - \boldsymbol{\omega}}$$

with an understanding, between grown-ups, that the indeterminate form 0/0 in the zero slot calls for special evaluation. See that — it looks just like the scalar case. (Suggestions of how better to express these understandings are welcome.)

The situation for

$$\mathbf{S} = \mathbf{1} + \boldsymbol{\omega}^p + \boldsymbol{\omega}^{2p} + \cdots + \boldsymbol{\omega}^{(q-1)p}$$

is a little different. As above we find

$$(\mathbf{1} - \boldsymbol{\omega}^p)\mathbf{S} = 1 - \boldsymbol{\omega}^{pq}\,.$$

Suppose that $pq = N$. Then the right hand side is the zero vector. Now,

$$\boldsymbol{\omega}^p = (1, \omega^p, \omega^{2p}, \ldots, \omega^{(N-1)p}) = (1, \omega^{N/q}, \omega^{2N/q}, \ldots, \omega^{(N-1)N/q})\,,$$

and so $\mathbf{1} - \boldsymbol{\omega}^p$ will have zeros in the slots 0, q, $2q$, …, $(p-1)q$ and nowhere else. Therefore $\mathbf{S}$ must have zeros *other* than in these slots, and we see that the value of $\mathbf{S}$ is q in each of the slots 0, q, $2q$, …, $(p-1)q$. This shows that

$$\mathbf{1} + \boldsymbol{\omega}^p + \boldsymbol{\omega}^{2p} + \cdots + \boldsymbol{\omega}^{(q-1)p} = q\boldsymbol{\delta}_0 + q\boldsymbol{\delta}_q + q\boldsymbol{\delta}_{2q} + \ldots + q\boldsymbol{\delta}_{(p-1)q}\,,$$

or more compactly

$$\sum_{k=0}^{q-1} \boldsymbol{\omega}^{kp} = q\sum_{k=0}^{p-1} \boldsymbol{\delta}_{kq}\,, \quad \text{where } pq = N\,.$$

Note that this includes the special case when $p = 1$ and $q = N$.

We'll only use the first geometric sum formula and not the one just above. The second comes up in defining and working with a discrete Ш function.[13]

7.15 Appendix: The Discrete Rect and its DFT

We defined the discrete rect function to be indexed from $-N/2 + 1$ to $N/2$, so we assume here that N is even. Suppose p is also even where $0 < p < N/2$. For any real number α, define

$$\underline{\Pi}_p^\alpha = \alpha(\boldsymbol{\delta}_{p/2} + \boldsymbol{\delta}_{-p/2}) + \sum_{k=-p/2+1}^{p/2-1} \boldsymbol{\delta}_k \,.$$

Why the extra parameter α in setting the value at the endpoints $\pm p/2$? In the continuous case one also encounters different normalizations of the rect function at the points of discontinuity, but it hardly makes a difference in any formulas and calculations; most of the time there's an integral involved and changing the value of a function at a few points has no effect on the value of the integral. Not so in the discrete case, where sums instead of integrals are the operators that one encounters. So with an eye toward flexibility in applications, we're allowing an α.

For work on digital filters, $\underline{\Pi}_p^\alpha$ is typically (part of) a transfer function and we want to know the impulse response, i.e., the inverse DFT. By duality it suffices to work with the forward transform, so we'll find $\underline{\mathcal{F}}\,\underline{\Pi}_p^\alpha$.

Just as $\underline{\Pi}_p^\alpha$ comes in two parts, so too does its Fourier transform:. The first part is easy:

$$\underline{\mathcal{F}}\,(\alpha(\boldsymbol{\delta}_{p/2} + \boldsymbol{\delta}_{-p/2})) = \alpha(\boldsymbol{\omega}^{p/2} + \boldsymbol{\omega}^{-p/2}) = 2\alpha\,\mathrm{Re}\{\boldsymbol{\omega}^{p/2}\} = 2\alpha\cos\left(\frac{\pi p}{N}[-\frac{N}{2} + 1 : \frac{N}{2}]\right) .$$

The second part takes more work:

$$\underline{\mathcal{F}}\left(\sum_{k=-p/2+1}^{p/2-1} \boldsymbol{\delta}_k\right) = \sum_{k=-p/2+1}^{p/2-1} \boldsymbol{\omega}^k = \sum_{k=0}^{p/2-1} \left(\boldsymbol{\omega}^k + \boldsymbol{\omega}^{-k}\right) - \mathbf{1} = 2\,\mathrm{Re}\left(\sum_{k=0}^{p/2-1} \boldsymbol{\omega}^k\right) - \mathbf{1}$$

Take the zero slot first. We find directly that

$$\underline{\mathcal{F}}\,\underline{\Pi}_p^\alpha[0] = 2\alpha + p - 1 \,.$$

For the nonzero slots we use the formula for the geometric series,

$$\sum_{k=0}^{p/2-1} \boldsymbol{\omega}^k = \frac{p}{2}\boldsymbol{\delta}_0 + \frac{1 - \boldsymbol{\omega}^{p/2}}{1 - \boldsymbol{\omega}} \,.$$

Now, with the understanding that we're omitting the zero slot,

$$\frac{1}{1 - \boldsymbol{\omega}} = \frac{1}{2}\mathbf{1} + \frac{1}{2}\frac{1 + \boldsymbol{\omega}}{1 - \boldsymbol{\omega}} = \frac{1}{2}\mathbf{1} - \frac{i}{2}\frac{\cos(\frac{\pi}{N}[-\frac{N}{2} + 1 : \frac{N}{2}])}{\sin(\frac{\pi}{N}[-\frac{N}{2} + 1 : \frac{N}{2}])} \,.$$

[13] For those interested, I have extra notes on this, together with a discussion of a discrete sampling formula that we're developing for applications to medical imaging problems. Current stuff.

Thus

$$\begin{aligned}
2\,\mathrm{Re}\left(\sum_{k=0}^{p/2-1}\boldsymbol{\omega}^k\right)-\mathbf{1} &= 2\,\mathrm{Re}\left(\left(\frac{1}{2}\mathbf{1}-\frac{i\cos(\frac{\pi}{N}[-\frac{N}{2}+1:\frac{N}{2}])}{2\sin(\frac{\pi}{N}[-\frac{N}{2}+1:\frac{N}{2}])}\right)(1-\boldsymbol{\omega}^{p/2})\right)-\mathbf{1}\\
&= -\,\mathrm{Re}\{\boldsymbol{\omega}^{p/2}\}-\frac{\cos(\frac{\pi}{N}[-\frac{N}{2}+1:\frac{N}{2}])}{\sin(\frac{\pi}{N}[-\frac{N}{2}+1:\frac{N}{2}])}\,\mathrm{Im}\{\boldsymbol{\omega}^{p/2}\}\\
&= -\cos(\frac{\pi p}{N}[-\frac{N}{2}+1:\frac{N}{2}])+\frac{\cos(\frac{\pi}{N}[-\frac{N}{2}+1:\frac{N}{2}])\sin(\frac{\pi p}{N}[-\frac{N}{2}+1:\frac{N}{2}])}{\sin(\frac{\pi}{N}[-\frac{N}{2}+1:\frac{N}{2}])}\\
&= \frac{\sin(\frac{\pi(p-1)}{N}[-\frac{N}{2}+1:\frac{N}{2}])}{\sin(\frac{\pi}{N}[-\frac{N}{2}+1:\frac{N}{2}])}\,.
\end{aligned}$$

Combining this with the first part of the calculation gives

$$\underline{\mathcal{F}}\,\underline{\Pi}_p^\alpha=\begin{cases}2\alpha+p-1 & \text{in slot } 0\\ 2\alpha\cos(\frac{\pi p}{N}[-\frac{N}{2}+1:\frac{N}{2}])+\frac{\sin(\frac{\pi(p-1)}{N}[-\frac{N}{2}+1:\frac{N}{2}])}{\sin(\frac{\pi}{N}[-\frac{N}{2}+1:\frac{N}{2}])} & \text{otherwise}\end{cases}$$

Not to put too fine a point on it, but slot 0 of

$$\frac{\sin(\frac{\pi(p-1)}{N}[-\frac{N}{2}+1:\frac{N}{2}])}{\sin(\frac{\pi}{N}[-\frac{N}{2}+1:\frac{N}{2}])}$$

does make sense as a limit and its value is $p-1$. I wouldn't object to writing the formula for $\underline{\mathcal{F}}\,\underline{\Pi}_p^\alpha$ simply as

$$\underline{\mathcal{F}}\,\underline{\Pi}_p^\alpha=2\alpha\cos(\frac{\pi p}{N}[-\frac{N}{2}+1:\frac{N}{2}])+\frac{\sin(\frac{\pi(p-1)}{N}[-\frac{N}{2}+1:\frac{N}{2}])}{\sin(\frac{\pi}{N}[-\frac{N}{2}+1:\frac{N}{2}])}$$

At a point $m\in\mathbf{Z}$,

$$\underline{\mathcal{F}}\,\underline{\Pi}_p^\alpha[m]=2\alpha\cos(\frac{\pi pm}{N})+\frac{\sin(\frac{\pi(p-1)m}{N})}{\sin(\frac{\pi m}{N})}\,.$$

Since p is even, and $p-1$ is odd, we observe that $\underline{\mathcal{F}}\,\underline{\Pi}_p^\alpha$ is periodic of period N, which it had better be. (Because $p-1$ is odd both the numerator and the denominator of the second term change sign if m is replaced by $m+N$, while, because p is even, the cosine term is unchanged.)

The most common choices of α are 0, 1, and 1/2. The corresponding Fourier transforms are

$$\underline{\mathcal{F}}\,\underline{\Pi}_p^0=\frac{\sin(\frac{(p-1)\pi}{N}[-\frac{N}{2}+1:\frac{N}{2}])}{\sin(\frac{\pi}{N}[-\frac{N}{2}+1:\frac{N}{2}])}\,,\quad \underline{\mathcal{F}}\,\underline{\Pi}_p^0[0]=p-1$$

$$\underline{\mathcal{F}}\,\underline{\Pi}_p^1=\frac{\sin(\frac{(p+1)\pi}{N}[-\frac{N}{2}+1:\frac{N}{2}])}{\sin(\frac{\pi}{N}[-\frac{N}{2}+1:\frac{N}{2}])}\,,\quad \underline{\mathcal{F}}\,\underline{\Pi}_p^1[0]=p+1$$

(use the addition formula for sines to write
$2\cos(\pi pm/N)\sin(\pi m/N)+\sin\pi(p-1)m/N)=\sin(\pi(p+1)m/N))$

$$\begin{aligned}
\underline{\mathcal{F}}\,\underline{\Pi}_p^{1/2} &= \tfrac{1}{2}(\underline{\mathcal{F}}\,\underline{\Pi}_p^1+\underline{\mathcal{F}}\,\underline{\Pi}_p^0)\\
&= \frac{\cos(\frac{\pi}{N}[-\frac{N}{2}+1:\frac{N}{2}])\sin(\frac{\pi p}{N}[-\frac{N}{2}+1:\frac{N}{2}])}{\sin(\frac{\pi}{N}[-\frac{N}{2}+1:\frac{N}{2}])}\,,\quad \underline{\mathcal{F}}\,\underline{\Pi}_p^{1/2}[0]=p
\end{aligned}$$

This last formula is the one we had earlier in the notes for the impulse response of the discrete lowpass filter. In the notation there, $p/2=m_c$ and

$$\mathbf{h}[m]=\frac{\cos(\pi m/N)\sin(2\pi mm_c/N)}{\sin(\pi m/N)}\,.$$

Chapter 8

n-dimensional Fourier Transform

8.1 Space, the Final Frontier

To quote Ron Bracewell from p. 119 of his book *Two-Dimensional Imaging*, "In two dimensions phenomena are richer than in one dimension." True enough, working in two dimensions offers many new and rich possibilities. Contemporary applications of the Fourier transform are just as likely to come from problems in two, three, and even higher dimensions as they are in one — imaging is one obvious and important example. To capitalize on the work we've already done, however, as well as to highlight differences between the one-dimensional case and higher dimensions, we want to mimic the one-dimensional setting and arguments as much as possible. It is a measure of the naturalness of the fundamental concepts that the extension to higher dimensions of the basic ideas and the mathematical definitions that we've used so far proceeds almost automatically. However much we'll be able to do in class and in these notes, you should be able to read more on your own with some assurance that you won't be reading anything too much different from what you've already read.

Notation The higher dimensional case looks most like the one-dimensional case when we use vector notation. For the sheer thrill of it, I'll give many of the definitions in n dimensions, but to raise the comfort level we'll usually look at the special case of two dimensions in more detail; two and three dimensions are where most of our examples will come from.

We'll write a point in $\mathbf{R}^n$ as an n-tuple, say

$$\mathbf{x} = (x_1, x_2, \ldots, x_n)\,.$$

Note that we're going back to the usual indexing from 1 to n. (And no more periodic extensions of the n-tuples either!) We'll be taking Fourier transforms and may want to assign a physical meaning to our variables, so we often think of the x_i's as coordinates in space, with the dimension of length, and $\mathbf{x}$ as the "spatial variable". We'll then also need an n-tuple of "frequencies", and without saying yet what "frequency" means, we'll (typically) write

$$\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_n)$$

for those variables "dual to $\mathbf{x}$". Recall that the dot product of vectors in $\mathbf{R}^n$ is given by

$$\mathbf{x} \cdot \boldsymbol{\xi} = x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n\,.$$

The geometry of $\mathbf{R}^n$ is governed by the dot product, and using it will greatly help our understanding as well as streamline our notation.

8.1.1 The Fourier transform

We started this course with Fourier series and periodic phenomena and went on from there to define the Fourier transform. There's a place for Fourier series in higher dimensions, but, carrying all our hard won experience with us, we'll proceed directly to the higher dimensional Fourier transform. I'll save Fourier series for a later section that includes a really interesting application to random walks.

How shall we define the Fourier transform? We consider real- or complex-valued functions f defined on $\mathbf{R}^n$, and write $f(\mathbf{x})$ or $f(x_1, \ldots, x_n)$, whichever is more convenient in context. The Fourier transform of $f(\mathbf{x})$ is the function $\mathcal{F}f(\boldsymbol{\xi})$, or $\hat{f}(\boldsymbol{\xi})$, defined by

$$\mathcal{F}f(\boldsymbol{\xi}) = \int_{\mathbf{R}^n} e^{-2\pi i \mathbf{x}\cdot} f(\mathbf{x})\, d\mathbf{x}\,.$$

The inverse Fourier transform of a function $g(\boldsymbol{\xi})$ is

$$\mathcal{F}^{-1}g(\mathbf{x}) = \int_{\mathbf{R}^n} e^{2\pi i \mathbf{x}\cdot} g(\boldsymbol{\xi})\, d\boldsymbol{\xi}\,.$$

The Fourier transform, or the inverse transform, of a real-valued function is (in general) complex valued.

The exponential now features the dot product of the vectors $\mathbf{x}$ and $\boldsymbol{\xi}$; this is the key to extending the definitions from one dimension to higher dimensions and making it look like one dimension. The integral is over all of $\mathbf{R}^n$, and as an n-fold multiple integral all the x_j's (or ξ_j's for $\mathcal{F}^{-1}$) go from $-\infty$ to ∞. Realize that because the dot product of two vectors is a number, we're integrating a scalar function, not a vector function. Overall, the shape of the definitions of the Fourier transform and the inverse transform are *the same* as before.

The kinds of functions to consider and how they enter into the discussion — Schwartz functions, L^1, L^2, etc. — is entirely analogous to the one-dimensional case, and so are the definitions of these types of functions. Because of that we don't have to redo distributions et al. (good news), and I'll seldom point out when this aspect of the general theory is (or must be) invoked.

Written out in coordinates, the definition of the Fourier transform reads:

$$\mathcal{F}f(\xi_1, \xi_2, \ldots, \xi_n) = \int_{\mathbf{R}^n} e^{-2\pi i(x_1\xi_1 + \cdots + x_n\xi_n)} f(x_1, \ldots, x_n)\, dx_1 \ldots dx_n\,,$$

so for two dimensions,

$$\mathcal{F}f(\xi_1, \xi_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi i(x_1\xi_1 + x_2\xi_2)} f(x_1, x_2)\, dx_1\, dx_2\,.$$

The coordinate expression is manageable in the two-dimensional case, but I hope to convince you that it's almost always *much* better to use the vector notation in writing formulas, deriving results, and so on.

Arithmetic with vectors, including the dot product, is pretty much just like arithmetic with numbers. Consequently, all of the familiar algebraic properties of the Fourier transform are present in the higher dimensional setting. We won't go through them all, but, for example,

$$\mathcal{F}f(-\boldsymbol{\xi}) = \int_{\mathbf{R}^n} e^{-2\pi i \mathbf{x}\cdot(-)} f(\mathbf{x})\, d\mathbf{x} = \int_{\mathbf{R}^n} e^{2\pi i \mathbf{x}\cdot} f(\mathbf{x})\, d\mathbf{x} = \mathcal{F}^{-1}f(\boldsymbol{\xi})\,,$$

which is one way of stating the duality between the Fourier and inverse Fourier transforms. Here, recall that if $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)$ then

$$-\boldsymbol{\xi} = (-\xi_1, \ldots, -\xi_n)\,.$$

To be neater, we again use the notation

$$f^-(\boldsymbol{\xi}) = f(-\boldsymbol{\xi})\,,$$

and with this definition the duality results read exactly as in the one-dimensional case:

$$\mathcal{F}f^- = (\mathcal{F}f)^-, \quad (\mathcal{F}f)^- = \mathcal{F}^{-1}f$$

In connection with these formulas, I have to point out that changing variables, one of our prized techniques in one dimension, can be more complicated for multiple integrals. We'll approach this on a need to know basis.

It's still the case that the complex conjugate of the integral is the integral of the complex conjugate, so when $f(\mathbf{x})$ is real valued,

$$\mathcal{F}f(-\xi) = \overline{\mathcal{F}f(\xi)}\,.$$

Finally, evenness and oddness are defined exactly as in the one-dimensional case. That is:

$f(\mathbf{x})$ is *even* if $f(-\mathbf{x}) = f(\mathbf{x})$, or without writing the variables, if $f^- = f$.

$f(\mathbf{x})$ is *odd* $f(-\boldsymbol{\xi}) = -f(\boldsymbol{\xi})$, or $f^- = -f$.

Of course, we no longer have quite the easy geometric interpretations of evenness and oddness in terms of a graph in the higher dimensional case as we have in the one-dimensional case. But as algebraic properties of a function, these conditions do have the familiar consequences for the higher dimensional Fourier transform, e.g., if $f(\mathbf{x})$ is even then $\mathcal{F}f(\boldsymbol{\xi})$ is even, if $f(\mathbf{x})$ is real and even then $\mathcal{F}f(\boldsymbol{\xi})$ is real and even, *etc.* You could write them all out. I won't.

Soon enough we'll calculate the Fourier transform of some model functions, but first let's look a little bit more at the complex exponentials in the definition and get a better sense of what "the spectrum" means in higher dimensions.

Harmonics, periodicity, and spatial frequencies The complex exponentials are again the building blocks — the harmonics — for the Fourier transform and its inverse in higher dimensions. Now that they involve a dot product, is there anything special we need to know?

As mentioned just above, we tend to view $\mathbf{x} = (x_1, \ldots, x_n)$ as a *spatial* variable and $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)$ as a *frequency* variable. It's not hard to imagine problems where one would want to specify n spatial dimensions each with the unit of distance, but it's not so clear what an n-tuple of frequencies should mean. One thing we can say is that if the spatial variables $(x_1, \ldots, x_n)$ do have the dimension of distance then the corresponding frequency variables $(\xi_1, \ldots, \xi_n)$ have the dimension 1/distance. For then

$$\mathbf{x} \cdot \boldsymbol{\xi} = x_1\xi_1 + \cdots + x_n\xi_n$$

is dimensionless and $\exp(-2\pi i \mathbf{x} \cdot \boldsymbol{\xi})$ makes sense. This corresponds to dimensions of time and 1/time in the one-dimensional time domain and frequency domain picture.

For some further insight let's look at the two-dimensional case. Consider

$$\exp(\pm 2\pi i \mathbf{x} \cdot \boldsymbol{\xi}) = \exp(\pm 2\pi i (x_1\xi_1 + x_2\xi_2))\,.$$

(It doesn't matter for the following discussion whether we take + or − in the exponent.) The exponent equals 1 whenever $\mathbf{x}\cdot\boldsymbol{\xi}$ is an integer, that is, when

$$\xi_1 x_1 + \xi_2 x_2 = n, \quad n \text{ an integer}.$$

With $\boldsymbol{\xi} = (\xi_1, \xi_2)$ fixed this is a condition on (x_1, x_2), and one says that the complex exponential has *zero phase* whenever $\xi_1 x_1 + \xi_2 x_2$ is an integer. This terminology comes from optics.

There's a natural geometric interpretation of the zero phase condition that's very helpful in understanding the most important properties of the complex exponential. For a fixed $\boldsymbol{\xi}$ the equations

$$\xi_1 x_1 + \xi_2 x_2 = n$$

determine a family of parallel lines in the (x_1, x_2)-plane (or in the *spatial domain* if you prefer that phrase). Take $n = 0$. Then the condition on x_1 and x_2 is

$$\xi_1 x_1 + \xi_2 x_2 = 0$$

and we recognize this as the equation of a line through the origin with (ξ_1, ξ_2) as a normal vector to the line.[1] (Remember your vectors!) Then (ξ_1, ξ_2) is a normal to *each* of the parallel lines in the family. One could also describe the geometry of the situation by saying that the lines each make an angle θ with the x_1-axis satisfying

$$\tan\theta = \frac{\xi_2}{\xi_1},$$

but I think it's much better to think in terms of normal vectors to specify the direction — the vector point of view generalizes readily to higher dimensions, as we'll discuss.

Furthermore, the family of lines $\xi_1 x_1 + \xi_2 x_2 = n$ are evenly spaced as n varies; in fact, the distance between the line $\xi_1 x_1 + \xi_2 x_2 = n$ and the line $\xi_1 x_1 + \xi_2 x_2 = n + 1$ is

$$\text{distance} = \frac{1}{\|\boldsymbol{\xi}\|} = \frac{1}{\sqrt{\xi_1^2 + \xi_2^2}}.$$

I'll let you derive that. This is our first hint, in two dimensions, of a reciprocal relationship between the spatial and frequency variables:

- The spacing of adjacent lines of zero phase is the reciprocal of the length of the frequency vector.

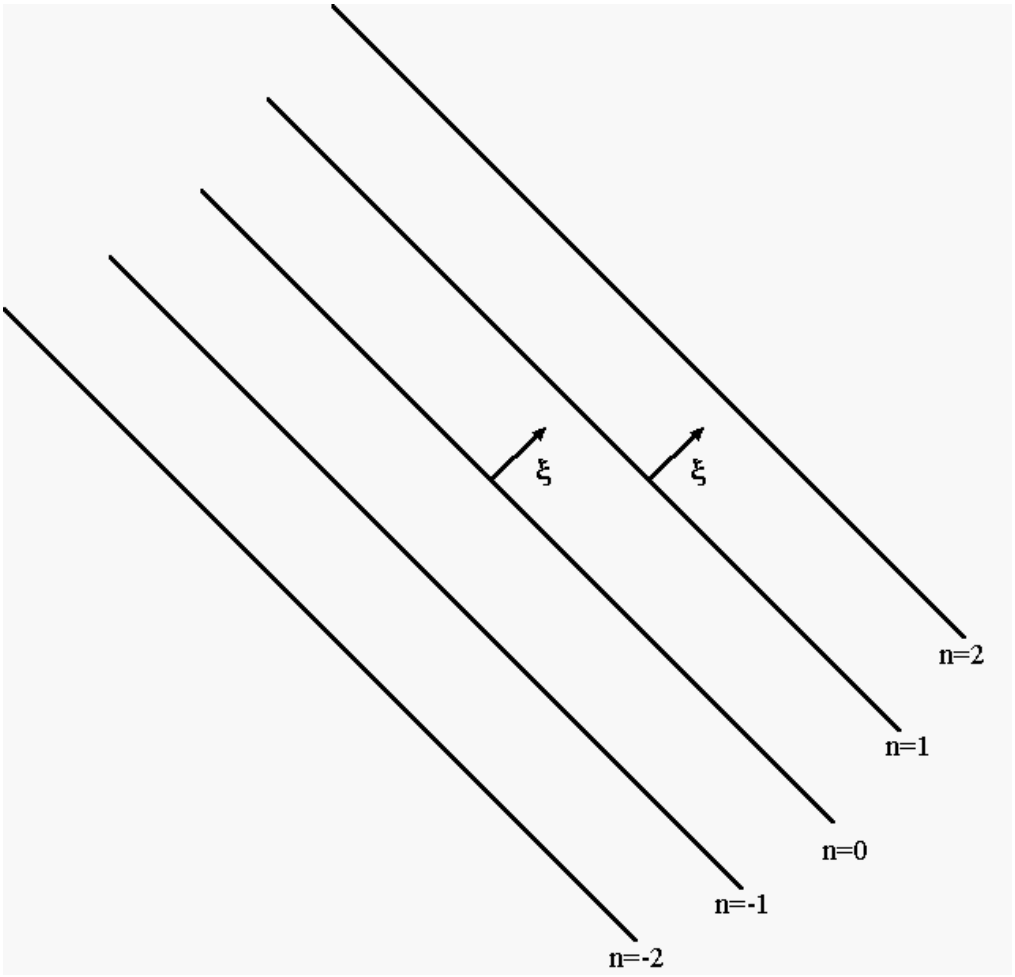
Drawing the family of parallel lines with a fixed normal $\boldsymbol{\xi}$ also gives us some sense of the periodic nature of the harmonics $\exp(\pm 2\pi i\, \mathbf{x}\cdot\boldsymbol{\xi})$. The frequency vector $\boldsymbol{\xi} = (\xi_1, \xi_2)$, as a normal to the lines, determines how the harmonic is oriented, so to speak, and the magnitude of $\boldsymbol{\xi}$, or rather its reciprocal, $1/\sqrt{\xi_1^2 + \xi_2^2}$ determines the period of the harmonic. To be precise, start at any point (a, b) and move in the direction of the *unit* normal, $\boldsymbol{\xi}/\|\boldsymbol{\xi}\|$. That is, move from (a, b) along the line

$$\mathbf{x}(t) = (x_1(t), x_2(t)) = (a, b) + t\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|} \quad \text{or} \quad x_1(t) = a + t\frac{\xi_1}{\|\boldsymbol{\xi}\|},\ x_2(t) = b + t\frac{\xi_2}{\|\boldsymbol{\xi}\|}$$

at unit speed. The dot product of $\mathbf{x}(t)$ and $\boldsymbol{\xi}$ is

$$\mathbf{x}(t)\cdot\boldsymbol{\xi} = (x_1(t), x_2(t))\cdot(\xi_1, \xi_2) = a\xi_1 + b\xi_2 + t\frac{\xi_1^2 + \xi_2^2}{\|\boldsymbol{\xi}\|} = a\xi_1 + b\xi_2 + t\|\boldsymbol{\xi}\|,$$

[1] Note that (ξ_1, ξ_2) isn't assumed to be a unit vector, so it's not the unit normal.

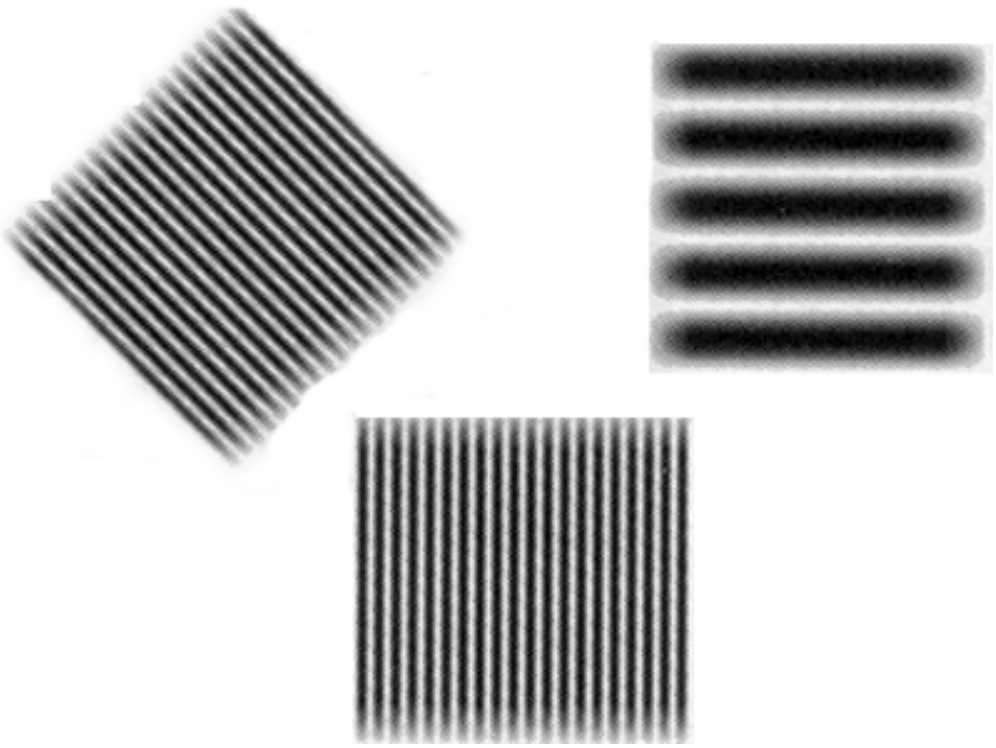


and the complex exponential is a function of t along the line:

$$\exp(\pm 2\pi i\, \mathbf{x}\cdot\boldsymbol{\xi}) = \exp(\pm 2\pi i(a\xi_1 + b\xi_2))\, \exp(\pm 2\pi i t\|\boldsymbol{\xi}\|)\,.$$

The factor $\exp(\pm 2\pi i(a\xi_1 + b\xi_2))$ doesn't depend on t and the factor $\exp(\pm 2\pi i t\|\boldsymbol{\xi}\|)$ is periodic with period $1/\|\boldsymbol{\xi}\|$, the spacing between the lines of zero phase. Now, if ξ_1 or ξ_2 is large, then the spacing of the lines is close and, by the same token, if ξ_1 and ξ_2 are small then the lines are far apart. Thus although "frequency" is now a vector quantity we still tend to speak in terms of a "high frequency" harmonic, when the lines of zero phase are spaced close together and a "low frequency" harmonic when the lines of zero phase are spaced far apart ("high" and "low" are relatively speaking, of course). Half way between the lines of zero phase, when $t = 1/2\|\boldsymbol{\xi}\|$, we're on lines where the exponential is -1, so 180° out of phase with the lines of zero phase.

One often sees pictures like the following.



Here's what you're looking at: The function $e^{2\pi i \mathbf{x}\cdot}$ is complex valued, but consider its real part

$$\begin{aligned}\operatorname{Re} e^{2\pi i \mathbf{x}\cdot} &= \tfrac{1}{2}\big(e^{2\pi i \mathbf{x}\cdot} + e^{-2\pi i \mathbf{x}\cdot}\big) \\ &= \cos 2\pi i \mathbf{x}\cdot\boldsymbol{\xi} = \cos 2\pi(\xi_1 x_1 + \xi_2 x_2)\end{aligned}$$

which has the same periodicity and same lines of zero phase as the complex exponential. Put down white stripes where $\cos 2\pi(\xi_1 x_1 + \xi_2 x_2) \geq 0$ and black stripes where $\cos 2\pi(\xi_1 x_1 + \xi_2 x_2) < 0$, or, if you want to get fancy, use a gray scale to go from pure white on the lines of zero phase, where the cosine is 1, down to pure black on the lines 180° out of phase, where the cosine is -1, and back up again. This gives a sense of a periodically varying intensity, and the slowness or rapidity of the changes in intensity indicate low or high spatial frequencies.

The spectrum The Fourier transform of a function $f(x_1, x_2)$ finds the spatial frequencies (ξ_1, ξ_2). The set of all spatial frequencies is called the *spectrum*, just as before. The inverse transform recovers the function from its spectrum, adding together the corresponding spatial harmonics, each contributing an amount $\mathcal{F}f(\xi_1, \xi_2)$. As mentioned above, when $f(x_1, x_2)$ is real we have

$$\mathcal{F}f(-\xi_1, -\xi_2) = \overline{\mathcal{F}f(\xi_1, \xi_2)}\,,$$

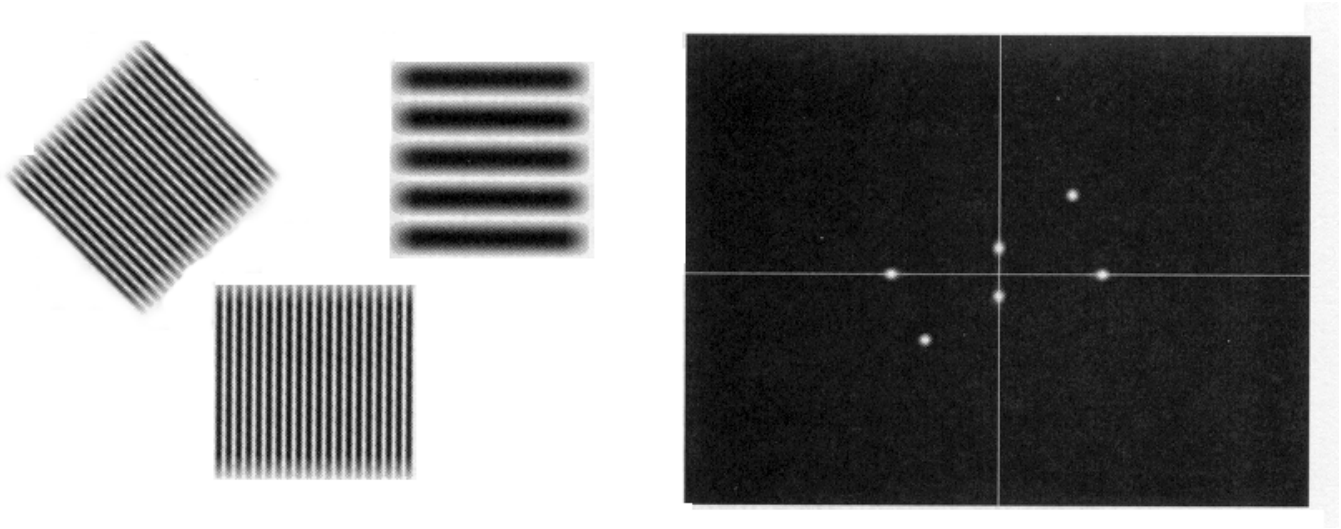
so that if a particular $\mathcal{F}f(\xi_1, \xi_2)$ is not zero then there is also a contribution from the "negative frequency" $(-\xi_1, -\xi_2)$. Thus for a real signal, the spectrum, as a set of points in the (ξ_1, ξ_2)-plane, is symmetric about the origin.[2] If we think of the exponentials of corresponding positive and negative frequency vectors adding up to give the signal then we're adding up (integrating) a bunch of cosines and the signal really does seem to be made of a bunch of a stripes with different spacings, different orientations, and different intensities

[2] *N.b.*: It's not the *values* $\mathcal{F}f(\xi_1, \xi_2)$ that are symmetric, just the set of points (ξ_1, ξ_2) of contributing frequencies.

(the magnitudes $|\mathcal{F}f(\xi_1, \xi_2)|$). It may be hard to imagine that an image, for example, is such a sum of stripes, but, then again, why is music the sum of a bunch of sine curves?

In the one-dimensional case we are used to drawing a picture of the magnitude of the Fourier transform to get some sense of how the energy is distributed among the different frequencies. We can do a similar thing in the two-dimensional case, putting a bright (or colored) dot at each point (ξ_1, ξ_2) that is in the spectrum, with a brightness proportional to the magnitude $|\mathcal{F}f(\xi_1, \xi_2)|$. This, the *energy spectrum* or the *power spectrum*, *is* symmetric about the origin because $|\mathcal{F}f(\xi_1, \xi_2)| = |\mathcal{F}f(-\xi_1, -\xi_2)|$.

Here are pictures of the spatial harmonics we showed before and their respective spectra.



Which is which? The stripes have an orientation (and a spacing) determined by $\boldsymbol{\xi} = (\xi_1, \xi_2)$ which is normal to the stripes. The horizontal stripes have a normal of the form $(0, \xi_2)$ and they are of lower frequency so ξ_2 is small. The vertical stripes have a normal of the form $(\xi_1, 0)$ and are of a higher frequency so ξ_1 is large, and the oblique stripes have a normal of the form (ξ, ξ) with a spacing about the same as for the vertical stripes

Here's a more interesting example.[3]

For the picture of the woman, what is the function we are taking the Fourier transform *of*? The function $f(x_1, x_2)$ is the intensity of light at each point (x_1, x_2) — that's what a black-and-white image *is* for the purposes of Fourier analysis. Incidentally, because the dynamic range (the range of intensities) can be so large in images it's common to light up the pixels in the spectral picture according to the *logarithm* of the intensity.

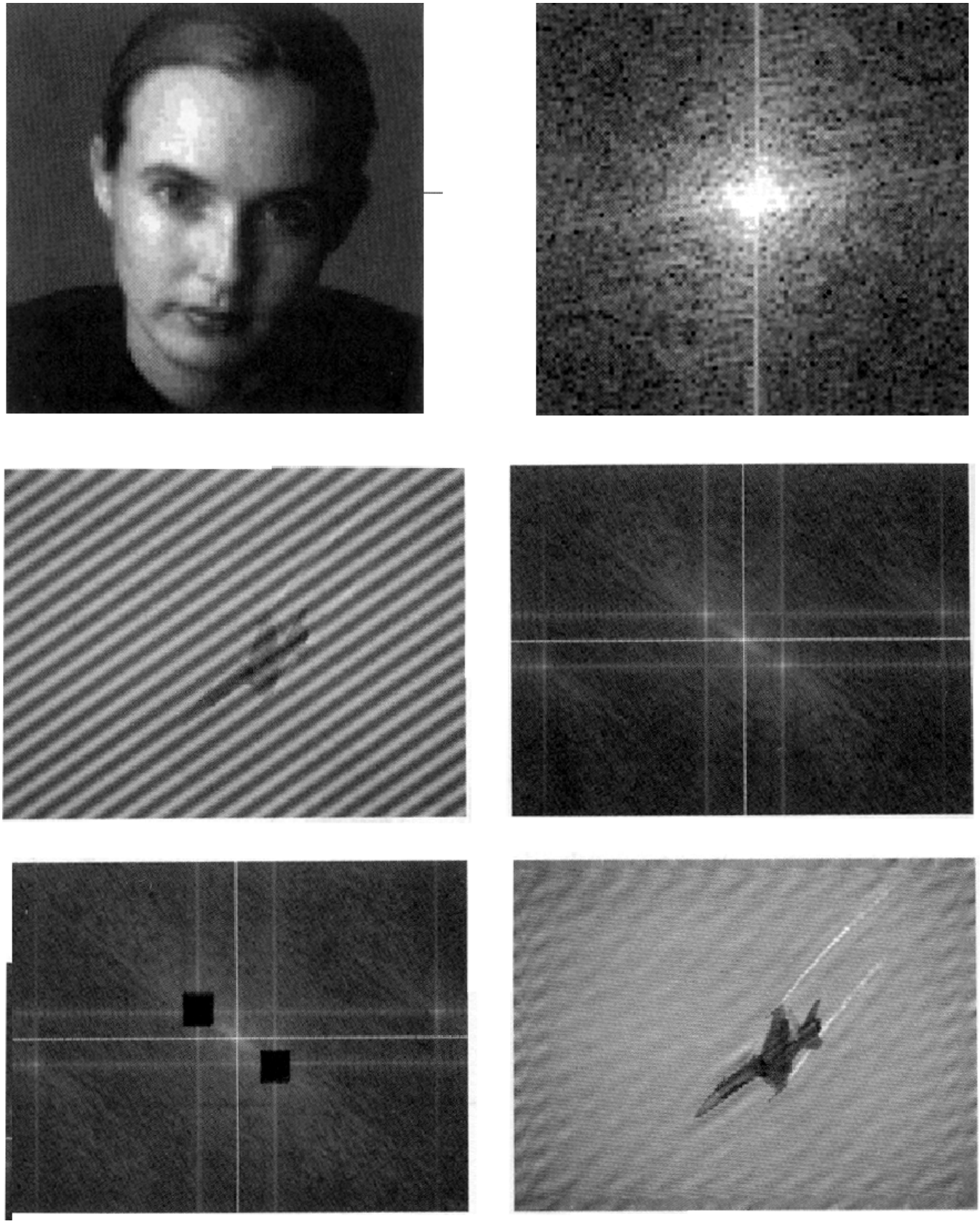
Here's a natural application of filtering in the frequency domain for an image.

The first picture shows periodic noise that appears quite distinctly in the frequency spectrum. We eliminate those frequencies and take the inverse transform to show the plane more clearly.[4]

Finally, there are reasons to *add* things to the spectrum as well as take them away. An important and relatively new application of the Fourier transform in imaging is *digital watermarking*. Watermarking is an old technique to authenticate printed documents. Within the paper an image is imprinted (somehow — I don't know how this is done!) that only becomes visible if held up to a light or dampened by water. The

[3] I showed this picture to the class a few years ago and someone yelled : "That's Natalie!"

[4] All of these examples are taken from the book *Digital Image Processing* by G. Baxes.



idea is that someone trying to counterfeit the document will not know of or cannot replicate the watermark, but that someone who knows where to look can easily verify its existence and hence the authenticity of the

document. The newer US currency now uses watermarks, as well as other anticounterfeiting techniques.

For electronic documents a *digital watermark* is added by adding to the spectrum. Insert a few extra harmonics here and there and keep track of what you added. This is done in a way to make the changes in the image undetectable (you hope) and so that no one else could possibly tell what belongs in the spectrum and what you put there (you hope). If the receivers of the document know where to look in the spectrum they can find your mark and verify that the document is legitimate.

Higher dimensions In higher dimensions the words to describe the harmonics and the spectrum are pretty much the same, though we can't draw the pictures[5]. The harmonics are the complex exponentials $e^{\pm 2\pi i \mathbf{x}\cdot}$ and we have n spatial frequencies, $\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_n)$. Again we single out where the complex exponentials are equal to 1 (zero phase), which is when $\boldsymbol{\xi} \cdot \mathbf{x}$ is an integer. In three-dimensions a given (ξ_1, ξ_2, ξ_3) defines a family $\boldsymbol{\xi} \cdot \mathbf{x}$ = integer of parallel planes (of zero phase) in (x_1, x_2, x_3)-space. The normal to any of the planes is the vector $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)$ and adjacent planes are a distance $1/\|\boldsymbol{\xi}\|$ apart. The exponential is periodic in the direction $\boldsymbol{\xi}$ with period $1/\|\boldsymbol{\xi}\|$. In a similar fashion, in n dimensions we have families of parallel hyperplanes ($(n-1)$-dimensional "planes") with normals $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)$, and distance $1/\|\boldsymbol{\xi}\|$ apart.

8.1.2 Finding a few Fourier transforms: separable functions

There are times when a function $f(x_1, \ldots, x_n)$ of n variables can be written as a product of n functions of one-variable, as in

$$f(x_1, \ldots, x_n) = f_1(x_1) f_2(x_2) \cdots f_n(x_n)\,.$$

Attempting to do this is a standard technique in finding special solutions of partial differential equations — there it's called the method of *separation of variables*. When a function can be factored in this way, its Fourier transform can be calculated as the product of the Fourier transform of the factors. Take $n = 2$ as a representative case:

$$\begin{aligned}
\mathcal{F}f(\xi_1, \xi_2) &= \int_{\mathbf{R}^n} e^{-2\pi i \mathbf{x}\cdot} f(\mathbf{x})\, d\mathbf{x} \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-2\pi i (x_1\xi_1 + x_2\xi_2)} f(x_1, x_2)\, dx_1\, dx_2 \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-2\pi i \xi_1 x_1} e^{-2\pi i \xi_2 x_2} f_1(x_1) f_2(x_2)\, dx_1\, dx_2 \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{-2\pi i \xi_1 x_1} f_1(x)\, dx_1 \right) e^{-2\pi i \xi_2 x_2} f_2(x_2)\, dx_2 \\
&= \mathcal{F}f_1(\xi_1) \int_{-\infty}^{\infty} e^{-2\pi i \xi_2 x_2} f_2(x_2)\, dx_2 \\
&= \mathcal{F}f_1(\xi_1)\, \mathcal{F}f_2(\xi_2)
\end{aligned}$$

In general, if $f(x_1, x_2, \ldots, x_n) = f_1(x_1) f_2(x_2) \cdots f_n(x_n)$ then

$$\mathcal{F}f(\xi_1, x_2, \ldots \xi_n) = \mathcal{F}f_1(\xi_1) \mathcal{F}f_2(\xi_2) \cdots \mathcal{F}f_n(\xi_n)\,.$$

If you really want to impress your friends and confound your enemies, you can invoke *tensor products* in this context. In mathematical parlance the separable signal f is the tensor product of the functions f_i and

[5] Any computer graphics experts out there care to add color and 3D-rendering to try to draw the spectrum?

one writes

$$f = f_1 \otimes f_2 \otimes \cdots \otimes f_n \,,$$

and the formula for the Fourier transform as

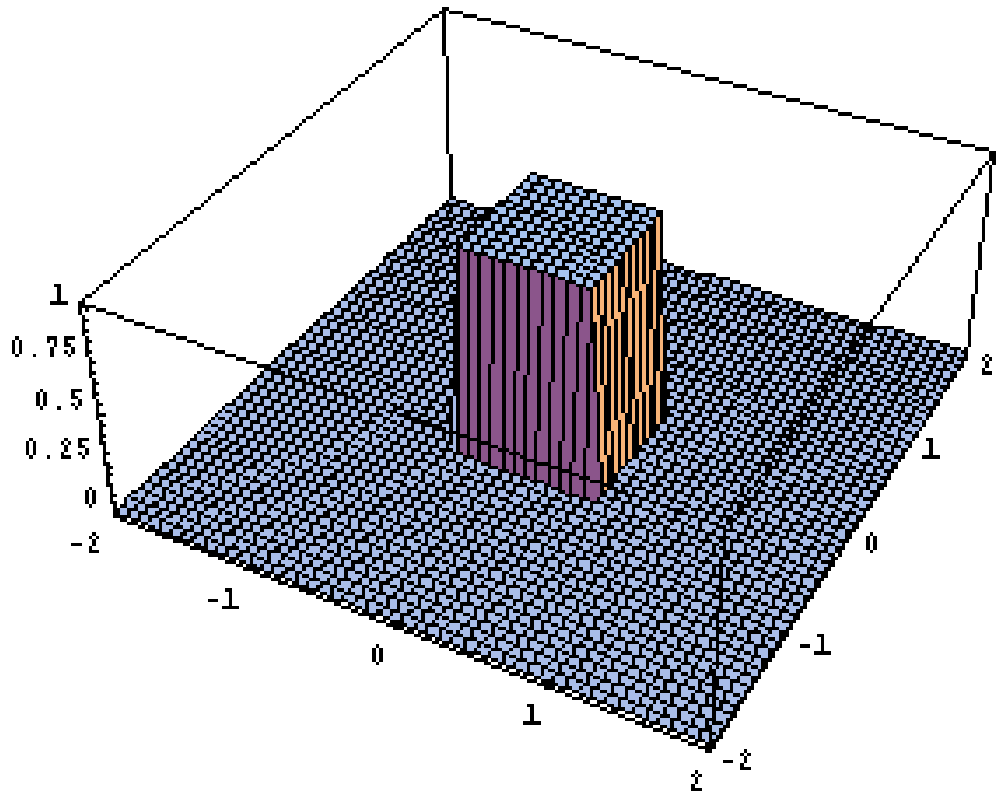
$$\mathcal{F}(f_1 \otimes f_2 \otimes \cdots \otimes f_n) = \mathcal{F}f_1 \otimes \mathcal{F}f_2 \otimes \cdots \otimes \mathcal{F}f_n \,.$$

People run in terror from the $\otimes$ symbol. Cool.

Higher dimensional rect functions The simplest, useful example of a function that fits this description is a version of the rect function in higher dimensions. In two dimensions, for example, we want the function that has the value 1 on the square of side length 1 centered at the origin, and has the value 0 outside this square. That is,

$$\Pi(x_1, x_2) = \begin{cases} 1 & -\frac{1}{2} < x_1 < \frac{1}{2} \,,\ -\frac{1}{2} < x_2 < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

You can fight it out how you want to define things on the edges. Here's a graph.



We can factor $\Pi(x_1, x_2)$ as the product of two one-dimensional rect functions:

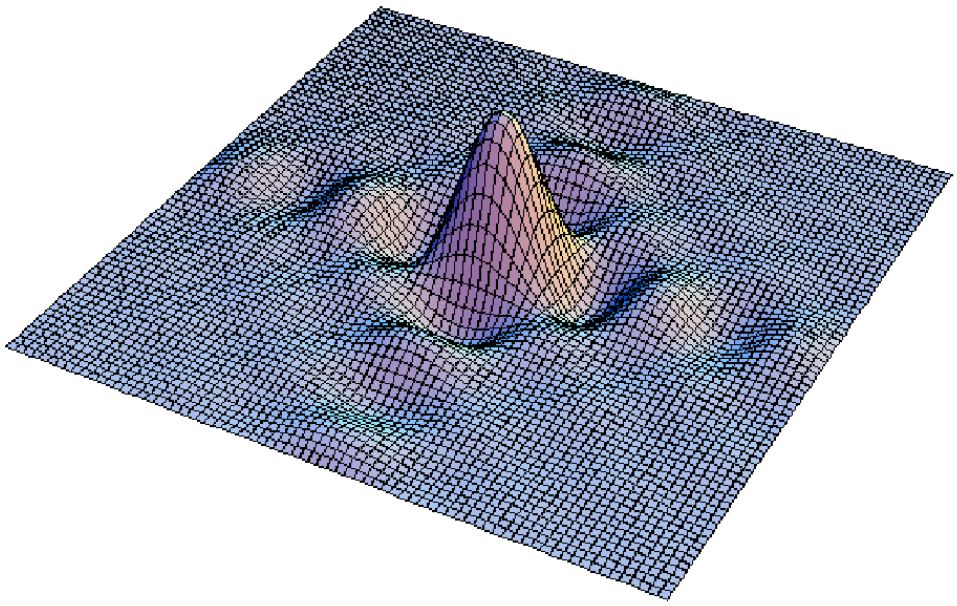
$$\Pi(x_1, x_2) = \Pi(x_1)\Pi(x_2) \,.$$

(I'm using the same notation for the rect function in one or more dimensions because, in this case, there's little chance of confusion.) The reason that we can write $\Pi(x_1, x_2)$ this way is because it is identically 1 if *all* the coordinates are between $-1/2$ and $1/2$ and it is zero otherwise — so it's zero if *any* of the coordinates is outside this range. That's exactly what happens for the product $\Pi(x_1)\Pi(x_2)$.

For the Fourier transform of the 2-dimensional Π we then have

$$\mathcal{F}\Pi(\xi_1, \xi_2) = \operatorname{sinc}\xi_1 \operatorname{sinc}\xi_2 \,.$$

Here's what the graph looks like.



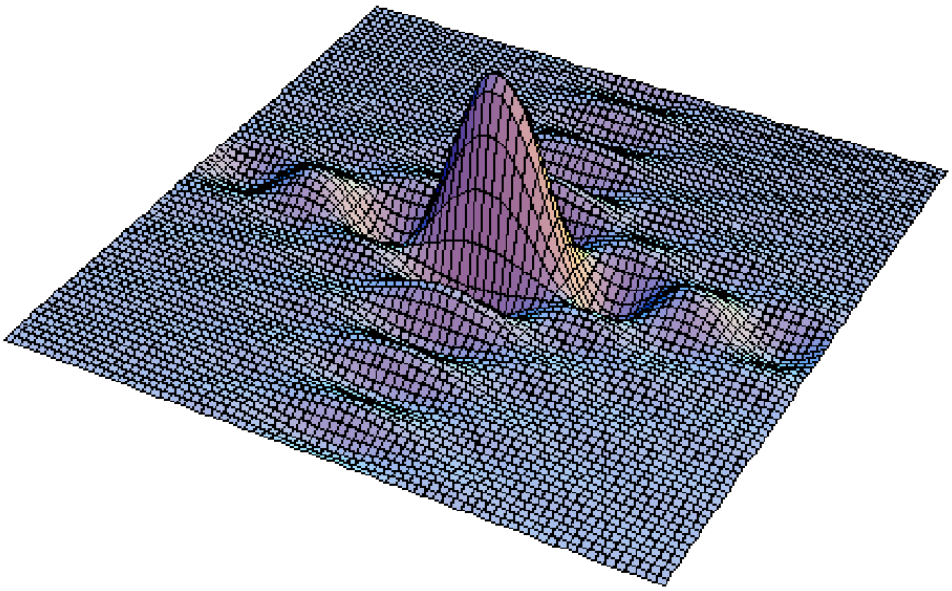
A helpful feature of factoring the rect function this way is the ability, easily, to change the widths in the different coordinate directions. For example, the function which is 1 in the rectangle $-a_1/2 < x_1 < a_1/2$, $-a_2/2 < x_2 < a_2/2$ and zero outside that rectangle is (in appropriate notation)

$$\Pi_{a_1 a_2}(x_1, x_2) = \Pi_{a_1}(x_1)\Pi_{a_2}(x_2) \,.$$

The Fourier transform of this is

$$\mathcal{F}\Pi_{a_1 a_2}(\xi_1, \xi_2) = (a_1 \operatorname{sinc} a_1\xi_1)(a_2 \operatorname{sinc} a_2\xi_2) \,.$$

Here's a plot of $(2\operatorname{sinc}2\xi_1)(4\operatorname{sinc}4\xi_2)$. You can see how the shape has changed from what we had before.



The direct generalization of the (basic) rect function to n dimensions is

$$\Pi(x_1, x_2, \ldots, x_n) = \begin{cases} 1 & -\frac{1}{2} < x_k < \frac{1}{2}\,, \quad k = 1, \ldots, n \\ 0 & \text{otherwise} \end{cases}$$

which factors as

$$\Pi(x_1, x_2, \ldots, x_n) = \Pi(x_1)\Pi(x_2)\cdots\Pi(x_n)\,.$$

For the Fourier transform of the n-dimensional Π we then have

$$\mathcal{F}\Pi(\xi_1, \xi_2, \ldots, \xi_n) = \operatorname{sinc}\xi_1 \operatorname{sinc}\xi_2 \cdots \operatorname{sinc}\xi_n\,.$$

It's obvious how to modify higher-dimensional Π to have different widths on different axes.

Gaussians Another good example of a separable function — one that often comes up in practice — is a Gaussian. By analogy to the one-dimensional case, the most natural Gaussian to use in connection with Fourier transforms is

$$g(\mathbf{x}) = e^{-\pi|\mathbf{x}|^2} = e^{-\pi(x_1^2 + x_2^2 + \cdots + x_n^2)}\,.$$

This factors as a product of n one-variable Gaussians:

$$g(x_1, \ldots, x_n) = e^{-\pi(x_1^2 + x_2^2 + \cdots + x_n^2)} = e^{-\pi x_1^2} e^{-\pi x_2^2} \cdots e^{-\pi x_n^2} = h(x_1)h(x_2)\cdots h(x_n)\,,$$

where

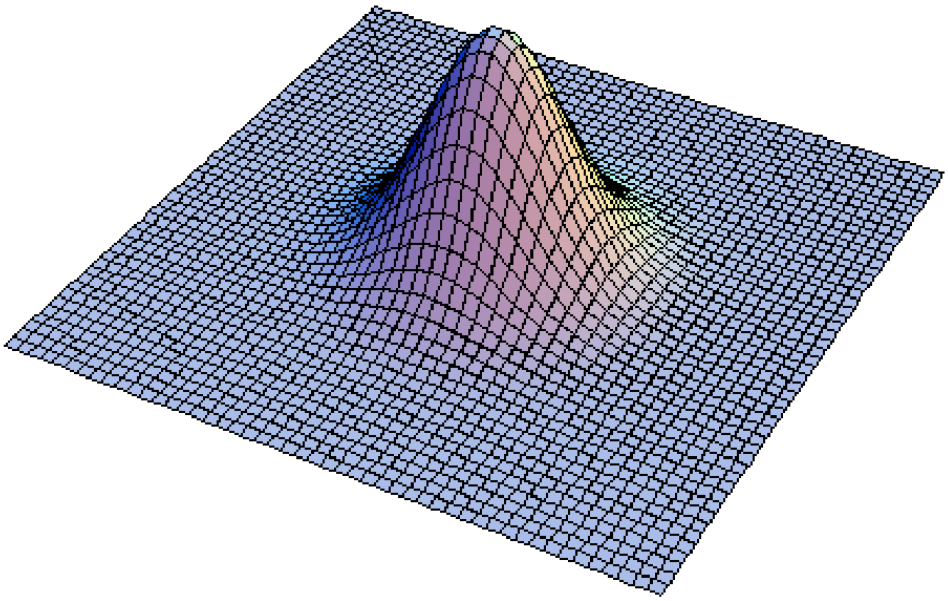
$$h(x_k) = e^{-\pi x_k^2}\,.$$

Taking the Fourier transform and applying the one-dimensional result (and reversing the algebra that we did above) gets us

$$\mathcal{F}g(\boldsymbol{\xi}) = e^{-\pi\xi_1^2} e^{-\pi\xi_2^2} \cdots e^{-\pi\xi_n^2} = e^{-\pi(\xi_1^2 + \xi_2^2 + \cdots + \xi_n^2)} = e^{-\pi||^2}\,.$$

As for one dimension, we see that g is its own Fourier transform.

Here's a plot of the two-dimensional Gaussian.



8.2 Getting to Know Your Higher Dimensional Fourier Transform

You already know a lot about the higher dimensional Fourier transform because you already know a lot about the one-dimensional Fourier transform — that's the whole point. Still, it's useful to collect a few of the basic facts. If some result corresponding to the one-dimensional case isn't mentioned here, that doesn't mean it doesn't hold, or isn't worth mentioning — it only means that the following is a very quick and very partial survey. Sometimes we'll work in $\mathbf{R}^n$, for any n, and sometimes just in $\mathbf{R}^2$; nothing should be read into this for or against $n = 2$.

8.2.1 Linearity

Linearity is obvious:

$$\mathcal{F}(\alpha f + \beta g)(\boldsymbol{\xi}) = \alpha \mathcal{F} f(\boldsymbol{\xi}) + \beta \mathcal{F} g(\boldsymbol{\xi}) \, .$$

8.2.2 Shifts

In one dimension a shift in time corresponds to a phase change in frequency. The statement of this is the shift theorem:

- If $f(x) \rightleftharpoons F(s)$ then $f(x \pm b) \rightleftharpoons e^{\pm 2\pi i s b} F(s)$.

It looks a little slicker (to me) if we use the delay operator $(\tau_b f)(x) = f(x - b)$, for then we can write

$$\mathcal{F}(\tau_b f)(s) = e^{-2\pi i s b} \mathcal{F} f(s) \, .$$

(Remember, τ_b involves $-b$.) Each to their own taste.

The shift theorem in higher dimensions can be made to look just like it does in the one-dimensional case. Suppose that a point $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is shifted by a displacement $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ to $\mathbf{x} + \mathbf{b} = (x_1 + b_1, x_2 + b_2, \ldots, x_n + b_n)$. Then the effect on the Fourier transform is:

- **The Shift Theorem** If $f(\mathbf{x}) \rightleftharpoons F(\boldsymbol{\xi})$ then $f(\mathbf{x} \pm \mathbf{b}) \rightleftharpoons e^{\pm 2\pi i \mathbf{b}\cdot} F(\boldsymbol{\xi})$.

Vectors replace scalars and the dot product replaces multiplication, but the formulas look much the same.

Again we can introduce the delay operator, this time "delaying" by a vector:

$$\tau_{\mathbf{b}} f(\mathbf{x}) = f(\mathbf{x} - \mathbf{b}) \,,$$

and the shift theorem then takes the form

$$\mathcal{F}(\tau_{\mathbf{b}} f)(\boldsymbol{\xi}) = e^{-2\pi i \mathbf{b}\cdot} \mathcal{F} f(\boldsymbol{\xi}) \,.$$

(Remember, $\tau_{\mathbf{b}}$ involves a $-\mathbf{b}$.) Each to their own taste, again.

If you're more comfortable writing things out in coordinates, the result, in two dimensions, would read:

$$\mathcal{F} f(x_1 \pm b_1, x_2 \pm b_2) = e^{2\pi i(\pm \xi_1 b_1 \pm \xi_2 b_2)} \mathcal{F} f(\xi_1, \xi_2) \,.$$

The only advantage in writing it out this way (and you certainly wouldn't do so for any dimension higher than two) is a more visible reminder that in shifting (x_1, x_2) to $(x_1 \pm b_1, x_2 \pm b_2)$ we shift the variables independently, so to speak. This independence is also (more) visible in the Fourier transform if we break up the dot product and multiply the exponentials:

$$\mathcal{F} f(x_1 \pm b_1, x_2 \pm b_2) = e^{\pm 2\pi i \xi_1 b_1} e^{\pm 2\pi i \xi_2 b_2} \mathcal{F} f(\xi_1, \xi_2) \,.$$

The derivation of the shift theorem is pretty much as in the one-dimensional case, but let me show you how the change of variable works. We'll do this for $n = 2$, and, yes, we'll write it out in coordinates. Let's just take the case when we're adding b_1 and b_2. First off

$$\mathcal{F}(f(x_1 + b_2, x_2 + b_2)) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-2\pi i(x_1\xi_1 + x_2\xi_2)} f(x_1 + b_1, x_2 + b_2)\, dx_1\, dx_2$$

We want to make a change of variable, turning $f(x_1 + b_1, x_2 + b_2)$ into $f(u, v)$ by the substitutions $u = x_1 + b_1$ and $v = x_2 + b_2$ (or equivalently $x_1 = u - b_1$ and $x_2 = v - b_2$). You have two choices at this point. The general change of variables formula for a multiple integral (stay with it for just a moment) immediately produces.

$$\begin{aligned}
&\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-2\pi i(x_1\xi_1 + x_2\xi_2)} f(x_1 + b_1, x_2 + b_2)\, dx_1\, dx_2 \\
&\qquad = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-2\pi i((u-b_1)\xi_1 + (v-b_2)\xi_2)} f(u, v)\, du\, dv \\
&\qquad = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{2\pi i b_1 \xi_1} e^{2\pi i b_2 \xi_2} e^{-2\pi i(u\xi_2 + v\xi_2)} f(u, v)\, du\, dv \\
&\qquad = e^{2\pi i(b_1\xi_1 + b_2\xi_2)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-2\pi i(u\xi_2 + v\xi_2)} f(u, v)\, du\, dv \\
&\qquad = e^{2\pi i(b_1\xi_1 + b_2\xi_2)} \mathcal{F} f(\xi_1, \xi_2) \,,
\end{aligned}$$

and there's our formula.

If you know the general change of variables formula then the shift formula *and* its derivation really are just like the one-dimensional case, but this doesn't do you much good if you don't know the change of variables formula for a multiple integral. So, for completeness, let me show you an alternative derivation that works because the change of variables $u = x_1 + b_1$, $v = x_2 + b_2$ changes x_1 and x_2 separately.

$$\begin{aligned}
\mathcal{F}f(x_1+b_2, x_2+b_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi i(x_1\xi_1+x_2\xi_2)} f(x_1+b_1, x_2+b_2)\,dx_1\,dx_2 \\
&= \int_{-\infty}^{\infty} e^{2\pi i x_1\xi_1}\left(\int_{-\infty}^{\infty} e^{2\pi i x_2\xi_2} f(x_1+b_1, x_2+b_2)\,dx_2\right)dx_1 \\
&= \int_{-\infty}^{\infty} e^{2\pi i x_1\xi_1}\left(\int_{-\infty}^{\infty} e^{-2\pi i(v-b_2)\xi_2} f(x_1+b_1, v)\,dv\right)dx_1 \\
&\quad \text{(substituting } v = x_2 + b_2\text{)} \\
&= e^{2\pi i b_2\xi_2}\int_{-\infty}^{\infty} e^{-2\pi i x_1\xi_1}\left(\int_{-\infty}^{\infty} e^{-2\pi i v\xi_2} f(x_1+b_1, v)\,dv\right)dx_1 \\
&= e^{2\pi i b_2\xi_2}\int_{-\infty}^{\infty} e^{-2\pi i v\xi_2}\left(\int_{-\infty}^{\infty} e^{-2\pi i x_1\xi_1} f(x_1+b_1, v)\,dx_1\right)dv \\
&= e^{2\pi i b_2\xi_2}\int_{-\infty}^{\infty} e^{-2\pi i v\xi_2}\left(\int_{-\infty}^{\infty} e^{-2\pi i(u-b_1)\xi_1} f(u, v)\,du\right)dv \\
&\quad \text{(substituting } u = x_1 + b_1\text{)} \\
&= e^{2\pi i b_2\xi_2} e^{2\pi i b_1\xi_1}\int_{-\infty}^{\infty} e^{-2\pi i v\xi_2}\left(\int_{-\infty}^{\infty} e^{-2\pi i u\xi_1} f(u, v)\,du\right)dv \\
&= e^{2\pi i b_2\xi_2} e^{2\pi i b_1\xi_1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi i(u\xi_1+v\xi_2)} f(u, v)\,du\,dv \\
&= e^{2\pi i b_2\xi_2} e^{2\pi i b_1\xi_1}\,\mathcal{F}f(\xi_1, \xi_2) \\
&= e^{2\pi i(b_2\xi_2+b_1\xi_1)}\,\mathcal{F}f(\xi_1, \xi_2)\,.
\end{aligned}$$

And there's our formula, again.

The good news is, we've certainly derived the shift theorem! The bad news is, you may be saying to yourself: "This is not what I had in mind when you said the higher dimensional case is just like the one-dimensional case." I don't have a quick comeback to that, except that I'm trying to make honest statements about the similarities and the differences in the two cases and, if you want, you can assimilate the formulas and just skip those derivations in the higher dimensional case that bug your sense of simplicity. I will too, mostly.

8.2.3 Stretches

There's really only one stretch theorem in higher dimensions, but I'd like to give two versions of it. The first version can be derived in a manner similar to what we did for the shift theorem, making separate changes of variable. This case comes up often enough that it's worth giving it its own moment in the sun. The second version (which includes the first) needs the general change of variables formula for the derivation.

- **Stretch Theorem, first version**

$$\mathcal{F}(f(a_1x_1, a_2x_2)) = \frac{1}{|a_1||a_2|}\mathcal{F}(f)\left(\frac{\xi_1}{a_1}, \frac{\xi_2}{a_2}\right)\,.$$

There is an analogous statement in higher dimensions.

I'll skip the derivation.

The reason that there's a second version of the stretch theorem is because there's something new that can be done by way of transformations in higher dimensions that doesn't come up in the one-dimensional setting. We can look at a *linear change of variables* in the spatial domain. In two dimensions we write this as

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

or, written out,

$$\begin{aligned} u_1 &= ax_1 + bx_2 \\ u_2 &= cx_1 + dx_2 \end{aligned}$$

The simple, "independent" stretch is the special case

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} .$$

For a general linear transformation the coordinates can get mixed up together instead of simply changing independently.

A linear change of coordinates is not at all an odd a thing to do — think of linearly distorting an image, for whatever reason. Think also of rotation, which we'll consider below. Finally, a linear transformation as a linear change of coordinates isn't much good if you can't change the coordinates back. Thus it's natural to work only with invertible transformations here, i.e., those for which $\det A \neq 0$.

The general stretch theorem answers the question of what happens to the spectrum when the spatial coordinates change linearly — what is $\mathcal{F}(f(u_1, u_2)) = \mathcal{F}(f(ax_1 + bx_2, cx_1 + dx_2))$? The nice answer is most compactly expressed in matrix notation, in fact just as easily for n dimensions as for two. Let A be an $n \times n$ *invertible* matrix. We introduce the notation

$$A^{-\mathsf{T}} = (A^{-1})^{\mathsf{T}} ,$$

the transpose of the inverse of A. You can check that also $A^{-\mathsf{T}} = (A^{\mathsf{T}})^{-1}$, i.e., $A^{-\mathsf{T}}$ can be defined either as the transpose of the inverse or as the inverse of the transpose. ($A^{-\mathsf{T}}$ will also come up naturally when we apply the Fourier transform to lattices and "reciprocal lattices", i.e., to crystals.)

We can now state:

- **Stretch Theorem, general version**

$$\mathcal{F}(f(A\mathbf{x})) = \frac{1}{|\det A|} \mathcal{F}f(A^{-\mathsf{T}}\boldsymbol{\xi}) .$$

There's another way of writing this that you might prefer, depending (as always) on your tastes. Using $\det A^{\mathsf{T}} = \det A$ and $\det A^{-1} = 1/\det A$ we have

$$\frac{1}{|\det A|} = |\det A^{-\mathsf{T}}|$$

so the formula reads

$$\mathcal{F}(f(A\mathbf{x})) = |\det A^{-\mathsf{T}}| \, \mathcal{F}f(A^{-\mathsf{T}}\boldsymbol{\xi}) .$$

Finally, I'm of a mind to introduce the general *scaling* operator defined by

$$(\sigma_A f)(\mathbf{x}) = f(A\mathbf{x})\,,$$

where A is an invertible $n \times n$ matrix. Then I'm of a mind to write

$$\mathcal{F}(\sigma_A f)(\boldsymbol{\xi}) = \frac{1}{|\det A|}\mathcal{F}f(A^{-\mathsf{T}}\boldsymbol{\xi})\,.$$

Your choice. I'll give a derivation of the general stretch theorem in Section 8.2.7.

Let's look at the two-dimensional case in a little more detail. To recover the first version of the stretch theorem we apply the general version to the diagonal matrix

$$A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \quad \text{with} \quad \det A = a_1 a_2 \neq 0\,.$$

Then

$$A^{-1} = \begin{pmatrix} 1/a_1 & 0 \\ 0 & 1/a_2 \end{pmatrix} \;\Rightarrow\; A^{-\mathsf{T}} = \begin{pmatrix} 1/a_1 & 0 \\ 0 & 1/a_2 \end{pmatrix}.$$

This gives

$$\mathcal{F}(f(a_1x_1, a_2x_2)) = \mathcal{F}(f(A\mathbf{x})) = \frac{1}{|\det A|}\mathcal{F}f(A^{-\mathsf{T}}\boldsymbol{\xi}) = \frac{1}{|a_1|\,|a_2|}\mathcal{F}f\left(\frac{\xi_1}{a_1}, \frac{\xi_2}{a_2}\right).$$

Works like a charm.

An important special case of the stretch theorem is when A is a rotation matrix:

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

A rotation matrix is *orthogonal*, meaning that $AA^{\mathsf{T}} = I$:

$$AA^{\mathsf{T}} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \cos^2\theta + \sin^2\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus $A^{-1} = A^{\mathsf{T}}$ so that

$$A^{-\mathsf{T}} = (A^{-1})^{\mathsf{T}} = (A^{\mathsf{T}})^{\mathsf{T}} = A\,.$$

Also

$$\det A = \cos^2\theta + \sin^2\theta = 1\,.$$

The consequence of all of this for the Fourier transform is that if A is a rotation matrix then

$$\mathcal{F}(f(A\mathbf{x})) = \mathcal{F}f(A\boldsymbol{\xi}),\,.$$

In words:

- A rotation in the spatial domain corresponds to an identical rotation in the frequency domain.

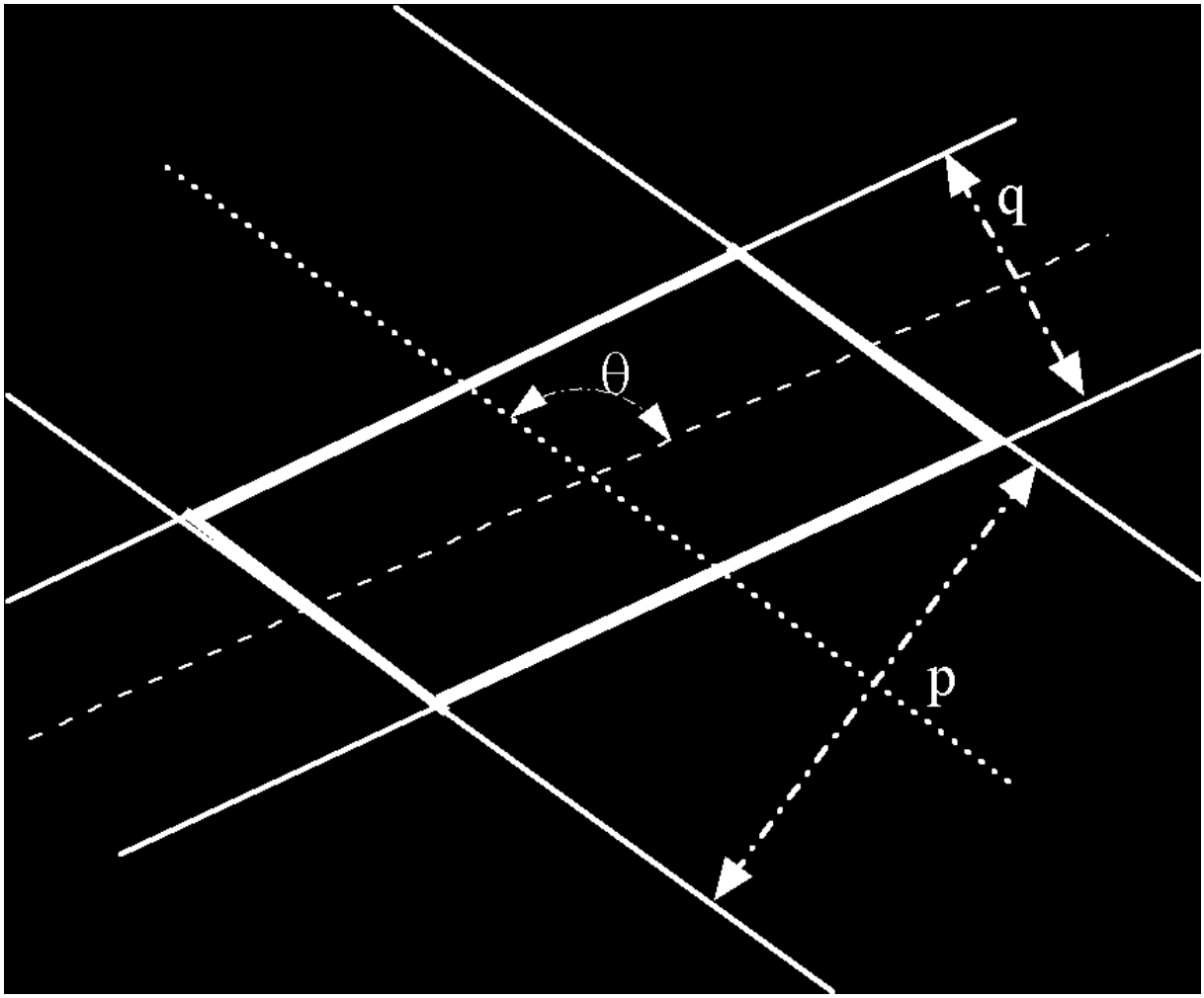
This result is used all the time in imaging problems.

Finally, it's worth knowing that for a 2×2 matrix we can write down $A^{-\mathsf{T}}$ explicitly:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad \text{so the transpose of this is} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-\mathsf{T}} = \frac{1}{\det A}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$$

This jibes with what we found for a rotation matrix.

The indicator function for a parallelogram As an exercise in using the stretch theorem you can show the following. Consider a parallelogram centered at $(0, 0)$:



One set of data that describes the parallelogram are the distances between sides, p and q, and the vectors that give the directions of the sides. Let $\mathbf{u}$ be a unit vector in the direction of the sides that are p apart and let $\mathbf{v}$ be a unit vector in the direction of the sides that are q apart.

The *indicator* function P for the parallelogram is the function that is equal to 1 on the parallelogram and equal to 0 outside the parallelogram. The Fourier transform of P can be shown to be

$$\mathcal{F}P(\boldsymbol{\xi}) = \frac{pq}{|\sin\theta|}\,\mathrm{sinc}\left(\frac{p(\mathbf{u}\cdot\boldsymbol{\xi})}{\sin\theta}\right)\mathrm{sinc}\left(\frac{q(\mathbf{v}\cdot\boldsymbol{\xi})}{\sin\theta}\right).$$

Shift and stretch As an example of using the general formula, let's combine a shift with a stretch and show:

$$\mathcal{F}(f(A\mathbf{x}+\mathbf{b})) = \exp(2\pi i\mathbf{b}\cdot A^{-\mathsf{T}}\boldsymbol{\xi})\frac{1}{|\det A|}\mathcal{F}f(A^{-\mathsf{T}}\boldsymbol{\xi})$$

(I think the exponential is a little crowded to write it as e to a power here.) Combining shifts and stretches seems to cause a lot of problems for people (even in one dimension), so let me do this in several ways.

As a first approach, and to keep the operations straight, write

$$g(\mathbf{x}) = f(\mathbf{x}+b)\,,$$

and then

$$f(A\mathbf{x} + b) = g(A\mathbf{x})\,.$$

Using the stretch theorem first,

$$\mathcal{F}(g(A\mathbf{x})) = \frac{1}{|\det A|}\mathcal{F}g(A^{-\mathsf{T}}\boldsymbol{\xi})$$

Applying the shift theorem next gives

$$(\mathcal{F}g)(A^{-\mathsf{T}}\boldsymbol{\xi}) = \exp(2\pi i\mathbf{b}\cdot A^{-\mathsf{T}}\boldsymbol{\xi})\mathcal{F}f((A^{-\mathsf{T}}\boldsymbol{\xi})\,.$$

Putting these together gives the final formula for $\mathcal{F}(f(A\mathbf{x}+\mathbf{b}))$.

Another way around is instead to write

$$g(\mathbf{x}) = f(A\mathbf{x})$$

and then

$$f(A\mathbf{x}+\mathbf{b}) = f(A(\mathbf{x}+A^{-1}\mathbf{b})) = g(\mathbf{x}+A^{-1}\mathbf{b})\,.$$

Now use the shift theorem first to get

$$\mathcal{F}(g(\mathbf{x}+A^{-1}\mathbf{b})) = \exp(2\pi iA^{-1}\mathbf{b}\cdot\boldsymbol{\xi})\,(\mathcal{F}g)(\boldsymbol{\xi}) = \exp(2\pi i\mathbf{b}\cdot A^{-\mathsf{T}}\boldsymbol{\xi})\,(\mathcal{F}g)(\boldsymbol{\xi})\,.$$

The stretch theorem comes next and it produces

$$\mathcal{F}g(\boldsymbol{\xi}) = \mathcal{F}(f(A\mathbf{x})) = \frac{1}{|\det A|}\mathcal{F}f(A^{-\mathsf{T}}\boldsymbol{\xi})\,.$$

This agrees with what we had before, as if there was any doubt.

Finally, by popular demand, I do this one more time by expressing $f(A\mathbf{x}+\mathbf{b})$ using the delay and scaling operators. It's a question of which comes first, and parallel to the first derivation above we can write:

$$f(A\mathbf{x}+\mathbf{b}) = \sigma_A(\tau_{-\mathbf{b}}f)(\mathbf{x}) = (\sigma_A\tau_{-\mathbf{b}}f)(\mathbf{x})\,,$$

which we verify by

$$(\sigma_A\tau_{-\mathbf{b}}f)(\mathbf{x}) = (\tau_{-\mathbf{b}}f)(A\mathbf{x}) = f(A\mathbf{x}+\mathbf{b})\,.$$

And now we have

$$\mathcal{F}(\sigma_A(\tau_{-\mathbf{b}}f))(\boldsymbol{\xi}) = \frac{1}{|\det A|}\mathcal{F}(\tau_{-\mathbf{b}}f)(A^{-\mathsf{T}}\boldsymbol{\xi}) = \frac{1}{|\det A|}\exp(2\pi iA^{-\mathsf{T}}\boldsymbol{\xi}\cdot\mathbf{b})\mathcal{F}f(A^{-\mathsf{T}}\boldsymbol{\xi})\,.$$

I won't give a second version of the second derivation.

8.2.4 Convolution

What about convolution? For two real-valued functions f and g on $\mathbf{R}^n$ the definition is

$$(f*g)(\mathbf{x}) = \int_{\mathbf{R}^n} f(\mathbf{x}-\mathbf{y})g(\mathbf{y})\,d\mathbf{y}\,.$$

Written out in coordinates this looks much more complicated. For $n=2$, for example,

$$(f*g)(x_1,x_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1-y_1,x_2-y_2)g(y_1,y_2)\,dy_1\,dy_2\,.$$

The intelligent person would not write out the corresponding coordinatized formula for higher dimensions unless absolutely pressed. The intelligent person would also not try too hard to flip, drag or otherwise visualize a convolution in higher dimensions. The intelligent person would be happy to learn, however, that once again

$$\mathcal{F}(f*g)(\boldsymbol{\xi}) = \mathcal{F}f(\boldsymbol{\xi})\mathcal{F}g(\boldsymbol{\xi}) \quad\text{and}\quad \mathcal{F}(fg)(\boldsymbol{\xi}) = (\mathcal{F}f*\mathcal{F}g)(\boldsymbol{\xi})\,.$$

The typical interpretations of convolution — smoothing, averaging, etc. — continue to apply, when applied by an intelligent person.

8.2.5 A little δ now, more later

We'll see that things get more interesting in higher dimensions for delta functions, but the definition of the plain vanilla δ is the same as before. To give the distributional definition, I'll pause, just for a moment, to define what it means for a function of several variables to be a Schwartz function.

Schwartz functions The theory and practice of tempered distributions works the same in higher dimensions as it does in one. The basis of the treatment is via the Schwartz functions as the class of test functions. The condition that a function of several variables be rapidly decreasing is that all partial derivatives (including mixed partial derivatives) decrease faster than any power of any of the coordinates. This can be stated in any number of equivalent forms. One way is to require that

$$|\mathbf{x}|^p \, |\partial^q \varphi(\mathbf{x})| \to 0 \quad \text{as} \quad |\mathbf{x}| \to \infty \, .$$

I'll explain the funny notation — it's an example of the occasional awkwardness that sets in when writing formulas in higher dimensions. p is a positive integer, so that just gives a power of $|\mathbf{x}|$, and q is a *multi-index*. This means that $q = (q_1, \dots, q_n)$, each q_i a positive integer, so that ∂^q is supposed to mean

$$\frac{\partial^{q_1 + \cdots + q_n}}{(\partial x_1)^{q_1} (\partial x_2)^{q_2} \cdots (\partial x_n)^{q_n}} \, .$$

There's no special font used to indicate multi-indices — you just have to intuit it.

From here, the definitions of tempered distributions, the Fourier transform of a tempered distribution, and everything else, goes through just as before. Shall we leave it alone? I thought so.

δ in higher dimensions The δ-function is the distribution defined by the pairing

$$\langle \delta, \varphi \rangle = \varphi(0, \dots, 0) \quad \text{or} \quad \langle \delta, \varphi \rangle = \varphi(\mathbf{0}) \quad \text{in vector notation}$$

where $\varphi(x_1, , \dots, x_n)$ is a Schwartz function.[6] As is customary, we also write this in terms of integration as:

$$\int_{\mathbf{R}^n} \varphi(\mathbf{x}) \delta(\mathbf{x}) \, d\mathbf{x} = \varphi(\mathbf{0})$$

You can show that δ is even as a distribution (once you've reminded yourself what it means for a distribution to be even).

As before, one has

$$f(\mathbf{x}) \delta(\mathbf{x}) = f(\mathbf{0}) \delta(\mathbf{x}) \, ,$$

when f is a smooth function, and for convolution

$$(f * \delta)(\mathbf{x}) = f(\mathbf{x}) \, .$$

The shifted delta function $\delta(\mathbf{x} - \mathbf{b}) = \delta(x_1 - b_1, x_2 - b_2, , \dots, x_n - b_n)$ or $\delta_{\mathbf{b}} = \tau_{\mathbf{b}} \delta$, has the corresponding properties

$$f(\mathbf{x}) \delta(\mathbf{x} - \mathbf{b}) = f(\mathbf{b}) \delta(\mathbf{x} - \mathbf{b}) \quad \text{and} \quad f * \delta(\mathbf{x} - \mathbf{b}) = f(\mathbf{x} - \mathbf{b}) \, .$$

In some cases it is useful to know that we can "factor" the delta function into one-dimensional deltas, as in

$$\delta(x_1, x_2, \dots, x_n) = \delta_1(x_1) \delta_2(x_2) \cdots \delta_n(x_n) \, .$$

[6] Actually, δ is in a larger class than the tempered distributions. It is defined by the pairing $\langle \delta, \varphi \rangle = \varphi(0)$ when φ is any smooth function of compact support.

I've put subscripts on the δ's on the right hand side just to tag them with the individual coordinates — there are some advantages in doing this. Though it remains true, as a general rule, that multiplying distributions is not (and cannot be) defined, this is one case where it makes sense. The formula holds because of how each side acts on a Schwartz function.[7] Let's just check this in the two-dimensional case, and play a little fast and loose by writing the pairing as an integral. Then, on the one hand,

$$\int_{\mathbf{R}^2} \varphi(\mathbf{x})\delta(\mathbf{x})\, d\mathbf{x} = \varphi(0,0)$$

by definition of the 2-dimensional delta function. On the other hand,

$$\begin{aligned}\int_{\mathbf{R}^2} \varphi(x_1,x_2)\delta_1(x_1)\delta_2(x_2)\, dx_1\, dx_2 &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} \varphi(x_1,x_2)\delta_1(x_1)\, dx_1\right)\delta_2(x_2)\, dx_2 \\ &= \int_{-\infty}^{\infty} \varphi(0,x_2)\delta_2(x_2)\, dx_2 = \varphi(0,0).\end{aligned}$$

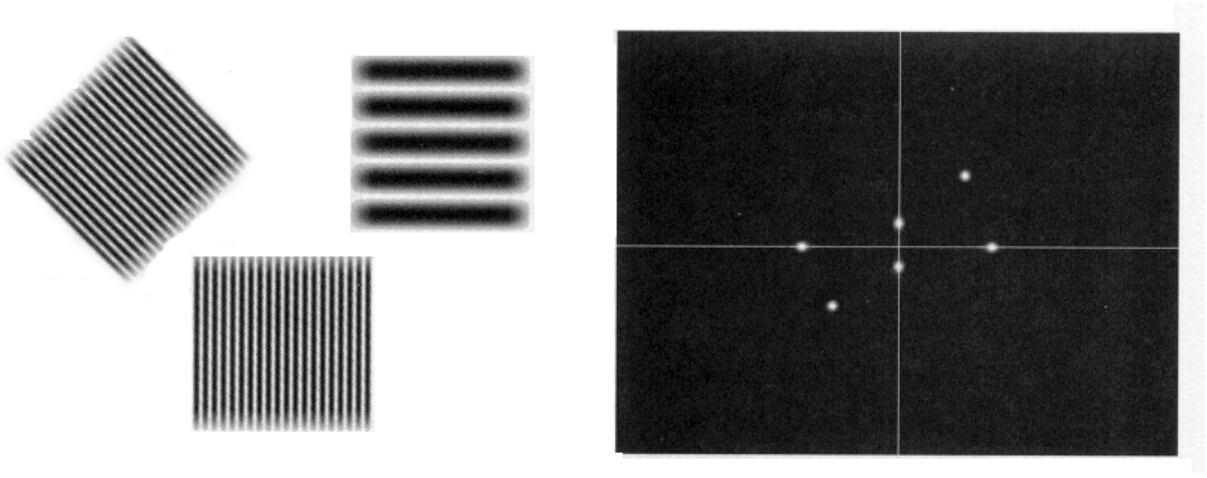
So $\delta(x_1,x_2)$ and $\delta_1(x_1)\delta_2(x_2)$ have the same effect when integrated against a test function.

The Fourier transform of δ And finally — the Fourier transform of the delta function is, of course, 1 (that's the constant function 1). The argument is the same as in the one-dimensional case. By duality, the Fourier transform of 1 is δ. One can then shift to get

$$\delta(\mathbf{x}-\mathbf{b}) \rightleftharpoons e^{-2\pi i \mathbf{b}\cdot} \quad \text{or} \quad \mathcal{F}\delta_{\mathbf{b}} = e^{-2\pi i \mathbf{b}\cdot}\,.$$

You can now see (again) where those symmetrically paired dots come from in looking at the spectral picture for alternating black and white stripes. It comes from the Fourier transforms of $\cos(2\pi\,\mathbf{x}\cdot\boldsymbol{\xi}_0) = \operatorname{Re}\exp(2\pi i\,\mathbf{x}\cdot\boldsymbol{\xi}_0)$ for $\boldsymbol{\xi}_0 = (\xi_1,0)$, $\boldsymbol{\xi}_0 = (0,\xi_2)$, and $\boldsymbol{\xi}_0 = (\xi_3,\xi_3)$, since

$$\mathcal{F}\cos(2\pi\,\mathbf{x}\cdot\boldsymbol{\xi}_0) = \tfrac{1}{2}(\delta(\boldsymbol{\xi}-\boldsymbol{\xi}_0) + \delta(\boldsymbol{\xi}+\boldsymbol{\xi}_0))\,.$$



[7] The precise way to do this is through the use of tensor products of distributions, something we have not discussed, and will not.

Scaling delta functions Recall how a one-dimensional delta function scales:

$$\delta(ax) = \frac{1}{|a|}\delta(x)\,.$$

Writing a higher dimensional delta function as a product of one-dimensional delta functions we get a corresponding formula. In two dimensions:

$$\begin{aligned}\delta(a_1x_1, a_2x_2) &= \delta_1(a_1x_1)\delta_2(a_2x_2)\\ &= \frac{1}{|a_1|}\delta_1(x_1)\frac{1}{|a_2|}\delta_2(x_2)\\ &= \frac{1}{|a_1|\,|a_2|}\delta_1(x_1)\delta_2(x_2) = \frac{1}{|a_1a_2|}\delta(x_1, x_2),\end{aligned}$$

and in n-dimensions

$$\delta(a_1x_1, \ldots, a_nx_n) = \frac{1}{|a_1\cdots a_n|}\delta(x_1, \ldots, x_n)\,.$$

It's also possible (and useful) to consider $\delta(A\mathbf{x})$ when A is an invertible matrix. The result is

$$\delta(A\mathbf{x}) = \frac{1}{|\det A|}\delta(\mathbf{x})\,.$$

See Section 8.2.7 for a derivation of this. This formula bears the same relationship to the preceding formula as the general stretch theorem bears to the first version of the stretch theorem.

8.2.6 The Fourier transform of a radial function

For use in many applications, we're going to consider one further aspects of the 2-dimensional case. A function on $\mathbf{R}^2$ is *radial* (also called *radially symmetric* or *circularly symmetric*) if it depends only on the distance from the origin. In polar coordinates the distance from the origin is denoted by r, so to say that a function is radial is to say that it depends only on r (and that it does not depend on θ, writing the usual polar coordinates as (r, θ)).

The definition of the Fourier transform is set up in Cartesian coordinates, and it's clear that we'll be better off writing it in polar coordinates if we work with radial functions. This is actually *not* so straightforward, or, at least, it involves introducing some special functions to write the formulas in a compact way.

We have to convert

$$\int_{\mathbf{R}^2} e^{-2\pi i\mathbf{x}\cdot}f(\mathbf{x})\,d\mathbf{x} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi i(x_1\xi_1+x_2\xi_2)}f(x_1, x_2)\,dx_1\,dx_2$$

to polar coordinates. There are several steps: To say that $f(\mathbf{x})$ is a radial function means that it becomes $f(r)$. To describe all of $\mathbf{R}^2$ in the limits of integration, we take r going from 0 to ∞ and θ going from 0 to 2π. The area element $dx_1\,dx_2$ becomes $r\,dr\,d\theta$. Finally, the problem is the inner product $\mathbf{x}\cdot\boldsymbol{\xi} = x_1\xi_1+x_2\xi_2$ in the exponential and how to write it in polar coordinates. If we identify $(x_1, x_2) = (r, \theta)$ (varying over the (x_1, x_2)-plane) and put $(\xi_1, \xi_2) = (\rho, \phi)$ (fixed in the integral) then

$$\mathbf{x}\cdot\boldsymbol{\xi} = \|\mathbf{x}\|\,\|\boldsymbol{\xi}\|\cos(\theta-\phi) = r\rho\cos(\theta-\phi)\,.$$

The Fourier transform of f is thus

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi i\mathbf{x}\cdot}f(\mathbf{x})\,d\mathbf{x} = \int_0^{2\pi}\int_0^{\infty} f(r)e^{-2\pi ir\rho\cos(\theta-\phi)}\,r\,dr\,d\theta\,.$$

There's more to be done. First of all, because $e^{-2\pi i r\rho\cos(\theta-\phi)}$ is periodic (in θ) of period 2π, the integral

$$\int_0^{2\pi} e^{-2\pi i r\rho\cos(\theta-\phi)}\,d\theta$$

does not depend on ϕ.[8] Consequently,

$$\int_0^{2\pi} e^{-2\pi i r\rho\cos(\theta-\phi)}\,d\theta = \int_0^{2\pi} e^{-2\pi i r\rho\cos\theta}\,d\theta\,.$$

The next step is to define ourselves out of trouble. We introduce the function

$$J_0(a) = \frac{1}{2\pi}\int_0^{2\pi} e^{-ia\cos\theta}\,d\theta\,.$$

We give this integral a name, $J_0(a)$, because, try as you might, there is no simple closed form expression for it, so we take the integral as defining a new function. It is called the zero order Bessel function of the first kind. Sorry, but Bessel functions, of whatever order and kind, always seem to come up in problems involving circular symmetry; ask any physicist.

Incorporating J_0 into what we've done,

$$\int_0^{2\pi} e^{-2\pi i r\rho\cos\theta}\,d\theta = 2\pi J_0(2\pi r\rho)$$

and the Fourier transform of $f(r)$ is

$$2\pi\int_0^{\infty} f(r)J_0(2\pi r\rho)\,r\,dr$$

Let's summarize:

- If $f(\mathbf{x})$ is a radial function then its Fourier transform is

$$F(\rho) = 2\pi\int_0^{\infty} f(r)J_0(2\pi r\rho)\,r dr$$

- In words, the important conclusion to take away from this is that the Fourier transform of a radial function is also radial.

The formula for $F(\rho)$ in terms of $f(r)$ is sometimes called the zero order *Hankel transform* of $f(r)$ but, again, we understand that it is nothing other than the Fourier transform of a radial function.

Circ and Jinc A useful radial function to define, sort of a radially symmetric analog of the rectangle function, is

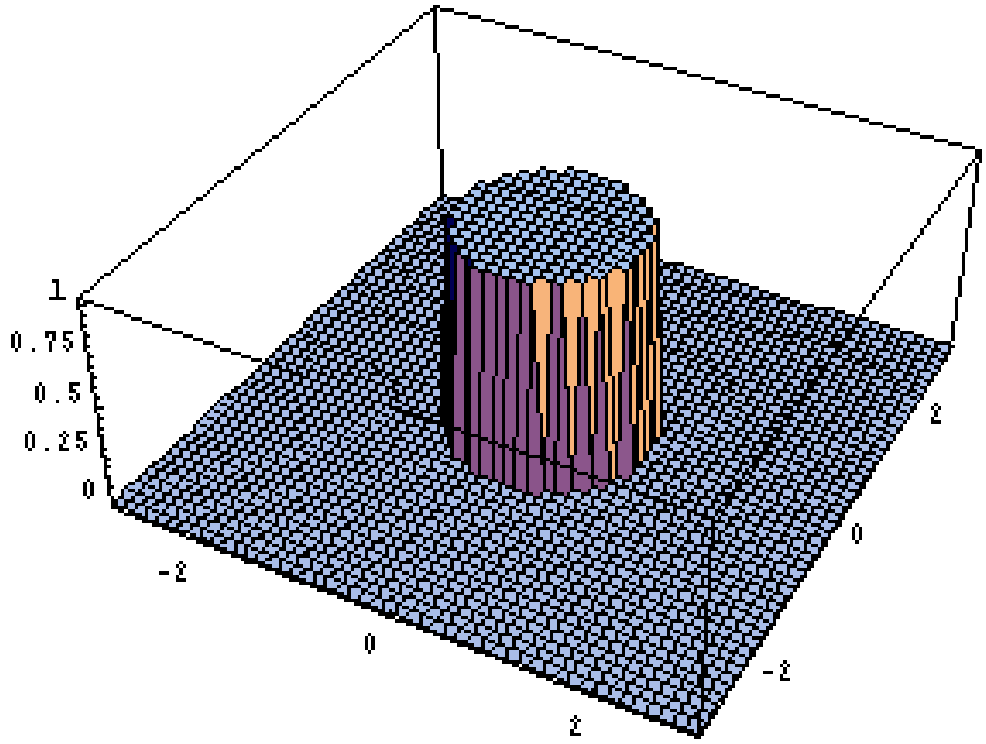
$$\mathrm{circ}(r) = \begin{cases} 1 & r < 1 \\ 0 & r \geq 1 \end{cases}$$

(And one can argue about the value at the rim $r = 1$.) Here's the graph.

[8] We've applied this general fact implicitly or explicitly on earlier occasions when working with periodic functions, namely if g is periodic with period 2π then

$$\int_0^{2\pi} g(\theta-\phi)\,d\theta = \int_0^{2\pi} g(\theta)\,d\theta$$

Convince yourself of this; for instance let $G(\phi) = \int_0^{2\pi} g(\theta-\phi)\,d\theta$ and show that $G''(\phi) \equiv 0$. Hence $G(\phi)$ is constant, so $G(\phi) = G(0)$.



For its Fourier transform the limits of integration on r go only from 0 to 1, and so we have simply

$$\mathcal{F}\text{circ}(\rho) = 2\pi \int_0^1 J_0(2\pi r\rho)\, r\, dr\,.$$

We make a change of variable, $u = 2\pi r\rho$. Then $du = 2\pi\rho dr$ and the limits of integration go from $u = 0$ to $u = 2\pi\rho$. The integral becomes

$$\mathcal{F}\text{circ}(\rho) = \frac{1}{2\pi\rho^2} \int_0^{2\pi\rho} uJ_0(u)\, du\,.$$

We write the integral this way because, you will now be ecstatic to learn, there is an identity that brings in the first-order Bessel function of the first kind. That identity goes

$$\int_0^x uJ_0(u)\, du = xJ_1(x)\,.$$

In terms of J_1 we can now write

$$\mathcal{F}\text{circ}(\rho) = \frac{J_1(2\pi\rho)}{\rho}$$

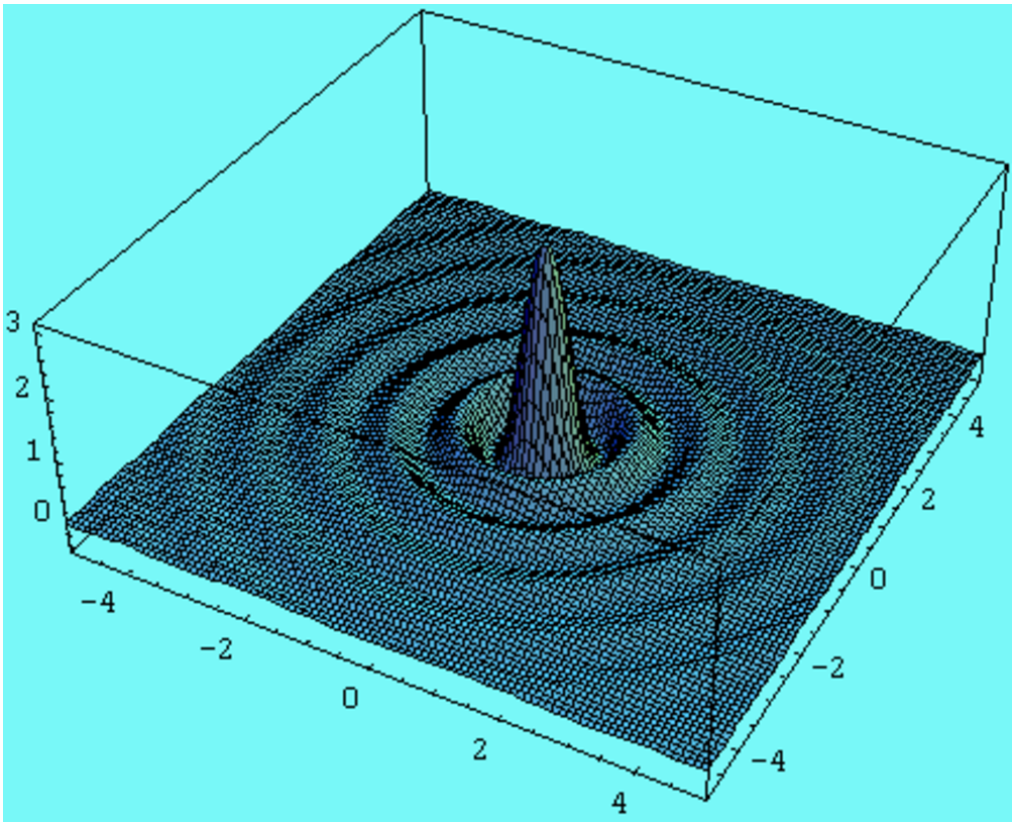
It is customary to introduce the jinc function, defined by

$$\text{jinc}(\rho) = \frac{J_1(\pi\rho)}{2\rho}\,.$$

In terms of this,

$$\mathcal{F}\text{circ}(\rho) = 4\,\text{jinc}(2\rho)\,.$$

The graph of $\mathcal{F}$circ is:



I could plot this because Bessel functions are so common (really) that they are built into many mathematical software packages, such as Matlab or Mathematica. If you think the jinc function looks like some kind of radially symmetric version of the sinc function you'd be right. But it's not obvious just how one goes from sinc to jinc, and we'll have to pass on this.[9]

8.2.7 A Derivation of the General Stretch Theorem

The general stretch theorem says that if A is an invertible $n \times n$ matrix then

$$\mathcal{F}(f(A\mathbf{x})) = \frac{1}{|\det A|}\mathcal{F}f(A^{-\mathsf{T}}\boldsymbol{\xi})\,.$$

To derive this let's start with the left hand side:

$$\mathcal{F}(f(A\mathbf{x})) = \int_{\mathbf{R}^n} e^{-2\pi i\cdot\mathbf{x}} f(A\mathbf{x})\,d\mathbf{x}\,.$$

Our object is to make a change of variable, $\mathbf{u} = A\mathbf{x}$. For this, we need to use the change of variables formula for multiple integrals. In the form we need it, we can state:

> If A is an invertible $n \times n$ matrix and $\mathbf{u} = A\mathbf{x}$ then
>
> $$\int_{\mathbf{R}^n} g(A\mathbf{x})\,|\det A|\,d\mathbf{x} = \int_{\mathbf{R}^n} g(\mathbf{u})\,d\mathbf{u}\,.$$
>
> for an integrable function g.

[9] There's a symmetrization process at work involving repeated convolutions. I have notes on this...

Want to feel good (or at least OK) about this in a familiar setting? Take the case $n = 1$. Then

$$\int_{-\infty}^{\infty} g(ax)\,|a|dx = \int_{-\infty}^{\infty} g(u)\,du\,,$$

making the substitution $u = ax$. The transformation $u = ax$ of $\mathbf{R}$ scales lengths, and the scaling factor is a. ($du = a\,dx$). That's if a is positive; the absolute value of a is in there in case a is negative — thus "sense reversing". In n-dimensions the transformation $\mathbf{u} = A\mathbf{x}$ scales n-dimensional *volumes*, and the scaling factor is $\det A$. ($d\mathbf{u} = \det A\,d\mathbf{x}$.) The absolute value $|\det A|$ is in there because a matrix A with $\det A > 0$ is sense preserving on $\mathbf{R}^n$, and it is sense reversing if $\det A < 0$. Thus, in general,

$$d\mathbf{u} = |\det A|\,d\mathbf{x}$$

so the substitution $\mathbf{u} = A\mathbf{x}$ leads right to the formula

$$\int_{\mathbf{R}^n} g(A\mathbf{x})\,|\det A|\,d\mathbf{x} = \int_{\mathbf{R}^n} g(\mathbf{u})\,d\mathbf{u}\,.$$

To apply this to the Fourier transform of $f(A\mathbf{x})$ we have

$$\begin{aligned}
\int_{\mathbf{R}^n} e^{-2\pi i\xi\cdot\mathbf{x}} f(A\mathbf{x})\,d\mathbf{x} &= \int_{\mathbf{R}^n} e^{-2\pi i\xi\cdot A^{-1}(A\mathbf{x})}\, f(A\mathbf{x})\frac{1}{|\det A|}|\det A|\,d\mathbf{x} \\
&= \frac{1}{|\det A|}\int_{\mathbf{R}^n} e^{-2\pi i\xi\cdot A^{-1}(A\mathbf{x})}\, f(A\mathbf{x})\,|\det A|\,d\mathbf{x} \quad \text{(now substitute } \mathbf{u} = A\mathbf{x}) \\
&= \frac{1}{|\det A|}\int_{\mathbf{R}^n} e^{-2\pi i\xi\cdot A^{-1}\mathbf{u}}\, f(\mathbf{u})\,d\mathbf{u}
\end{aligned}$$

If you think this looks complicated *imagine* writing it out in coordinates!

Next we use an identity for what happens to the dot product when there's a matrix operating on one of the vectors, namely, for a matrix B and any vectors $\boldsymbol{\xi}$ and $\mathbf{u}$,

$$\boldsymbol{\xi}\cdot B\mathbf{u} = B^{\mathsf{T}}\boldsymbol{\xi}\cdot\mathbf{u}\,.$$

We take $B = A^{-1}$ and then

$$\boldsymbol{\xi}\cdot A^{-1}\mathbf{u} = A^{-\mathsf{T}}\boldsymbol{\xi}\cdot\mathbf{u}\,.$$

With this:

$$\frac{1}{|\det A|}\int_{\mathbf{R}^n} e^{-2\pi i\xi\cdot A^{-1}\mathbf{u}}\, f(\mathbf{u})\,d\mathbf{u} = \frac{1}{|\det A|}\int_{\mathbf{R}^n} e^{-2\pi iA^{-\mathsf{T}}\xi\cdot\mathbf{u}} f(\mathbf{u})\,d\mathbf{u}.$$

But this last integral is exactly $\mathcal{F}(f)(A^{-\mathsf{T}}\boldsymbol{\xi})$. We have shown that

$$\mathcal{F}(f(A\mathbf{x})) = \frac{1}{|\det A|}\mathcal{F}(f)(A^{-\mathsf{T}}\boldsymbol{\xi})\,,$$

as desired.

Scaling the delta function The change of variables formula also allows us to derive

$$\delta(A\mathbf{x}) = \frac{1}{|\det A|}\delta(\mathbf{x})\,.$$

Writing the pairing of $\delta(A\mathbf{x})$ with a test function φ via integration —not strictly legit, but it helps to organize the calculation —leads to

$$\begin{aligned}\int_{\mathbf{R}^n} \delta(A\mathbf{x})\varphi(\mathbf{x})\,d\mathbf{x} &= \int_{\mathbf{R}^n} \delta(A\mathbf{x})\varphi(A^{-1}A\mathbf{x})\frac{1}{|\det A|}|\det A|\,d\mathbf{x} \\ &= \frac{1}{|\det A|}\int_{\mathbf{R}^n} \delta(\mathbf{u})\varphi(A^{-1}\mathbf{u})\,d\mathbf{u} \quad \text{(making the change of variables } \mathbf{u} = A\mathbf{x}) \\ &= \frac{1}{|\det A|}\,\varphi(A^{-1}\mathbf{0}) \quad \text{(by how the delta function acts)} \\ &= \frac{1}{|\det A|}\,\varphi(0) \quad (A^{-1}\mathbf{0} = \mathbf{0} \text{ because } A^{-1} \text{ is linear})\end{aligned}$$

Thus $\delta(A\mathbf{x})$ has the same effect as $\dfrac{1}{|\det A|}\delta$ when paired with a test function, so they must be equal.

8.3 Higher Dimensional Fourier Series

It's important to know that most of the ideas and constructions for Fourier series carry over directly to periodic functions in two, three, or higher dimensions. Here we want to give just the basic setup so you can see that the situation, and even the notation, is very similar to what we've already encountered. After that we'll look at a fascinating problem where higher dimensional Fourier series are central to the solution, but in a far from obvious way.

Periodic Functions The definition of periodicity for real-valued functions of several variables is much the same as for functions of one variable except that we allow for different periods in different slots. To take the two-dimensional case, we say that a function $f(x_1, x_2)$ is (p_1, p_2)-periodic if

$$f(x_1 + p_1, x_2) = f(x_1, x_2) \quad \text{and} \quad f(x_1, x_2 + p_2) = f(x_1, x_2)$$

for all x_1 and x_2. It follows that

$$f(x_1 + p_1, x_2 + p_2) = f(x_1, x_2)$$

and more generally that

$$f(x_1 + n_1p_1, x_2 + n_2p_2) = f(x_1, x_2)$$

for all integers n_1, n_2.

There's a small but important point associated with the definition of periodicity having to do with properties of $f(x_1, x_2)$ "one variable at a time" or "both variables together". The condition

$$f(x_1 + n_1p_1, x_2 + n_2p_2) = f(x_1, x_2)$$

for all integers n_1, n_2 *can* be taken as the definition of periodicity, but the condition $f(x_1 + p_1, x_2 + p_2) = f(x_1, x_2)$ alone is *not* the appropriate definition. The former implies that $f(x_1 + p_1, x_2) = f(x_1, x_2)$ and $f(x_1, x_2 + p_2) = f(x_1, x_2)$ by taking (n_1, n_2) to be $(1, 0)$ and $(0, 1)$, respectively, and this "independent periodicity" is what we want. The latter condition does not imply independent periodicity.

For our work now it's enough to assume that the period in each variable is 1, so the condition is

$$f(x_1 + 1, x_2) = f(x_1, x_2) \quad \text{and} \quad f(x_1, x_2 + 1) = f(x_1, x_2)\,,$$

or

$$f(x_1 + n_1, x_2 + n_2) = f(x_1, x_2) \quad \text{for all integers } n_1,\ n_2\,.$$

If we use vector notation and write $\mathbf{x}$ for (x_1, x_2) and (why not) $\mathbf{n}$ for the pair (n_1, n_2) of integers, then we can write the condition as

$$f(\mathbf{x} + \mathbf{n}) = f(\mathbf{x})\,,$$

and, except for the typeface, it looks like the one-dimensional case.

Where is $f(x_1, x_2)$ defined? For a periodic function (of period 1) it is enough to know the function for $x_1 \in [0, 1]$ and $x_2 \in [0, 1]$. We write this as

$$(x_1, x_2) \in [0, 1]^2\,.$$

We can thus consider $f(x_1, x_2)$ to be defined on $[0, 1]^2$ and then extended to be defined on all of $\mathbf{R}^2$ via the periodicity condition.

We can consider periodicity of functions in any dimension. To avoid conflicts with other notation, in this discussion I'll write the dimension as d rather than n. Let $\mathbf{x} = (x_1, x_2, \ldots, x_d)$ be a vector in $\mathbf{R}^d$ and let $\mathbf{n} = (n_1, n_2, \ldots, n_d)$ be an d-tuple of integers. Then $f(\mathbf{x}) = f(x_1, x_2, \ldots, x_d)$ is periodic (of period 1 in each variable) if

$$f(\mathbf{x} + \mathbf{n}) = f(\mathbf{x}) \quad \text{for all } \mathbf{n}\,.$$

In this case we consider the natural domain of $f(\mathbf{x})$ to be $[0, 1]^d$, meaning the set of points $(x_1, x_2, \ldots, x_d)$ where $0 \le x_j \le 1$ for each $j = 1, 2, \ldots, d$.

Complex exponentials, again What are the building blocks for periodic functions in higher dimensions? We simply multiply simple complex exponentials of one variable. Taking again the two-dimensional case as a model, the function

$$e^{2\pi i x_1} e^{2\pi i x_2}$$

is periodic with period 1 in each variable. Note that once we get beyond one dimension it's not so helpful to think of periodicity "in time" and to force yourself to write the variable as t.

In d dimensions the corresponding exponential is

$$e^{2\pi i x_1}\, e^{2\pi i x_2} \cdots e^{2\pi i x_d}$$

You may be tempted to use the usual rules and write this as

$$e^{2\pi i x_1}\, e^{2\pi i x_2} \cdots e^{2\pi i x_d} = e^{2\pi i (x_1 + x_2 + \cdots + x_d)}\,.$$

Don't do that yet.

Higher harmonics, Fourier series, et al. Can a periodic function $f(x_1, x_2, \ldots, x_d)$ be expressed as a Fourier series using multidimensional complex exponentials? The answer is yes and the formulas and theorems are virtually identical to the one-dimensional case. First of all, the natural setting is $L^2([0, 1]^d)$. This is the space of square integrable functions:

$$\int_{[0,1]^d} |f(\mathbf{x})|^2\, d\mathbf{x} < \infty$$

This is meant as a multiple integral, e.g., in the case $d = 2$ the condition is

$$\int_0^1 \int_0^1 |f(x_1, x_2)|^2 \, dx_1 \, dx_2 < \infty \, .$$

The inner product of two (complex-valued) functions is

$$(f, g) = \int_0^1 \int_0^1 f(x_1, x_2)\overline{g(x_1, x_2)} \, dx_1 \, dx_2 \, .$$

I'm not going to relive the greatest hits of Fourier series in the higher dimensional setting. The only thing I want us to know now is *what the expansions look like*. It's nice — watch. Let's do the two-dimensional case as an illustration. The general higher harmonic is of the form

$$e^{2\pi i n_1 x_1} e^{2\pi i n_2 x_2} \, ,$$

where n_1 and n_2 are integers. We would then imagine writing the Fourier series expansion as

$$\sum_{n_1, n_2} c_{n_1 n_2} e^{2\pi i n_1 x_1} e^{2\pi i n_2 x_2} \, ,$$

where the sum is over all integers n_1, n_2. More on the coefficients in a minute, but first let's find a more attractive way of writing such sums.

Instead of working with the product of separate exponentials, it's *now* time to combine them and see what happens:

$$\begin{aligned} e^{2\pi i n_1 x_1} e^{2\pi i n_2 x_2} &= e^{2\pi i (n_1 x_1 + n_2 x_2)} \\ &= e^{2\pi i \, \mathbf{n} \cdot \mathbf{x}} \quad \text{(dot product in the exponent!)} \end{aligned}$$

where we use vector notation and write $\mathbf{n} = (n_1, n_2)$. The Fourier series expansion then looks like

$$\sum_{\mathbf{n}} c_{\mathbf{n}} e^{2\pi i \mathbf{n} \cdot \mathbf{x}} \, .$$

The dot product in two dimensions has replaced ordinary multiplication in the exponent in one dimension, but the formula *looks the same*. The sum has to be understood to be over all points (n_1, n_2) with integer coefficients. We mention that this set of points in $\mathbf{R}^2$ is called the two-dimensional *integer lattice*, written $\mathbf{Z}^2$. Using this notation we would write the sum as

$$\sum_{\mathbf{n} \in \mathbf{Z}^2} c_{\mathbf{n}} e^{2\pi i \, \mathbf{n} \cdot \mathbf{x}} \, .$$

What are the coefficients? The argument we gave in one dimension extends easily to two dimensions (and more) and one finds that the coefficients are given by

$$\begin{aligned} \int_0^1 \int_0^1 e^{-2\pi i n_1 x_1} e^{-2\pi i n_2 x_2} f(x_1, x_2) \, dx_1 \, dx_2 &= \int_0^1 \int_0^1 e^{-2\pi i (n_1 x_1 + n_2 x_2)} f(x_1, x_2) \, dx_1 \, dx_2 \\ &= \int_{[0,1]^2} e^{-2\pi i \, \mathbf{n} \cdot \mathbf{x}} f(\mathbf{x}) \, d\mathbf{x} \end{aligned}$$

Thus the Fourier coefficients $\hat{f}(\mathbf{n})$ are defined by the integral

$$\hat{f}(\mathbf{n}) = \int_{[0,1]^2} e^{-2\pi i \, \mathbf{n} \cdot \mathbf{x}} f(\mathbf{x}) \, d\mathbf{x}$$

It should now come as no shock that the Fourier series for a periodic function $f(\mathbf{x})$ in $\mathbf{R}^d$ is

$$\sum_{\mathbf{n}} \hat{f}(\mathbf{n}) e^{2\pi i\, \mathbf{n}\cdot\mathbf{x}}\,,$$

where the sum is over all points $\mathbf{n} = (n_1, n_2, \ldots, n_d)$ with integer entries. (This set of points is the integer lattice in $\mathbf{R}^d$, written $\mathbf{Z}^d$.) The Fourier coefficients are defined to be

$$\hat{f}(\mathbf{n}) = \int_{[0,1]^d} e^{-2\pi i\, \mathbf{n}\cdot\mathbf{x}} f(\mathbf{x})\, d\mathbf{x}\,.$$

Coming up next is an *extremely* cool example of higher dimensional Fourier series in action. Later we'll come back to higher dimensional Fourier series and their application to crystallography.

8.3.1 The eternal recurrence of the same?

For this example we need to make some use of notions from probability, but nothing beyond what we used in discussing the Central Limit Theorem in Chapter 3. For this excursion, and your safe return, you will need:

- To remember what "probability" means.
- To know that for independent events the probabilities multiply, i.e., $\text{Prob}(A, B) = \text{Prob}(A)\,\text{Prob}(B)$, meaning that the probability of A *and* B occuring (together) is the product of the separate probabilities of A and B occuring.
- To use *expected value*, which we earlier called the *mean*.

Though the questions we'll ask may be perfectly natural, you may find the answers surprising.

Ever hear of a "random walk"? It's closely related to "Brownian motion" and can also be described as a "Markov process". We won't take either of these latter points of view, but if — or rather, *when* — you encounter these ideas in other courses, *you have been warned*.

Here's the setup for a random walk along a line:

> You're at home at the *origin* at time $n = 0$ and you take a step, left or right chosen with equal probability; flip a coin; — heads you move right, tails you move left. Thus at time $n = 1$ you're at one of the points $+1$ or -1. Again you take a step, left or right, chosen with equal probability. You're either back home at the origin or at ± 2. And so on.
>
> - As you take more and more steps, will you get home (to the origin)?
> - With what probability?

We can formulate the same question in two, three, or any number of dimensions. We can also tinker with the probabilities and assume that steps in some directions are more probable than in others, but we'll stick with the equally probable case.

---

[9] With apologies to F. Nietzsche

Random walks, Markov processes, et al. are used everyday by people who study queuing problems, for example. More recently they have been applied in mathematical finance. A really interesting treatment is the book *Random Walks and Electrical Networks* by P. Doyle and J. L. Snell.

To answer the questions it's necessary to give some precise definitions, and that will be helped by fixing some notation. Think of the space case $d = 3$ as an example. We'll write the location of a point with reference to Cartesian coordinates. Start at the origin and start stepping. Each step is by a unit amount in one of six possible directions, and the directions are chosen with equal probability, e.g., throw a single die and have each number correspond to one of six directions. Wherever you go, you get there by adding to where you are one of the six unit steps

$$(\pm 1, 0, 0), \quad (0, \pm 1, 0), \quad (0, 0, \pm 1)\,.$$

Denote any of these "elementary" steps, or more precisely the random process of choosing any of these steps, by **step**; to take a step is to choose one of the triples, above, and each choice is made with probability 1/6. Since we're interested in walks more than we are individual steps, let's add an index to **step** and write $\mathbf{step}_1$ for the choice in taking the first step, $\mathbf{step}_2$ for the choice in taking the second step, and so on. We're also assuming that each step is a new adventure — the choice at the n-th step is made independently of the previous $n-1$ steps. In d dimensions there are $2d$ directions each chosen with probability $1/2d$, and $\mathbf{step}_n$ is defined in the same manner.

The process $\mathbf{step}_n$ is a discrete random variable. To be precise:

- The domain of $\mathbf{step}_n$ is the set of all possible walks and the value of $\mathbf{step}_n$ on a particular walk is the n'th step in that walk.

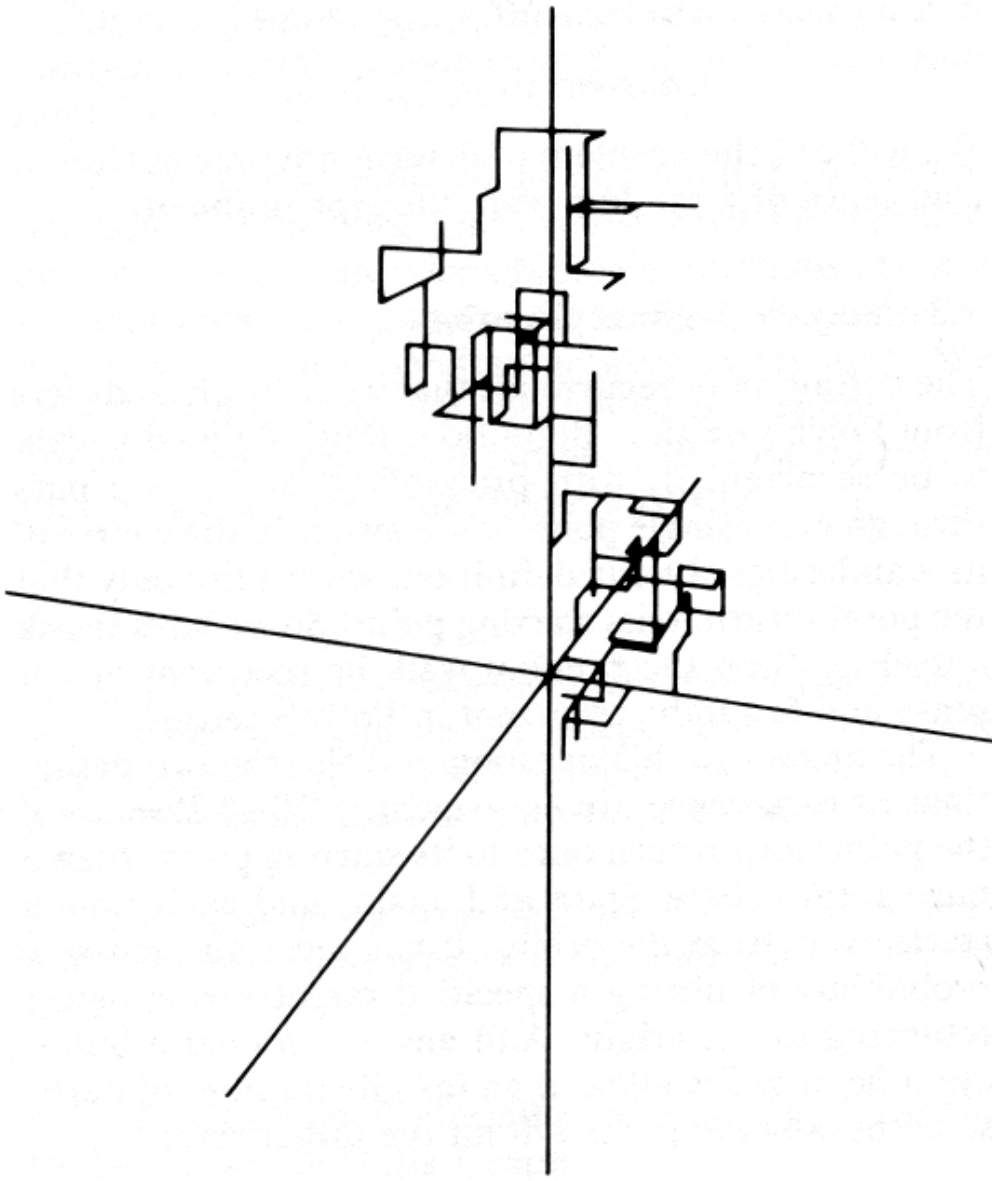
(Some people would call $\mathbf{step}_n$ a *random vector* since its values are d-tuples.) We're assuming that distribution of values of $\mathbf{step}_n$ is uniform (each particular step is taken with probability $1/2d$, in general) and that the steps are independent. Thus, in the parlance we've used in connection with the Central Limit Theorem, $\mathbf{step}_1$, $\mathbf{step}_2$, ..., $\mathbf{step}_n$ are independent, identically distributed random variables.

- The possible random walks of n steps are described exactly as

$$\mathbf{walk}_n = \mathbf{step}_1 + \mathbf{step}_2 + \cdots + \mathbf{step}_n, \quad \text{or, for short, just} \quad \mathbf{w}_n = \mathbf{s}_1 + \mathbf{s}_2 + \cdots + \mathbf{s}_n\,.$$

I'm using the vector notation for $\mathbf{w}$ and $\mathbf{s}$ to indicate that the action is in $\mathbf{R}^d$.

Here's a picture in $\mathbf{R}^3$.



After a walk of n steps, $n \geq 1$, you are at a lattice point in $\mathbf{R}^d$, i.e., a point with integer coordinates. We now ask two questions:

1. Given a particular lattice point $\mathbf{l}$, what is the probability after n steps that we are at $\mathbf{l}$?
2. How does $\mathbf{walk}_n$ behave as $n \to \infty$?

These famous questions were formulated and answered by G. Pólya in 1921. His brilliant analysis resulted in the following result.

Theorem In dimensions 1 and 2, with probability 1, the walker visits the origin infinitely often; in symbols

$$\text{Prob}(\mathbf{walk}_n = 0 \text{ infinitely often}) = 1\,.$$

In dimensions ≥ 3, with probability 1, the walker escapes to infinity:

$$\text{Prob}\left(\lim_{n\to\infty} |\mathbf{walk}_n| = \infty\right) = 1\,.$$

One says that a random walk along a line or in the plane is *recurrent* and that a random walk in higher dimensions is *transient*.

Here's the idea — very cunning and, frankly, rather unmotivated, but who can account for genius? For each $\mathbf{x} \in \mathbf{R}^d$ consider

$$\Phi_n = e^{2\pi i\, \mathbf{w}_n \cdot \mathbf{x}}\,,$$

where, as above, $\mathbf{w}_n$ is a walk of n steps. For a given n the possible values of $\mathbf{w}_n$, as a sum of steps corresponding to different walks, lie among the lattice points, and if $\mathbf{w}_n$ lands on a lattice point $\mathbf{l}$ then the value of Φ_n for that walk is $e^{2\pi i\, \mathbf{l}\cdot\mathbf{x}}$. What is the expected value of Φ_n over all walks of n steps? It is the mean, i.e., the weighted average of the values of Φ_n over the possible (random) walks of n steps, each value weighted by the probability of its occurrence. That is,

$$\text{Expected value of } \Phi_n = \sum_{\mathbf{l}} \text{Prob}(\mathbf{w}_n = \mathbf{l}) e^{2\pi i\, \mathbf{l}\cdot\mathbf{x}}\,.$$

This is actually a finite sum because in n steps we can have reached only a finite number of lattice points, or, put another way, $\text{Prob}(\mathbf{w}_n = \mathbf{l})$ is zero for all but finitely many lattice points $\mathbf{l}$.

From this expression you can see (finite) Fourier series coming into the picture, but put that off for the moment.[10] We can compute this expected value, based on our assumption that steps are equally probable and independent of each other. First of all, we can write

$$\Phi_n = e^{2\pi i\, \mathbf{w}_n\cdot\mathbf{x}} = e^{2\pi i(\mathbf{s}_1+\mathbf{s}_2+\cdots+\mathbf{s}_n)\cdot\mathbf{x}} = e^{2\pi i\, \mathbf{s}_1\cdot\mathbf{x}} e^{2\pi i\, \mathbf{s}_2\cdot\mathbf{x}} \cdots e^{2\pi i\, \mathbf{s}_n\cdot\mathbf{x}}\,.$$

So we want to find the expected value of the product of exponentials. At this point we could appeal to a standard result in probability, stating that the expected value of the product of *independent* random variables is the product of their expected values. You might be able to think about this directly, however: The expected value of $e^{2\pi i\, \mathbf{s}_1\cdot\mathbf{x}} e^{2\pi i \mathbf{s}_2\cdot\mathbf{x}} \cdots e^{2\pi i \mathbf{s}_n\cdot\mathbf{x}}$ is, as above, the weighted average of the values that the function assumes, weighted by the probabilities of those values occuring. In this case we'd be summing over all steps $s_1, s_2, \ldots, s_n$ of the values $e^{2\pi i s_1\cdot\mathbf{x}} e^{2\pi i s_2\cdot\mathbf{x}} \cdots e^{2\pi i s_n\cdot\mathbf{x}}$ weighted by the appropriate probabilities. But now the fact that the steps are independent means

$$\begin{aligned}\text{Prob}(\mathbf{s}_1 = s_1, \mathbf{s}_2 = s_2, \ldots, \mathbf{s}_n = s_n) &= \text{Prob}(\mathbf{s}_1 = s_1)\,\text{Prob}(\mathbf{s}_2 = s_2)\cdots\text{Prob}(\mathbf{s}_n = s_n)\\ &\qquad \text{(probabilities } \textit{multiply} \text{ for independent events)}\\ &= \frac{1}{(2d)^n},\end{aligned}$$

and then

$$\begin{aligned}\text{Expected value of } \Phi_n &= \text{Expected value of } e^{2\pi i s_1\cdot\mathbf{x}} e^{2\pi i s_2\cdot\mathbf{x}} \cdots e^{2\pi i s_n\cdot\mathbf{x}}\\ &= \sum_{s_1}\sum_{s_2}\cdots\sum_{s_n} \text{Prob}(\mathbf{s}_1 = s_1, \mathbf{s}_2 = s_2, \ldots, \mathbf{s}_n = s_n) e^{2\pi i\, s_1\cdot\mathbf{x}} e^{2\pi i s_2\cdot\mathbf{x}} \cdots e^{2\pi i s_n\cdot\mathbf{x}}\\ &= \sum_{s_1}\sum_{s_2}\cdots\sum_{s_n} \frac{1}{(2d)^n} e^{2\pi i\, s_1\cdot\mathbf{x}} e^{2\pi i\, s_2\cdot\mathbf{x}} \cdots e^{2\pi i\, s_n\cdot\mathbf{x}}\,.\end{aligned}$$

[10] Also, though it's not in the standard form, i.e., a power series, I think of Pólya's idea here as writing down a *generating function* for the sequence of probabilities $\text{Prob}(\mathbf{w}_n = \mathbf{l})$. For an appreciation of this kind of approach to a great variety of problems — pure and applied — see the book *Generatingfunctionology* by H. Wilf. The first sentence of Chapter One reads: "A generating function is a clothesline on which we hang up a sequence of numbers for display." Seems pretty apt for the problem at hand.

The sums go over all possible choices of s_1, s_2,...,s_n. Now, these sums are "uncoupled", and so the nested sum is the product of

$$\sum_{s_1} \frac{1}{2d} e^{2\pi i\, s_1\cdot \mathbf{x}} \sum_{s_2} \frac{1}{2d} e^{2\pi i\, s_2\cdot \mathbf{x}} \cdots \sum_{s_n} \frac{1}{2d} e^{2\pi i\, s_n\cdot \mathbf{x}}\,.$$

But the sums are, respectively, the expected values of $e^{2\pi i s_j\cdot \mathbf{x}}$, $j = 1, \ldots, n$, and *these expected values are all the same.* (The steps are independent and identically distributed). So all the sums are equal, say, to the first sum, and we may write

$$\text{Expected value of } \Phi_n = \Big(\frac{1}{2d}\sum_{s_1} e^{2\pi i\, s_1\cdot \mathbf{x}}\Big)^n$$

A further simplification is possible. The first step s_1, as a d-tuple has exactly one slot with a ± 1 and the rest 0's. Summing over these $2d$ possibilities allows us to combine "positive and negative terms". Check the case $d = 2$, for which the choices of s_1 are

$$(1, 0)\,, \quad (-1, 0)\,, \quad (0, 1)\,, \quad (0, -1)\,.$$

This leads to a sum with four terms:

$$\begin{aligned}\sum_{s_1} \frac{1}{2\cdot 2} e^{2\pi i\, s_1\cdot \mathbf{x}} &= \sum_{s_1} \frac{1}{2\cdot 2} e^{2\pi i\, s_1\cdot (x_1,x_2)} \\ &= \tfrac{1}{2}(\tfrac{1}{2}e^{2\pi i x_1} + \tfrac{1}{2}e^{-2\pi i x_1} + \tfrac{1}{2}e^{2\pi i x_2} + \tfrac{1}{2}e^{-2\pi i x_2}) \\ &= \tfrac{1}{2}(\cos 2\pi x_1 + \cos 2\pi x_2)\end{aligned}$$

The same thing happens in dimension d, and our final formula is

$$\sum_{\mathbf{l}} \text{Prob}(\mathbf{w}_n = \mathbf{l}) e^{2\pi i\, \mathbf{l}\cdot \mathbf{x}} = \Big(\frac{1}{d}(\cos 2\pi x_1 + \cos 2\pi x_2 + \cdots + \cos 2\pi x_d)\Big)^n\,.$$

Let us write

$$\phi_d(\mathbf{x}) = \frac{1}{d}(\cos 2\pi x_1 + \cos 2\pi x_2 + \cdots + \cos 2\pi x_d)\,.$$

Observe that $|\phi_d(\mathbf{x})| \le 1$, since $\phi_d(\mathbf{x})$ is the sum of d cosines by d and $|\cos 2\pi x_j| \le 1$ for $j = 1, 2, \ldots, d$.

This has been quite impressive already. But there's more! Let's get back to Fourier series and consider the sum of probabilities times exponentials, above, *as a function of* $\mathbf{x}$; i.e., let

$$f(\mathbf{x}) = \sum_{\mathbf{l}} \text{Prob}(\mathbf{w}_n = \mathbf{l})\, e^{2\pi i\, \mathbf{l}\cdot \mathbf{x}}\,.$$

This is a (finite) Fourier series for $f(\mathbf{x})$ and as such the coefficients *must be* the Fourier coefficients,

$$\text{Prob}(\mathbf{w}_n = \mathbf{l}) = \hat{f}(\mathbf{l})\,.$$

But according to our calculation, $f(\mathbf{x}) = \phi_d(\mathbf{x})^n$, and so this must also be the Fourier coefficient of $\phi_d(\mathbf{x})^n$, that is,

$$\text{Prob}(\mathbf{w}_n = \mathbf{l}) = \hat{f}(\mathbf{l}) = \widehat{(\phi_d)^n}(\mathbf{l}) = \int_{[0,1]^d} e^{-2\pi i\, \mathbf{l}\cdot \mathbf{x}} \phi_d(\mathbf{x})^n\, d\mathbf{x}\,.$$

In particular, the probability that the walker visits the origin, $\mathbf{l} = 0$, in n steps is

$$\text{Prob}(\mathbf{w}_n = 0) = \int_{[0,1]^d} \phi_d(\mathbf{x})^n\, d\mathbf{x}\,.$$

Then the expected number of times the walker visits the origin for a random walk of infinite length is

$$\sum_{n=0}^{\infty} \text{Prob}(\mathbf{w}_n = 0)\,,$$

and we can figure this out.[11] Here's how we do this. We'd like to say that

$$\begin{aligned}\sum_{n=0}^{\infty} \text{Prob}(\mathbf{w}_n = 0) &= \sum_{n=0}^{\infty} \int_{[0,1]^d} \phi_d(\mathbf{x})^n \, d\mathbf{x} \\ &= \int_{[0,1]^d} \left(\sum_{n=0}^{\infty} \phi_d(x)^n \right) d\mathbf{x} = \int_{[0,1]^d} \frac{1}{1-\phi_d(\mathbf{x})} \, d\mathbf{x}\end{aligned}$$

using the formula for adding a geometric series. The final answer is correct, but the derivation isn't quite legitimate: The formula for the sum of a geometric series is

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

provided that $|r|$ is strictly less than 1. In our application we know only that $|\phi_d(\mathbf{x})| \le 1$. To get around this difficulty, let $\alpha < 1$, and write

$$\begin{aligned}\sum_{n=0}^{\infty} \text{Prob}(\mathbf{w}_n = 0) &= \lim_{\alpha \to 1} \sum_{n=0}^{\infty} \alpha^n \, \text{Prob}(\mathbf{w}_n = 0) = \lim_{\alpha \to 1} \int_{[0,1]^d} \left(\sum_{n=0}^{\infty} \alpha^n \phi_d(x)^n \right) d\mathbf{x} \\ &= \lim_{\alpha \to 1} \int_{[0,1]^d} \frac{1}{1-\alpha\phi_d(\mathbf{x})} \, d\mathbf{x} = \int_{[0,1]^d} \frac{1}{1-\phi_d(\mathbf{x})} \, d\mathbf{x}\end{aligned}$$

(Pulling the limit inside the integral is justified by convergence theorems in the theory of Lebesgue integration, specifically, dominated convergence. Not to worry.)

- The crucial question now concerns the integral

$$\int_{[0,1]^d} \frac{1}{1-\phi_d(\mathbf{x})} \, d\mathbf{x}\,.$$

  Is it finite or infinite?

This depends on the dimension — and this is exactly where the dimension d enters the picture.

Using some calculus (think Taylor series) it is not difficult to show (I won't) that if $|\mathbf{x}|$ is small then

$$1 - \phi_d(\mathbf{x}) \sim c|\mathbf{x}|^2\,,$$

for a constant c. Thus

$$\frac{1}{1-\phi_d(\mathbf{x})} \sim \frac{1}{c|\mathbf{x}|^2}\,,$$

and the convergence of the integral we're interested in depends on that of the "power integral"

$$\int_{x \text{ small}} \frac{1}{|\mathbf{x}|^2} \, d\mathbf{x} \quad \text{in dimension } d\,.$$

It is an important mathematical fact of nature (something you should file away for future use) that

[11] For those more steeped in probability, here's a further argument why this sum is the expected number of visits to the origin. Let V_n be the random variable which is 1 if the walker returns to the origin in n steps and is zero otherwise. The expected value of V_n is then $\text{Prob}(\mathbf{w}_n = 0) \cdot 1$, the value of the function, 1, times the probability of that value occurring. Now set $V = \sum_{n=0}^{\infty} V_n$. The expected value of V is what we want and it is the sum of the expected values of the V_n, i.e. $\sum_{n=0}^{\infty} \text{Prob}(\mathbf{w}_n = 0)$.

- The power integral diverges for $d = 1, 2$.
- The power integral converges for $d \geq 3$

Let me illustrate why this is so for $d = 1, 2, 3$. For $d = 1$ we have an ordinary improper integral,

$$\int_0^a \frac{dx}{x^2}, \quad \text{for some small } a > 0\,,$$

and this diverges by direct integration. For $d = 2$ we have a double integral, and to check its properties we introduce polar coordinates (r, θ) and write

$$\int_{|\mathbf{x}| \text{ small}} \frac{dx_1\, dx_2}{x_1^2 + x_2^2} = \int_0^{2\pi}\int_0^a \frac{r\, dr\, d\theta}{r^2} = \int_0^{2\pi}\left(\int_0^a \frac{dr}{r}\right) d\theta\,.$$

The inner integral diverges. In three dimensions we introduce spherical coordinates (ρ, θ, φ), and something different happens. The integral becomes

$$\int_{|\mathbf{x}| \text{ small}} \frac{dx_1\, dx_2\, dx_3}{x_1^2 + x_2^2 + x_3^2} = \int_0^{\pi}\int_0^{2\pi}\int_0^a \frac{\rho^2 \sin\phi\, d\rho\, d\theta\, d\varphi}{\rho^2}\,.$$

This time the ρ^2 in the denominator cancels with the ρ^2 in the numerator and the ρ-integral is *finite*. The same phenomenon persists in higher dimensions, for the same reason (introducing higher dimensional polar coordinates).

Let's take stock. We have shown that

$$\text{Expected number of visits to the origin} = \sum_{n=0}^{\infty} \text{Prob}(\mathbf{w}_n = 0) = \int_{[0,1]^d} \frac{1}{1 - \phi_d(\mathbf{x})}\, d\mathbf{x}$$

and that this number is infinite in dimensions 1 and 2 and finite in dimension 3. From here we can go on to prove Pólya's theorem as he stated it:

> Prob($\mathbf{walk}_n = 0$ infinitely often) $= 1$ in dimensions 1 and 2.
>
> Prob($\lim_{n\to\infty} |\mathbf{walk}_n| = \infty$) $= 1$ in dimensions ≥ 3.

For the case $d \geq 3$, we know that the expected number of times that the walker visits the origin is finite. This can only be true if the actual number of visits to the origin is finite with probability 1. Now the origin is not special in any way, so the same must be true of any lattice point. But this means that for any $R > 0$ the walker eventually stops visiting the ball $|\mathbf{x}| \leq R$ of radius R with probability 1, and this is exactly saying that Prob($\lim_{n\to\infty} |\mathbf{walk}_n| = \infty$) $= 1$.

To settle the case $d \leq 2$ we formulate a lemma that you might find helpful in this discussion.[12]

> **Lemma** Let p_n be the probability that a walker visits the origin *at least* n times and let q_n be the probability that a walker visits the origin *exactly* n times. Then $p_n = p_1^n$ and $q_n = p_1^n(1-p_1)$

[12] We haven't had many lemmas in this class, but I think I can get away with one or two.

To show this we argue as follows. Note first that $p_0 = 1$ since the walker starts at the origin. Then

$$\begin{aligned}
p_{n+1} &= \text{Prob(visit origin at least } n+1 \text{ times)} \\
&= \text{Prob(visit origin at least } n+1 \text{ times given visit at least } n \text{ times)} \cdot \text{Prob(visit at least } n \text{ times)} \\
&= \text{Prob(visit origin at least 1 time given visit at least 0 times)} \cdot p_n \\
&\quad \text{(using independence and the definition of } p_n\text{)} \\
&= \text{Prob(visit at least 1 time)} \cdot p_n \\
&= p_1 \cdot p_n
\end{aligned}$$

From $p_0 = 1$ and $p_{n+1} = p_1 \cdot p_n$ it follows (by induction) that $p_n = p_1^n$.

For the second part,

$$\begin{aligned}
q_n &= \text{Prob(exactly } n \text{ visits to origin)} \\
&= \text{Prob(visits at least } n \text{ times)} - \text{Prob(visits at least } n+1 \text{ times)} \\
&= p_n - p_{n+1} = p_1^n(1-p_1)
\end{aligned}$$

Now, if p_1 were less than 1 then the expected number of visits to the origin would be

$$\begin{aligned}
\sum_{n=0}^{\infty} n q_n &= \sum_{n=0}^{\infty} n p_1^n (1-p_1) = (1-p_1) \sum_{n=0}^{\infty} n p_1^n \\
&= (1-p_1)\frac{p_1}{(1-p_1)^2} \quad \text{(Check that identity by differentiating identity } \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n\text{)} \\
&= \frac{p_1}{1-p_1} < \infty
\end{aligned}$$

But this contradicts the fact we established earlier, namely

$$\text{Expected visits to the origin} = \int_{[0,1]^2} \frac{1}{1-\phi_2(\mathbf{x})}\, d\mathbf{x} = \infty\,.$$

Thus we must have $p_1 = 1$, that is, the probability of returning to the origin is 1, and hence $\mathbf{walk}_n$ must equal 0 infinitely often with probability 1.

8.4 Ш, Lattices, Crystals, and Sampling

Our derivation of the sampling formula in Chapter **???** was a direct application and combination of the important properties of the Ш function,

$$Ш_p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kp)\,.$$

Without redoing the whole argument here, short as it is, let me remind you what it is about Ш that made things work.

- δ's being what they are, Ш_p is *the* tool to use for periodizing and for sampling:

$$(f * \text{Ш}_p)(t) = \sum_{k=-\infty}^{\infty} f(t - kp)$$

$$f(t)\text{Ш}_p(t) = \sum_{k=-\infty}^{\infty} f(kp)\delta(t - kp)\,.$$

- For the Fourier transform,

$$\mathcal{F}\text{Ш}_p = \frac{1}{p}\,\text{Ш}_{1/p}\,.$$

- It is through this property of the Fourier transform that periodizing in one domain corresponds to sampling in the other domain. Pay particular attention here to the reciprocity in spacing between Ш_p and its Fourier transform.

The sampling formula itself says that if $\mathcal{F}f(s)$ is identically 0 for $|s| \geq p/2$ then

$$f(t) = \sum_{k=-\infty}^{\infty} f\left(\frac{k}{p}\right)\text{sinc}(pt - k)\,.$$

We now want to see how things stand in two dimensions; there isn't much difference in substance between the two-dimensional case and higher dimensions, so we'll stick pretty much to the plane.

8.4.1 The two-dimensional Ш

The formula $\mathcal{F}\text{Ш}_p = (1/p)\text{Ш}_{1/p}$ depends crucially on the fact that Ш_p is a sum of impulses *at evenly spaced points* — this is an aspect of periodicity. We've already defined a two-dimensional δ, so to introduce a Ш that goes with it we need to define what "evenly spaced" means for points in $\mathbf{R}^2$. One way of spacing points evenly in $\mathbf{R}^2$ is to take all pairs (k_1, k_2), k_1, k_2 integers. The corresponding Ш-function is then defined to be

$$\text{Ш}(x_1, x_2) = \sum_{k_1,k_2=-\infty}^{\infty} \delta(x_1 - k_1, x_2 - k_2)\,.$$

Bracewell, and others, sometimes refer to this as the "bed of nails".

The points $\mathbf{k} = (k_1, k_2)$ with integer coordinates are said to form a *lattice* in the plane. We denote this particular lattice, called the *integer lattice*, by $\mathbf{Z}^2$; we'll have more general lattices in a short while. As a model of a physical system, you can think of such an array as a two-dimensional crystal, where there's an atom at every lattice point.

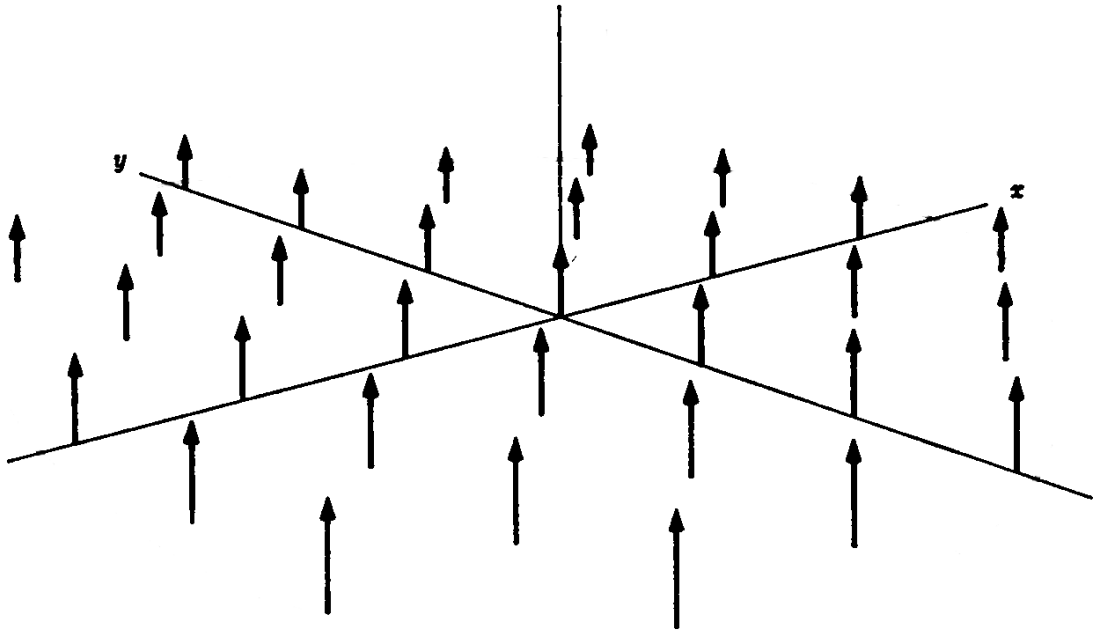
Since we prefer to write things in terms of vectors, another way to describe $\mathbf{Z}^2$ is to use the standard basis of $\mathbf{R}^2$, the vectors $\mathbf{e}_1 = (1, 0)$, $\mathbf{e}_2 = (0, 1)$, and write the points in the lattice as

$$\mathbf{k} = k_1\mathbf{e}_1 + k_2\mathbf{e}_2\,.$$

We can thus think of the elements of a lattice either as points or as vectors, and observe that the sum of two lattice points is another lattice point and that an integer multiple of a lattice point is another lattice point. The Ш-function can be written

$$\text{Ш}_{\mathbf{Z}^2}(\mathbf{x}) = \sum_{k_1,k_2=-\infty}^{\infty} \delta(\mathbf{x} - k_1\mathbf{e}_1 - k_2\mathbf{e}_2) = \sum_{\mathbf{k}\in\mathbf{Z}^2} \delta(\mathbf{x} - \mathbf{k})\,.$$

It is easy to show that $\text{Ш}_{\mathbf{Z}^2}$ is even.



Periodicity on $\mathbf{Z}^2$ and $\mathcal{F}\text{Ш}_{\mathbf{Z}^2}$ As in the one-dimensional case, $\text{Ш}_{\mathbf{Z}^2}$ is the tool to use to work with periodicity. If we form

$$\Phi(\mathbf{x}) = (\varphi * \text{Ш}_{\mathbf{Z}^2})(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbf{Z}^2} \varphi(\mathbf{x}-\mathbf{k})\,,$$

assuming that the sum converges, then Φ is periodic *on the lattice* $\mathbf{Z}^2$, or briefly, is $\mathbf{Z}^2$-periodic. This means that

$$\Phi(\mathbf{x}+\mathbf{n}) = \Phi(\mathbf{x})$$

for all $\mathbf{x}$ and for any lattice point $\mathbf{n} \in \mathbf{Z}^2$, and this is true because

$$\Phi(\mathbf{x}+\mathbf{n}) = \sum_{\mathbf{k}\in\mathbf{Z}^2} \varphi(\mathbf{x}+\mathbf{n}-\mathbf{k}) = \sum_{\mathbf{k}\in\mathbf{Z}^2} \varphi(\mathbf{x}-\mathbf{k}) = \Phi(\mathbf{x})\,;$$

the sum (or difference) of two lattice points, $\mathbf{n}-\mathbf{k}$, is a lattice point, so we're still summing over $\mathbf{Z}^2$ and we get back Φ.

Using periodicity, and the fact that $\mathbf{Z}^2$ is particularly "evenly spaced" as a set of points in $\mathbf{R}^2$ leads to the important and remarkable formula

$$\mathcal{F}\text{Ш}_{\mathbf{Z}^2} = \text{Ш}_{\mathbf{Z}^2}$$

corresponding precisely to the one-dimensional case. I'll put the details of the derivation of this in Section 8.4.4. It's also true that

$$\mathcal{F}^{-1}\text{Ш}_{\mathbf{Z}^2} = \text{Ш}_{\mathbf{Z}^2}$$

because $\text{Ш}_{\mathbf{Z}^2}$ is even.

At this point the most basic version of the two-dimensional sampling formula is already easily within reach. It's much more interesting, however, as well as ultimately much more useful to allow for some greater generality.

8.4.2 Lattices in general

$\mathbf{Z}^2$ isn't the only example of a set of evenly spaced points in the plane, though perhaps it's the example of the most evenly spaced points. It's easy to imagine "oblique" lattices, too. Not all crystals are square, after all, or even rectangular, and we want to be able to use general lattices to model crystals. We'll now consider such oblique arrangements, but be warned that the subject of lattices can go on forever; the effort here is to be brief to the point.

We adopt the vector point of view for defining a general lattice. Take any basis $\mathbf{u}_1$, $\mathbf{u}_2$ of $\mathbf{R}^2$ and consider all the points (or vectors) that are *integer* linear combinations of the two. These form:

$$\text{Lattice points} = \mathbf{p} = p_1\mathbf{u}_1 + p_2\mathbf{u}_2, \quad p_1, p_2 = 0, \pm 1, \pm 2, \dots$$

We'll denote such a lattice by $\mathcal{L}$. The sum and difference of two lattice points is again a lattice point, as is any integer times a lattice point.[13]

The vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ are said to be a *basis* for the lattice. Other vectors can also serve as a basis, and two bases for the same lattice are related by a 2×2 matrix with integer entries having determinant 1. (I won't go through the derivation of this.) The parallelogram determined by the basis vectors (any basis vectors) is called a *fundamental parallelogram* for the lattice, or, in crystallographers" terms, a *unit cell*. A fundamental parallelogram for $\mathbf{Z}^2$ is the square $0 \le x_1 < 1$, $0 \le x_2 < 1$.[14] By convention, one speaks of the *area of a lattice* in terms of the area of a fundamental parallelogram for the lattice, and we'll write

$$\text{Area}(\mathcal{L}) = \text{Area of a fundamental parallelogram}.$$

Two fundamental parallelograms for the same lattice have the same area since the bases are related by a 2×2 integer matrix with determinant 1 and the area scales by the determinant.

If we take the natural basis vectors $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$ for $\mathbf{R}^2$ we get the integer lattice $\mathbf{Z}^2$ as before. We can see that *any* lattice $\mathcal{L}$ can be *obtained* from $\mathbf{Z}^2$ via an invertible linear transformation A, the one that takes $\mathbf{e}_1$ and $\mathbf{e}_2$ to a basis $\mathbf{u}_1 = A\mathbf{e}_1$ and $\mathbf{u}_2 = A\mathbf{e}_2$ that defines $\mathcal{L}$. This is so precisely because A is linear: if

$$\mathbf{p} = p_1\mathbf{u}_1 + p_2\mathbf{u}_2, \quad p_1, p_2 \quad \text{integers},$$

is any point in $\mathcal{L}$ then

$$\mathbf{p} = p_1(A\mathbf{e}_1) + p_2(A\mathbf{e}_2) = A(p_1\mathbf{e}_1 + p_2\mathbf{e}_2),$$

showing that $\mathbf{p}$ is the image of a point in $\mathbf{Z}^2$. We write

$$\mathcal{L} = A(\mathbf{Z}^2)$$

A fundamental parallelogram for $\mathcal{L}$ is determined by $\mathbf{u}_1$ and $\mathbf{u}_2$, and so

$$\text{Area}(\mathcal{L}) = \text{Area of the parallelogram determined by } \mathbf{u}_1 \text{ and } \mathbf{u}_2 = |\det A|.$$

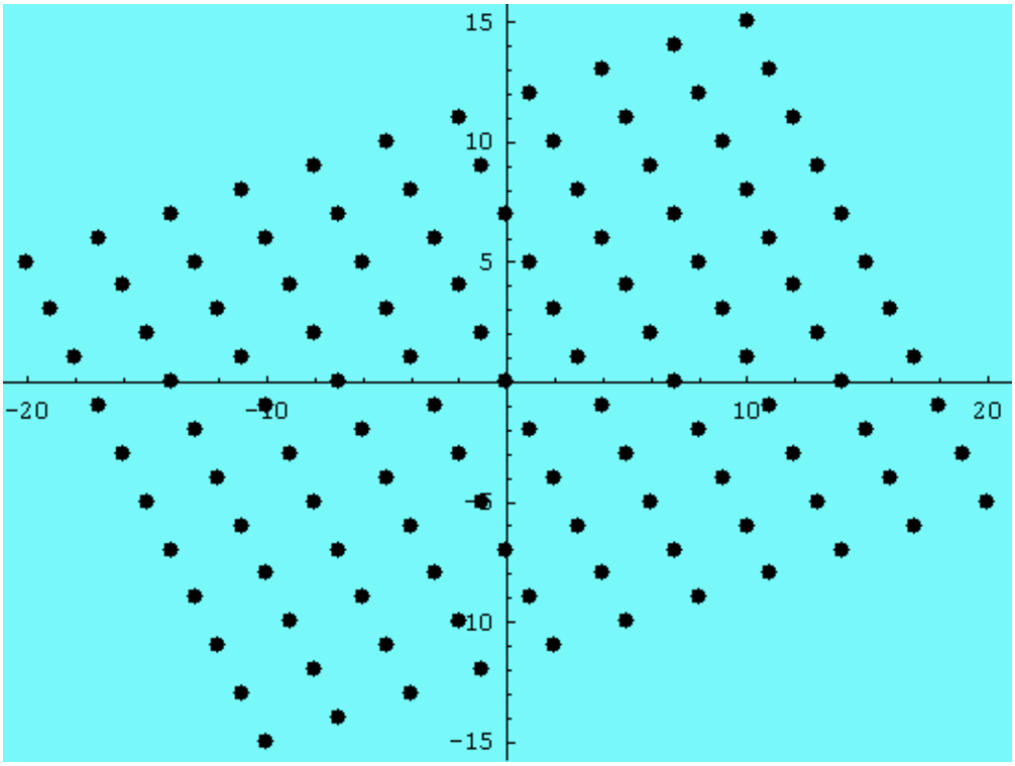
Here, for example, is the lattice obtained from $\mathbf{Z}^2$ by applying

$$A = \begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}$$

A basis is $\mathbf{u}_1 = (3, 1)$, $\mathbf{u}_2 = (-1, 2)$ (Draw the basis on the lattice!) The area of the lattice is 7.

[13] In mathematical terminology a lattice is a *module* over $\mathbf{Z}$; a module is like a vector space except that you can't divide by the scalars (the integers in this case) only add and multiply them. For a module, as opposed to a vector space, the scalars form a ring, not a field.

[14] It's a common convention to define a fundamental parallelogram to be "half open", including two sides ($x_1 = 0$ and $x_2 = 0$ in this case) and excluding two ($x_1 = 1$ and $x_2 = 1$). This won't be an issue for our work.



8.4.3 Ш for a lattice

It doesn't take a great leap in imagination to think about introducing Ш for a general lattice: If $\mathcal{L}$ is a lattice in $\mathbf{R}^2$ then the Ш function associated with $\mathcal{L}$ is

$$Ш_{\mathcal{L}}(\mathbf{x}) = \sum_{\mathbf{p}\in\mathcal{L}} \delta(\mathbf{x}-\mathbf{p})\,.$$

So there's your general "sum of delta functions at evenly spaced points". We could also write the definition as

$$Ш_{\mathcal{L}}(\mathbf{x}) = \sum_{k_1,k_2=-\infty}^{\infty} \delta(\mathbf{x}-k_1\mathbf{u}_1-k_2\mathbf{u}_2)\,.$$

As $\mathcal{L}$ can be obtained from $\mathbf{Z}^2$ via some linear transformation so too can $Ш_{\mathcal{L}}$ be expressed in terms of $Ш_{\mathbf{Z}^2}$. If $\mathcal{L} = A(\mathbf{Z}^2)$ then

$$Ш_{\mathcal{L}}(\mathbf{x}) = \sum_{\mathbf{p}\in\mathcal{L}} \delta(\mathbf{x}-\mathbf{p}) = \sum_{\mathbf{k}\in\mathbf{Z}^2} \delta(\mathbf{x}-A\mathbf{k})\,.$$

Next, using the formula for $\delta(A\mathbf{x})$ from earlier in this chapter,

$$\delta(\mathbf{x}-A\mathbf{k}) = \delta(A(A^{-1}\mathbf{x}-\mathbf{k})) = \frac{1}{|\det A|}\delta(A^{-1}\mathbf{x}-\mathbf{k})$$

Therefore

$$Ш_{\mathcal{L}}(\mathbf{x}) = \frac{1}{|\det A|}Ш_{\mathbf{Z}^2}(A^{-1}\mathbf{x})\,.$$

Compare this to our earlier formulas on how the one-dimensional Ш-function scales: With

$$Ш_p(x) = \sum_{k=-\infty}^{\infty} \delta(x-kp)$$

and

$$\text{Ш}(px) = \sum_{k=-\infty}^{\infty} \delta(px - k)$$

we found that

$$\text{Ш}(px) = \frac{1}{|p|}\text{Ш}_{1/p}(x)$$

Periodizing and sampling Periodizing with $\text{Ш}_{\mathcal{L}}$ via convolution results in a function that is periodic with respect to the lattice. If

$$\Phi(\mathbf{x}) = (\varphi * \text{Ш}_{\mathcal{L}})(\mathbf{x}) = \sum_{\mathbf{p}\in\mathcal{L}} \varphi(\mathbf{x} - \mathbf{p})$$

then

$$\Phi(\mathbf{x} + \mathbf{p}) = \Phi(\mathbf{x})$$

for all $\mathbf{x} \in \mathbf{R}^2$ and all $\mathbf{p} \in \mathcal{L}$. Another way of saying this is that Φ has two "independent" periods, one each in the directions of any pair of basis vectors for the lattice. Thus if $\mathbf{u}_1$, $\mathbf{u}_2$ are a basis for $\mathcal{L}$ then

$$\Phi(\mathbf{x} + k_1\mathbf{u}_1) = \Phi(\mathbf{x}) \quad \text{and} \quad \Phi(\mathbf{x} + k_2\mathbf{u}_2) = \Phi(\mathbf{x}), \quad k_1, k_2 \text{ any integers.}$$

$\text{Ш}_{\mathcal{L}}$ is also the tool to use for sampling on a lattice, for

$$(\varphi\text{Ш}_{\mathcal{L}})(\mathbf{x}) = \sum_{\mathbf{p}\in\mathcal{L}} \varphi(\mathbf{p})\delta(\mathbf{x} - \mathbf{p})\,.$$

We're almost ready to use this.

Dual lattices and $\mathcal{F}\text{Ш}_{\mathcal{L}}$ Of the many (additional) interesting things to say about lattices, the one that's most important for our concerns is how the Fourier transform of $\text{Ш}_{\mathcal{L}}$ depends on $\mathcal{L}$. This question leads to a fascinating phenomenon, one that is realized physically in x-ray diffraction images of crystals.

We mentioned earlier that for the integer lattice we have

$$\mathcal{F}\text{Ш}_{\mathbf{Z}^2} = \text{Ш}_{\mathbf{Z}^2}\,.$$

What about the Fourier transform of $\text{Ш}_{\mathcal{L}}$? We appeal to the general similarity theorem to obtain, for $\mathcal{L} = A\mathbf{Z}^2$,

$$\begin{aligned}
\mathcal{F}\text{Ш}_{\mathcal{L}}(\boldsymbol{\xi}) &= \frac{1}{|\det A|}\mathcal{F}(\text{Ш}_{\mathbf{Z}^2}(A^{-1}\mathbf{x})) \\
&= \frac{1}{|\det A|}\frac{1}{|\det A^{-1}|}\mathcal{F}\text{Ш}_{\mathbf{Z}^2}(A^{\mathsf{T}}\boldsymbol{\xi}) \\
&\qquad \text{(we just get } A^{\mathsf{T}} \text{ on the inside because we're already working with } A^{-1}) \\
&= \mathcal{F}\text{Ш}_{\mathbf{Z}^2}(A^{\mathsf{T}}\boldsymbol{\xi}) \\
&= \text{Ш}_{\mathbf{Z}^2}(A^{\mathsf{T}}\boldsymbol{\xi})
\end{aligned}$$

There's a much neater version of this last result, and one of genuine physical importance. But we need a new idea.

In crystallography it is common to introduce the *reciprocal lattice* associated to a given lattice. Given a lattice $\mathcal{L}$, the reciprocal lattice is the lattice $\mathcal{L}^*$ consisting of all points (or vectors) $\mathbf{q}$ such that

$$\mathbf{q} \cdot \mathbf{p} = \text{an integer for every } \mathbf{p} \text{ in the lattice } L.$$

In some other areas of applications, and in mathematics, the reciprocal lattice is known as the *dual lattice*. I'll show my heritage and generally use the term dual lattice.

> **Warning** People in crystallography, those in Material Sciences for example, use the reciprocal lattice all the time and define it this way. However, in some fields and for some applications the reciprocal lattice is normalized differently to require that $\mathbf{q} \cdot \mathbf{p}$ be an integer multiple of 2π. This alternate normalization is exactly tied up with the alternate ways of defining the Fourier transform, i.e., while we use $e^{-2\pi i \cdot \mathbf{x}}$, putting the 2π in the exponential, others do not put the 2π there and have to put a factor in front of the integral, and so on. I can do no more than to issue this warning and wish us all luck in sorting out the inconsistencies.

To develop the notion of the dual lattice a little, and to explain the terminology "reciprocal", suppose we get the lattice $\mathcal{L}$ from $\mathbf{Z}^2$ by applying an invertible matrix A to $\mathbf{Z}^2$. We'll show that the reciprocal lattice $\mathcal{L}^*$ of $\mathcal{L}$ is given by

$$\mathcal{L}^* = A^{-\mathsf{T}}(\mathbf{Z}^2)\,.$$

There's a maxim lurking here. Use of the Fourier transform always brings up "reciprocal" relations of some sort, and in higher dimensions more often than not:

- "Reciprocal" means inverse transpose.

Notice, by the way, that $(\mathbf{Z}^2)^* = \mathbf{Z}^2$, since A in this case is the identity, i.e., $\mathbf{Z}^2$ is "self-dual" as a lattice. This, coupled with the discussion to follow, is another reason for saying that $\mathbf{Z}^2$ wins the award for most evenly spaced points in $\mathbf{R}^2$.

Here's why $\mathcal{L}^* = A^{-\mathsf{T}}(\mathbf{Z}^2)$. Suppose $\mathbf{q} = A^{-\mathsf{T}}\mathbf{m}$ for some $\mathbf{m} = (m_1, m_2)$ in $\mathbf{Z}^2$. And suppose also, because $\mathcal{L} = A(\mathbf{Z}^2)$, that $\mathbf{p} = A\mathbf{m}'$ for some other $\mathbf{m}' = (m_1', m_2')$ in $\mathbf{Z}^2$. Then

$$\begin{aligned}
\mathbf{q} \cdot \mathbf{p} &= A^{-\mathsf{T}}\mathbf{m} \cdot A\mathbf{m}' \\
&= \mathbf{m} \cdot A^{-1}(A\mathbf{m}') \quad \text{(because of how matrices operate with the dot product)} \\
&= \mathbf{m} \cdot \mathbf{m}' = m_1 m_1' + m_2 m_2' \quad \text{(an integer)}
\end{aligned}$$

We want to draw two conclusions from the result that $\mathcal{L}^* = A^{-\mathsf{T}}(\mathbf{Z}^2)$. First, we see that

$$\text{Area}(\mathcal{L}^*) = |\det \mathrm{A}^{-\mathsf{T}}| = \frac{1}{|\det \mathrm{A}|} = \frac{1}{\text{Area}(\mathcal{L})}\,.$$

Thus the areas of $\mathcal{L}$ and $\mathcal{L}^*$ are reciprocals. This is probably the crystallographer's main reason for using the term reciprocal.

The second conclusion, and the second reason to use the term reciprocal, has to do with bases of $\mathcal{L}$ and of $\mathcal{L}^*$. With $\mathcal{L} = A(\mathbf{Z}^2)$ let

$$\mathbf{u}_1 = A\mathbf{e}_1, \quad \mathbf{u}_2 = A\mathbf{e}_2$$

be a basis for $\mathcal{L}$. Since $\mathcal{L}^* = A^{-\mathsf{T}}(\mathbf{Z}^2)$, the vectors

$$\mathbf{u}_1^* = A^{-\mathsf{T}}\mathbf{e}_1, \quad \mathbf{u}_2^* = A^{-\mathsf{T}}\mathbf{e}_2$$

are a basis for $\mathcal{L}^*$. They have a special property with respect to $\mathbf{u}_1$ and $\mathbf{u}_2$, namely

$$\mathbf{u}_i \cdot \mathbf{u}_j^* = \delta_{ij} \quad \text{(Kronecker delta)}\,.$$

This is simple to show, after all we've been through:

$$\mathbf{u}_i \cdot \mathbf{u}_j^* = A\mathbf{e}_i \cdot A^{-\mathsf{T}}\mathbf{e}_j = \mathbf{e}_i \cdot A^{\mathsf{T}}A^{-\mathsf{T}}\mathbf{e}_j = \mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}\,.$$

Now, in linear algebra — independent of any connection with lattices — bases $\{\mathbf{u}_1, \mathbf{u}_2\}$ and $\{\mathbf{u}_1^*, \mathbf{u}_2^*\}$ of $\mathbf{R}^2$ are called *dual* (or sometimes, *reciprocal*) if they satisfy

$$\mathbf{u}_i \cdot \mathbf{u}_j^* = \delta_{ij} \quad \text{(Kronecker delta)}\,.$$

We can therefore summarize what we've found by saying

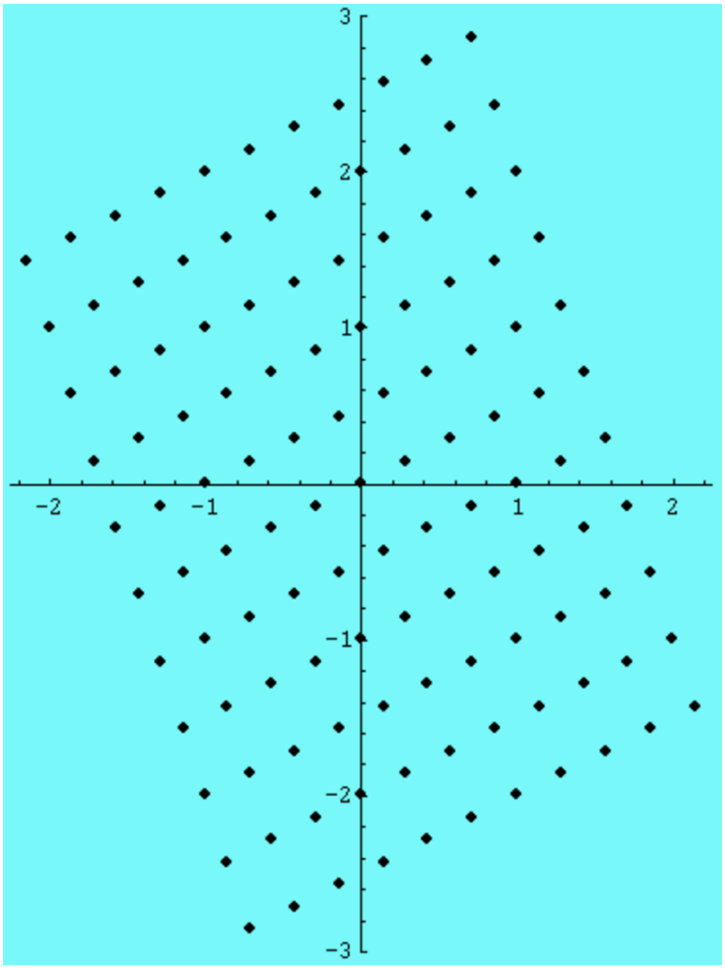
- If $\{\mathbf{u}_1, \mathbf{u}_2\}$ is a basis for a lattice $\mathcal{L}$ and if $\{\mathbf{u}_1^*, \mathbf{u}_2^*\}$ is the dual basis to $\{\mathbf{u}_1, \mathbf{u}_2\}$, then $\{\mathbf{u}_1^*, \mathbf{u}_2^*\}$ is a basis for the dual lattice $\mathcal{L}^*$.

Lots of words here, true, but it's worth your while understanding what we've just done. You're soon to see it all in action in the sampling formula.

Here's a picture of the dual lattice to the lattice pictured earlier. It's obtained from $\mathbf{Z}^2$ by applying

$$A^{-\mathsf{T}} = \begin{pmatrix} 2/7 & -1/7 \\ 1/7 & 3/7 \end{pmatrix}\,.$$

As the scales on the axes show, the dual lattice is, in this case, much more "compressed" than the original lattice. Its area is $1/7$.



Back to the Fourier transform. We showed that if $\mathcal{L} = A(\mathbf{Z}^2)$ then

$$\mathcal{F}\text{Ш}_{\mathcal{L}}(\boldsymbol{\xi}) = \text{Ш}_{\mathbf{Z}^2}(A^{\mathsf{T}}\boldsymbol{\xi})\,.$$

We want to call forth the reciprocal lattice. For this,

$$\begin{aligned}\text{Ш}_{\mathbf{Z}^2}(A^{\mathsf{T}}\boldsymbol{\xi}) &= \sum_{\mathbf{n}\in\mathbf{Z}^2} \delta(A^{\mathsf{T}}\boldsymbol{\xi} - \mathbf{n}) \\ &= \sum_{\mathbf{n}\in\mathbf{Z}^2} \delta(A^{\mathsf{T}}(\boldsymbol{\xi} - A^{-\mathsf{T}}\mathbf{n})) \\ &= \frac{1}{|\det A^{\mathsf{T}}|}\sum_{\mathbf{n}\in\mathbf{Z}^2} \delta(\boldsymbol{\xi} - A^{-\mathsf{T}}\mathbf{n}) = \frac{1}{|\det A|}\sum_{\mathbf{n}\in\mathbf{Z}^2} \delta(\boldsymbol{\xi} - A^{-\mathsf{T}}\mathbf{n})\,.\end{aligned}$$

But this last expression is exactly a sum over points in the reciprocal lattice $\mathcal{L}^*$. We thus have

$$\mathcal{F}(\text{Ш}_{\mathcal{L}})(\boldsymbol{\xi}) = \frac{1}{|\det A|}\text{Ш}_{\mathcal{L}^*}(\boldsymbol{\xi})\,.$$

Bringing in the areas of fundamental parallelograms for $\mathcal{L}$ and $\mathcal{L}^*$ we can write this either in the form

$$\mathcal{F}(\text{Ш}_{\mathcal{L}})(\boldsymbol{\xi}) = \text{Area}(\mathcal{L}^*)\text{Ш}_{\mathcal{L}^*}(\boldsymbol{\xi}) \quad \text{or} \quad \text{Area}(\mathcal{L})\mathcal{F}(\text{Ш}_{\mathcal{L}})(\boldsymbol{\xi}) = \text{Ш}_{\mathcal{L}^*}(\boldsymbol{\xi})\,.$$

Interchanging the roles of $\mathcal{L}$ and $\mathcal{L}^*$, we likewise have

$$\mathcal{F}(\text{Ш}_{\mathcal{L}^*})(\boldsymbol{\xi}) = \text{Area}(\mathcal{L})\text{Ш}_{\mathcal{L}}(\boldsymbol{\xi}) \quad \text{or} \quad \text{Area}(\mathcal{L}^*)\mathcal{F}(\text{Ш}_{\mathcal{L}^*})(\boldsymbol{\xi}) = \text{Ш}_{\mathcal{L}}(\boldsymbol{\xi})\,.$$

Formulas for the inverse Fourier transforms look just like these because the Ш's are even.

Compare these results to the formula in one dimension,

$$\mathcal{F}\text{Ш}_p = \frac{1}{p}\text{Ш}_{1/p}\,,$$

and now you'll see why I said "Pay particular attention here to the reciprocity in spacing between Ш_p and its Fourier transform" at the beginning of this section.

Higher dimensions Everything in the preceding discussion goes through in higher dimensions with *no significant changes*, e.g., "area" becomes "volume". The only reason for stating definitions and results in two-dimensions was to picture the lattices a little more easily. But, certainly, lattices in three dimensions are common in applications and provide a natural framework for understanding crystals, for example. Let's do that next.

8.4.4 The Poisson Summation Formula, again, and $\mathcal{F}\text{Ш}_{\mathbf{Z}^2}$

Back in Chapter 5 we derived the Poisson summation formula: if φ is a Schwartz function then

$$\sum_{k=-\infty}^{\infty} \mathcal{F}\varphi(k) = \sum_{k=-\infty}^{\infty} \varphi(k)\,.$$

It's a remarkable identity and it's the basis for showing that

$$\mathcal{F}\text{Ш} = \text{Ш}$$

for the one-dimensional Ш. In fact, the Poisson summation formula is *equivalent* to the Fourier transform identity.

The situation in higher dimensions is completely analogous. All that we need is a little bit on higher dimensional Fourier series, which we'll bring in here without fanfare; see the earlier section on "Higher dimensional Fourier series and random walks" for more background.

Suppose φ is a Schwartz function on $\mathbf{R}^2$. We periodize φ to be periodic on the integer lattice $\mathbf{Z}^2$ via

$$\Phi(\mathbf{x}) = (\varphi * \text{Ш}_{\mathbf{Z}^2})(\mathbf{x}) = \sum_{\mathbf{n}\in\mathbf{Z}^2} \varphi(\mathbf{x}-\mathbf{n})\,.$$

Then Φ has a two-dimensional Fourier series

$$\Phi(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbf{Z}^2} \widehat{\Phi}(\mathbf{k}) e^{2\pi i \mathbf{k}\cdot\mathbf{x}}\,.$$

Let's see what happens with the Fourier coefficients.

$$\begin{aligned}
\widehat{\Phi}(k_1,k_2) &= \int_0^1\int_0^1 e^{-2\pi i(k_1x_1+k_2x_2)}\Phi(x_1,x_2)\,dx_1\,dx_2 \\
&= \int_0^1\int_0^1 e^{-2\pi i(k_1x_1+k_2x_2)} \sum_{n_1,n_2=-\infty}^{\infty} \varphi(x_1-n_1,x_2-n_2)\,dx_1\,dx_2 \\
&= \sum_{n_1,n_2=-\infty}^{\infty} \int_0^1\int_0^1 e^{-2\pi i(k_1x_1+k_2x_2)}\varphi(x_1-n_1,x_2-n_2)\,dx_1\,dx_2\,.
\end{aligned}$$

Now we make the change of variables $u = x_1 - n_1$, $v = x_2 - n_2$. We can either do this "separately" (because the variables are changing separately) or together using the general change of variables formula.[15] Either way, the result is

$$\begin{aligned}
&\sum_{n_1,n_2=-\infty}^{\infty} \int_0^1 \int_0^1 e^{-2\pi i(k_1 x_1 + k_2 x_2)} \varphi(x_1 - n_1, x_2 - n_2)\, dx_1\, dx_2 \\
&\qquad = \sum_{n_1,n_2=-\infty}^{\infty} \int_{-n_1}^{1-n_1} \int_{-n_2}^{1-n_2} e^{-2\pi i(k_1(u+n_1)+k_2(v+n_2)} \varphi(u,v)\, du\, dv \\
&\qquad = \sum_{n_1,n_2=-\infty}^{\infty} \int_{-n_1}^{1-n_1} \int_{-n_2}^{1-n_2} e^{-2\pi i(k_1 n_1 + k_2 n_2)} e^{-2\pi i(k_1 u + k_2 v)} \varphi(u,v)\, du\, dv \\
&\qquad = \sum_{n_1,n_2=-\infty}^{\infty} \int_{-n_1}^{1-n_1} \int_{-n_2}^{1-n_2} e^{-2\pi i(k_1 u + k_2 v)} \varphi(u,v)\, du\, dv \\
&\qquad = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-2\pi i(k_1 u + k_2 v)} \varphi(u,v)\, du\, dv \\
&\qquad = \mathcal{F}\varphi(k_1, k_2)\,.
\end{aligned}$$

We have found, just as we did in one dimension, that the Fourier series for the $\mathbf{Z}^2$-periodization of φ is

$$\Phi(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbf{Z}^2} \mathcal{F}\varphi(\mathbf{k}) e^{2\pi i\, \mathbf{k}\cdot\mathbf{x}}\,.$$

We now evaluate $\Phi(\mathbf{0})$ in two ways, plugging $\mathbf{x} = \mathbf{0}$ into its definition as the periodization of φ and into its Fourier series. The result is

$$\sum_{\mathbf{k}\in\mathbf{Z}^2} \mathcal{F}\varphi(\mathbf{k}) = \sum_{\mathbf{k}\in\mathbf{Z}^2} \varphi(\mathbf{k})\,.$$

To wrap it all up, here's the derivation of

$$\mathcal{F}\text{Ш}_{\mathbf{Z}^2} = \text{Ш}_{\mathbf{Z}^2}$$

based on the Poisson summation formula. For any Schwartz function ψ,

$$\langle \mathcal{F}\text{Ш}_{\mathbf{Z}^2}, \psi\rangle = \langle \text{Ш}_{\mathbf{Z}^2}, \mathcal{F}\psi\rangle = \sum_{\mathbf{k}\in\mathbf{Z}^2} \mathcal{F}\psi(\mathbf{k}) = \sum_{\mathbf{k}\in\mathbf{Z}^2} \psi(\mathbf{k}) = \langle \text{Ш}_{\mathbf{Z}^2}, \psi\rangle\,.$$

8.5 Crystals

In a few paragraphs, here's one reason why all this stuff on dual lattices is so interesting. The physical model for a crystal is a three-dimensional lattice with atoms at the lattice points. An X-ray diffraction experiment scatters X-rays off the atoms in the crystal and results in spots on the X-ray film, of varying intensity, also located at lattice points. From this and other information the crystallographer attempts to deduce the structure of the crystal. The first thing the crystallographer has to know is that the lattice of spots on the film arising from diffraction is the *dual of the crystal lattice*. (In fact, it's more complicated

[15] Right here is where the property of $\mathbf{Z}^2$ as the "simplest" lattice comes in. If we were working with an "oblique" lattice we could not make such a simple change of variables. We would have to make a more general linear change of variables. This would lead to a more complicated result.

than that, for it is the projection onto the two-dimensional plane of the film of the three-dimensional dual lattice.)

We can explain this phenomenon — atoms on a lattice, spots on the dual lattice — via the Fourier transform. What the crystallographer ultimately wants to find is the electron density distribution for the crystal. The mathematical model for crystals puts a delta at each lattice point, one for each atom. If we describe the electron density distribution of a single atom by a function $\rho(\mathbf{x})$ then the electron density distribution of the crystal with atoms at points of a (three-dimensional) lattice $\mathcal{L}$ is

$$\rho_{\mathcal{L}}(\mathbf{x}) = \sum_{\mathbf{p}\in\mathcal{L}} \rho(\mathbf{x}-\mathbf{p}) = (\rho * \text{Ш}_{\mathcal{L}})(\mathbf{x})\,.$$

This is now a periodic function with three independent periods, one in the direction of each of the three basis vectors that determine $\mathcal{L}$. We worked with a one-dimensional version of this in Chapter 5.

The basic fact in X-ray crystallography is that the "scattered amplitude" of the X-rays diffracting off the crystal is proportional to the magnitude of the Fourier transform of the electron density distribution. This data, the results of X-ray diffraction, comes to us *directly in the frequency domain.* Now, we have

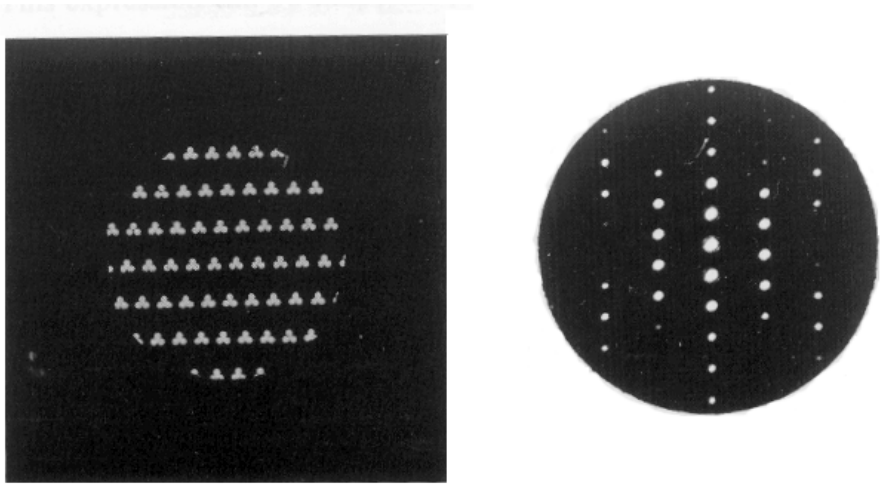
$$\mathcal{F}\rho_{\mathcal{L}}(\boldsymbol{\xi}) = \mathcal{F}\rho(\boldsymbol{\xi})\mathcal{F}\text{Ш}_{\mathcal{L}}(\boldsymbol{\xi}) = \mathcal{F}\rho(\boldsymbol{\xi})\,\text{Volume}(\mathcal{L}^*)\,\text{Ш}_{\mathcal{L}^*}(\boldsymbol{\xi})\,,$$

where $\mathcal{L}^*$ is the dual lattice. Taking this one more step,

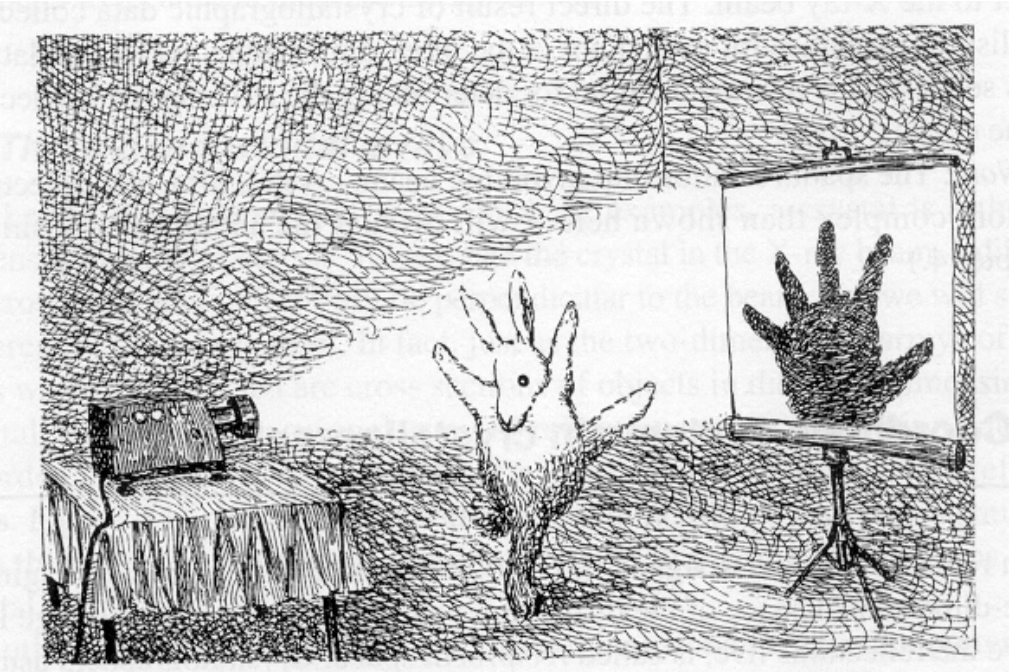
$$\mathcal{F}\rho_{\mathcal{L}}(\boldsymbol{\xi}) = \text{Volume}(\mathcal{L}^*) \sum_{\mathbf{q}\in\mathcal{L}^*} \mathcal{F}\rho(\mathbf{q})\delta(\boldsymbol{\xi}-\mathbf{q})\,.$$

The important conclusion is that the diffraction pattern has peaks at the lattice points of the *reciprocal* lattice. The picture is not complete, however. The intensities of the spots are related to the magnitude of the Fourier transform of the electron density distribution, but for a description of the crystal it is also necessary to determine the phases, and this is a hard problem.

Here's a picture of a macroscopic diffraction experiment. On the left is an array of pinholes and on the right is the diffraction pattern. The spots on the right are at the lattice points of the reciprocal lattice.



The goal of X-ray diffraction experiments is to determine the configuration of atoms from images of this type. Making the analysis even harder is that for 3D crystal lattices the images on an X-ray film are the projection onto the image plane of the 3D configuration. Just how difficult it may be to infer 3D structure from 2D projections is illustrated by a famous experiment: "Fun in Reciprocal Space" published in the distinguished American journal *The New Yorker.*



8.6 Bandlimited Functions on $\mathbf{R}^2$ and Sampling on a Lattice

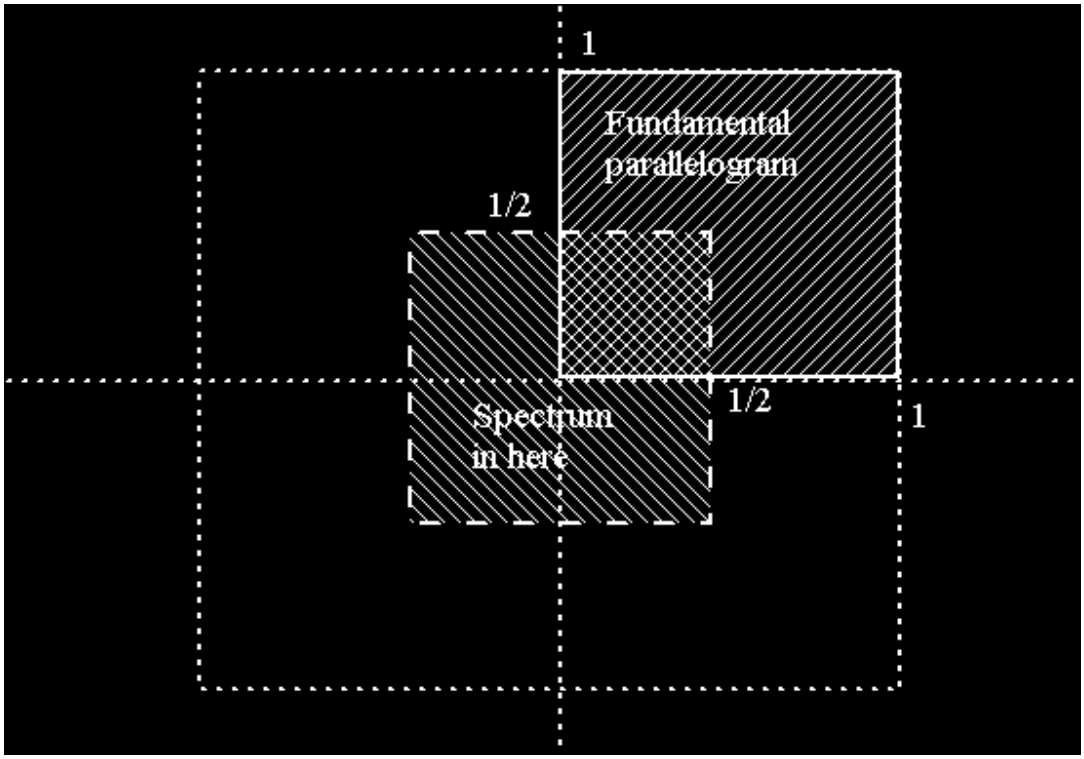
Let's develop the sampling formula in two dimensions. A function f on $\mathbf{R}^2$ is bandlimited if $\mathcal{F}f$ is identically zero outside of some bounded region. We always assume that f is real valued, and hence $\mathcal{F}f(-\boldsymbol{\xi}) = \overline{\mathcal{F}f(\boldsymbol{\xi})}$. Thus, as we've pointed out before, if $\mathcal{F}f(\boldsymbol{\xi}) \neq 0$ then $\mathcal{F}f(-\boldsymbol{\xi}) \neq 0$ and so, as a point set in $\mathbf{R}^2$, the spectrum is symmetric about the origin.

We want to derive a sampling formula associated with a lattice $\mathcal{L}$ by following the recipe of first periodizing $\mathcal{F}f$ via $\text{Ш}_{\mathcal{L}}$, then cutting off, and then taking the inverse Fourier transform. The result will be a sinc reconstruction of f from its sampled values — but just where those sampled values are is what's especially interesting and relevant to what we've just done.

To get the argument started we assume that the support of $\mathcal{F}f$ lies in a parallelogram. This parallelogram determines a fundamental parallelogram for a lattice $\mathcal{L}$, and the spectrum gets shifted parallel to itself and off itself through convolution with $\text{Ш}_{\mathcal{L}}$. This periodization is the first step and it's analogous to the one-dimensional case when the spectrum lies in an interval, say from $-p/2$ to $p/2$, and the spectrum gets shifted around and off itself through convolution with Ш_p. Recall that the crucial limitation is that the spectrum only goes up to $p/2$ and down to $-p/2$, while Ш_p has δ's spaced p apart. The spacing of the δ's is big enough to shift the spectrum off itself and no smaller spacing will do. Correspondingly in two dimensions, the parallelogram containing the spectrum determines a lattice with "big enough spacing" for a Ш based on the lattice to shift the spectrum off itself.

Using the general stretch theorem, we'll be able to get the general result by first deriving a special case, when the spectrum lies in a square. Suppose, then, that $\mathcal{F}f(\boldsymbol{\xi})$ is identically zero *outside* the (open) square $|\xi_1| < 1/2$, $|\xi_2| < 1/2$. We work with the integer lattice $\mathbf{Z}^2$ with basis $\mathbf{e}_1$ and $\mathbf{e}_2$. The (open) fundamental parallelogram for $\mathbf{Z}^2$ is $0 < \xi_1 < 1$, $0 < \xi_2 < 1$ and the spectrum is inside the center fourth of four copies of it, as pictured.

Periodizing $\mathcal{F}f$ by $\text{Ш}_{\mathbf{Z}^2}$ shifts the spectrum off itself, and no smaller rectangular lattice will do for this.



We then cut off by the two-dimensional rect function $\Pi(x_1, x_2) = \Pi(x_1)\Pi(x_2)$ and this gives back $\mathcal{F}f$:

$$\mathcal{F}f(\boldsymbol{\xi}) = \Pi(\xi_1)\,\Pi(\xi_2)(\mathcal{F}f * \text{Ш}_{\mathbf{Z}^2})(\boldsymbol{\xi})\,.$$

This is just as in the one-dimensional case, and now it's time to take the inverse Fourier transform. Using $\mathcal{F}\text{Ш}_{\mathbf{Z}^2} = \text{Ш}_{\mathbf{Z}^2}$, or rather $\mathcal{F}^{-1}\text{Ш}_{\mathbf{Z}^2} = \text{Ш}_{\mathbf{Z}^2}$, and invoking the convolution theorem we obtain

$$\begin{aligned} f(\mathbf{x}) = f(x_1, x_2) &= (\operatorname{sinc} x_1\ \operatorname{sinc} x_2) * (f(\mathbf{x}) \cdot \text{Ш}_{\mathbf{Z}^2}(\mathbf{x})) \\ &= (\operatorname{sinc} x_1\ \operatorname{sinc} x_2) * \Big(f(\mathbf{x}) \cdot \sum_{k_1,k_2=-\infty}^{\infty} \delta(\mathbf{x} - k_1\mathbf{e}_1 - k_2\mathbf{e}_2)\Big) \\ &= (\operatorname{sinc} x_1\ \operatorname{sinc} x_2) * \sum_{k_1,k_2=-\infty}^{\infty} f(k_1, k_2)\delta(x_1 - k_1, x_2 - k_2) \\ &= \sum_{k_1,k_2=-\infty}^{\infty} f(k_1, k_2)\operatorname{sinc}(x_1 - k_1)\operatorname{sinc}(x_2 - k_2)\,. \end{aligned}$$

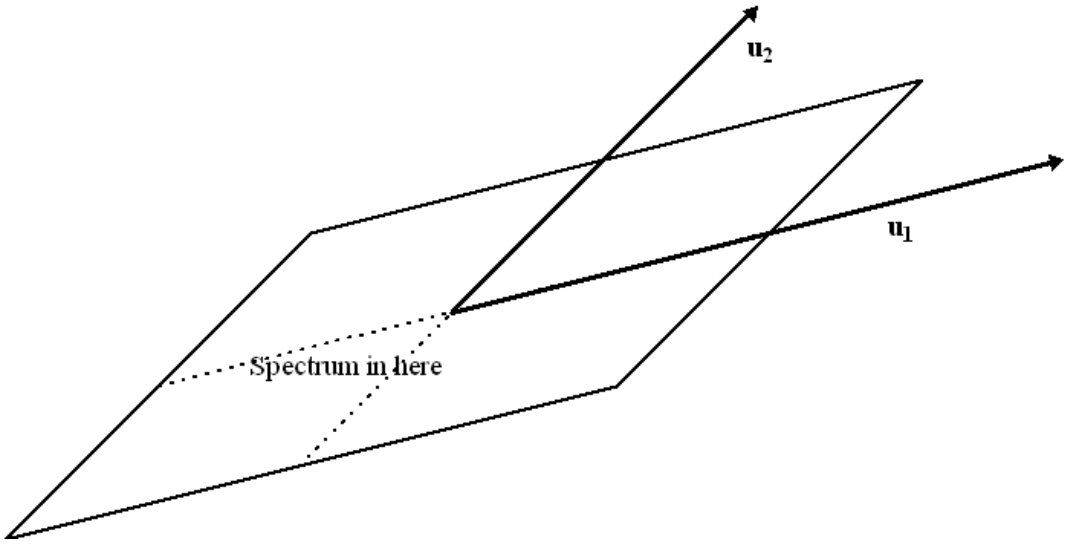
In solidarity with the general case soon to follow, let's write this "square sampling formula" as

$$f(\mathbf{x}) = \sum_{k_1,k_2=-\infty}^{\infty} f(k_1\mathbf{e}_1 + k_2\mathbf{e}_2)\operatorname{sinc}(\mathbf{x}\cdot\mathbf{e}_1 - k_1)\operatorname{sinc}(\mathbf{x}\cdot\mathbf{e}_2 - k_2)\,.$$

Now suppose that the spectrum of $\mathcal{F}f$ lies in the (open) parallelogram, as pictured, with $\mathbf{u}_1$ and $\mathbf{u}_2$ parallel to the sides and as long as the sides.

Let A be the 2×2 matrix that takes $\mathbf{e}_1$ to $\mathbf{u}_1$ and $\mathbf{e}_2$ to $\mathbf{u}_2$, so that A maps the lattice $\mathbf{Z}^2$ to the lattice $\mathcal{L}$ with basis $\mathbf{u}_1$ and $\mathbf{u}_2$. Let $B = A^{-\mathsf{T}}$ (hence $B^{-\mathsf{T}} = A$) and remember that B takes $\mathbf{Z}^2$ to the dual lattice $\mathcal{L}^*$ of $\mathcal{L}$. A basis for $\mathcal{L}^*$ (the dual basis to $\mathbf{u}_1$ and $\mathbf{u}_2$) is

$$\mathbf{u}_1^* = B\mathbf{e}_1, \quad \mathbf{u}_2^* = B\mathbf{e}_2\,.$$



Next let

$$g(\mathbf{x}) = f(B\mathbf{x})\,.$$

According to the general stretch theorem,

$$\mathcal{F}g(\boldsymbol{\xi}) = \frac{1}{|\det B|}\mathcal{F}f(B^{-\mathsf{T}}\boldsymbol{\xi}) = |\det A|\,\mathcal{F}f(A\boldsymbol{\xi})\,.$$

The determinant factor out front doesn't matter; what's important is that the spectrum of g is in the square $-1/2 < \xi_1 < 1/2$, $-1/2 < \xi_2 < 1/2$, since the corresponding points $A\boldsymbol{\xi}$ lie in the parallelogram containing the spectrum of f, i.e., $\mathcal{F}g$ is identically zero outside the square.

We now apply the square sampling formula to g to write

$$g(\mathbf{x}) = \sum_{k_1,k_2=-\infty}^{\infty} g(k_1\mathbf{e}_1 + k_2\mathbf{e}_2)\,\mathrm{sinc}(\mathbf{x}\cdot\mathbf{e}_1 - k_1)\,\mathrm{sinc}(\mathbf{x}\cdot\mathbf{e}_2 - k_2)$$

With $\mathbf{y} = B\mathbf{x}$ we can then say

$$\begin{aligned}
f(\mathbf{y}) &= \sum_{k_1,k_2=-\infty}^{\infty} f(B(k_1\mathbf{e}_1 + k_2\mathbf{e}_2))\,\mathrm{sinc}(B^{-1}\mathbf{y}\cdot\mathbf{e}_1 - k_1)\,\mathrm{sinc}(B^{-1}\mathbf{y}\cdot\mathbf{e}_2 - k_2)\\
&= \sum_{k_1,k_2=-\infty}^{\infty} f(k_1B\mathbf{e}_1 + k_2B\mathbf{e}_2)\,\mathrm{sinc}(A^{\mathsf{T}}\mathbf{y}\cdot\mathbf{e}_1 - k_1)\,\mathrm{sinc}(A^{\mathsf{T}}\mathbf{y}\cdot\mathbf{e}_2 - k_2)\\
&= \sum_{k_1,k_2=-\infty}^{\infty} f(k_1\mathbf{u}_1^* + k_2\mathbf{u}_2^*)\,\mathrm{sinc}(\mathbf{y}\cdot A\mathbf{e}_1 - k_1)\,\mathrm{sinc}(\mathbf{y}\cdot A\mathbf{e}_2 - k_2)\\
&= \sum_{k_1,k_2=-\infty}^{\infty} f(k_1\mathbf{u}_1^* + k_2\mathbf{u}_2^*)\,\mathrm{sinc}(\mathbf{y}\cdot\mathbf{u}_1 - k_1)\,\mathrm{sinc}(\mathbf{y}\cdot\mathbf{u}_2 - k_2)\,.
\end{aligned}$$

We're done. Change $\mathbf{y}$ to $\mathbf{x}$ for psychological comfort, and the "lattice sampling formula" says that

$$f(\mathbf{x}) = \sum_{k_1,k_2=-\infty}^{\infty} f(k_1\mathbf{u}_1^* + k_2\mathbf{u}_2^*)\,\mathrm{sinc}(\mathbf{x}\cdot\mathbf{u}_1 - k_1)\,\mathrm{sinc}(\mathbf{x}\cdot\mathbf{u}_2 - k_2). \tag{8.1}$$

This is a sinc reconstruction formula giving the function in terms of sample values on a lattice. But it's the dual lattice! Here's how to remember the highlights:

- The spectrum of f lies in a parallelogram, which determines a lattice with basis $\mathbf{u}_1$ and $\mathbf{u}_2$.
- That lattice determines a dual lattice (in the spatial domain) with dual basis $\mathbf{u}_1^*$ and $\mathbf{u}_2^*$.
- The sincs use data from the lattice, while the sample points are exactly the points in the *dual* lattice.

Look back at the one-dimensional sampling formula and tell yourself what you see of this picture.

Exercise What should we mean by "sampling rate" vis à vis the two-dimensional lattice sampling formula?

The next topics on this path would be to investigate aliasing and to consider the case of a finite spectrum and finite sampling. Another time, another class.

8.7 Naked to the Bone

Our final topic in the course will be a quick development of the use of the Fourier transform in medical imaging. We'll find that the two-dimensional Fourier transform is perfectly suited to the problem of recovering a *density* function — a function representing bones, internal organs, the whole lot — from the *projections* of that density obtained by passing parallel beams of X-rays through a two-dimensional cross section of the body. (For the discussion of the use of the Fourier transform I'm not making a distinction between the original methods of tomography using X-rays and those of magnetic resonance imaging.)

For an account of the history of medical imaging, I recommend the book *Naked to the Bone: Medical Imaging in the Twentieth Century* by Bettyann Kevles, from which I stole the title of this section.

Dimmer and dimmer What happens when light passes through murky water? It gets dimmer and dimmer the farther it goes, of course — this is not a trick question. If the water is the same murkiness throughout, meaning, for example, uniform density of stuff floating around in it, then it's natural to assume that the intensity of light decreases by the same *percent amount* per length of path traveled. Through absorption, scattering, etc., whatever intensity comes in, a certain percentage of that intensity goes out; over a given distance the murky water removes a percentage of light, and this percentage depends only on the distance traveled and not on where the starting and stopping points are.[16] We're assuming here that light is traveling in a straight line through the water.

Constant percent change characterizes exponential growth, or decay, so the attenuation of the intensity of light passing through a homogeneous medium is modeled by

$$I = I_0 e^{-\mu x},$$

where I_0 is the initial intensity, x is the distance traveled, and μ is a (positive) "murkiness constant". x has dimension of length and μ has dimension 1/length and units "murkiness/length". μ is constant because we assume that the medium is homogeneous. We know the value of I_0, and one measurement of x and I will determine μ. In fact, what we do is to put a detector at a known distance x and measure the intensity when it arrives at the detector.

[16] Optical fibers provide an interesting and important study in the progress of making something — glass in this case — less murky. In the **Appendix 8.12** I've attached a graph showing just how dramatic the progress has been.

Now suppose the water is not uniformly murky, but rather the light passes through a number of layers, each layer of uniform murkiness. If the i'th layer has murkiness constant μ_i and is of length Δx_i, and if there are n layers, then the intensity of light that reaches the detector can be modeled by

$$I = I_0 \exp\left(- \sum_{i=1}^{n} \mu_i \Delta x_i \right).$$

Clearly, if the murkiness is described by a function $\mu(x)$, then the intensity arriving at the detector is modeled by

$$I = I_0 \exp\left(- \int_L \mu(x)\, dx \right),$$

where L is the line the light travels along. It's common to call the number

$$p = \int_L \mu(x)\, dx = -\ln\left(\frac{I}{I_0}\right)$$

the *attenuation coefficient*.

Can we recover the density function $\mu(x)$ from knowledge of the intensity? Not so easily. Certainly not from a single reading — many arrangements of murkiness along the path could result in the same final intensity at the detector.

If we could vary the detector along the path and record the results then we would be able to determine $\mu(x)$. That is, if we could form

$$p(\xi) = \int_{\xi_0}^{\xi} \mu(x)\, dx\,,$$

as a *function* of a variable position ξ along the line (ξ_0 is some fixed starting point — the source) then we could find μ from p by finding the derivative p'. The trouble is moving the detector through the murky water along the path.

Tomography X-rays are light, too, and when they pass through murky stuff (your body) along a straight line they are attenuated and reach a detector on the other end at a reduced intensity. We can continue to assume that the attenuation, the decrease in intensity, is exponentially decreasing with the path length. The exponential of what? What do the X-rays pass through?

From the start we set this up as a two-dimensional problem. Take a planar slice through your body. The gunk in this two-dimensional slice — bones, organs, other tissue — is of *variable density*; let's say it's described by an unknown function $\mu(x_1, x_2)$. We consider $\mu(x_1, x_2)$ to be zero outside the section of the body. Take a line L through this slice — in the plane of the slice, the path that an X-ray would follow — and parameterize the line by $x_1(s)$, $x_2(s)$, where s is the arclength parameter going from s_0 to s_1. (The "arclength parameter" means that we move along the line at unit speed.) Then the density along the line is $\mu(x_1(s), x_2(s))$ and the attenuation of the X-ray intensity along the line is

$$I = I_0 \exp\left(- \int_{s_0}^{s_1} \mu(x_1(s), x_2(s))\, ds \right)$$

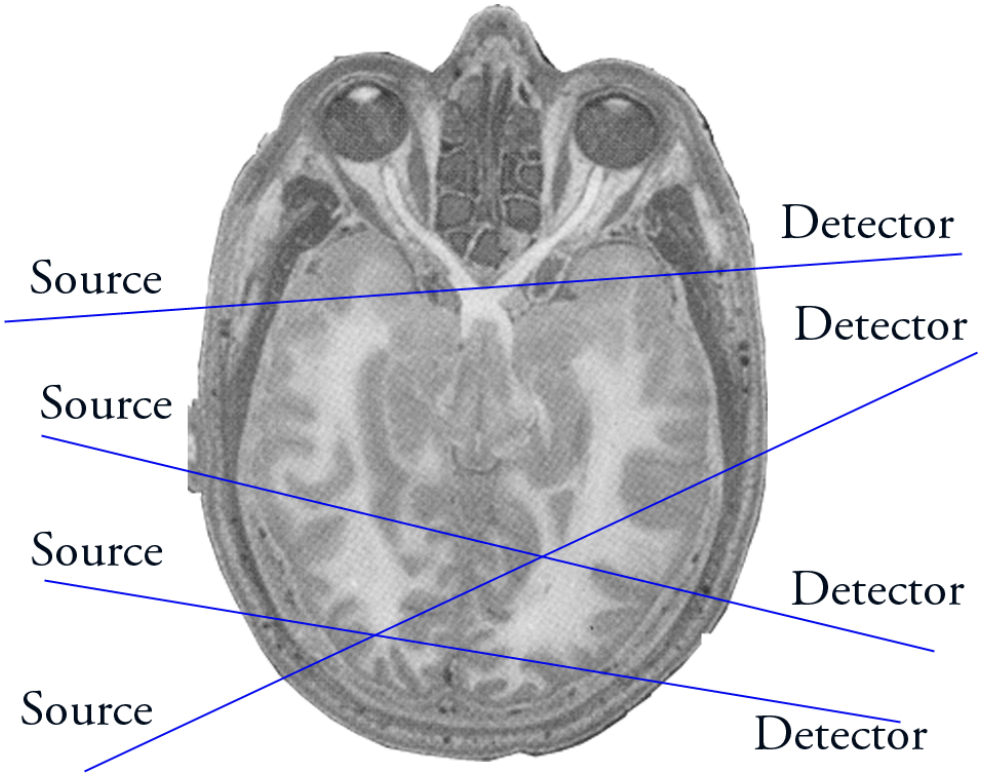
Instead of writing out the parameters and limits, we often write the integral simply as

$$\int_L \mu(x_1, x_2)\, ds\,.$$

We'll refer to this as a *line integral of μ along L*.

- The fundamental problem of tomography[17] is to determine the function $\mu(x, y)$ from these line integrals, taken over many lines through the region.

For example — what's inside?



In trying to solve this problem, what's not allowed is to move the detector through the body — that's not covered by HMO plans. What *is* allowed is to rotate the source (and the detector) to get X-rays circling around the two-dimensional cross-section of the body, and what we'll have are families of parallel X-rays. Before laying all this out, it pays to organize our study of the problem.

8.8 The Radon Transform

For each line L, cutting through the slice, the integral

$$\int_L \mu(x_1, x_2)\, ds$$

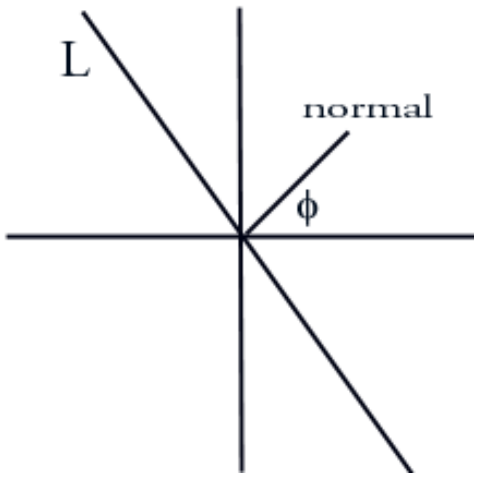
is a number. The operation "line determines number" thus defines a real-valued function of L. The whole subject of tomography is about this function. To work with it effectively we need to be able describe the set of all lines — not the (Cartesian) equation of a given line, but some kind of parametric description for the *collection* of lines. This will allow us to write the integral as a function of these parameters.

There are many ways to describe the collection of all lines in the plane. One that may seem most natural to you is to use the "slope-intercept" form for the equation of a line; a line can be written as $y = mx + b$ where m is the slope and b is the y-intercept. A line can thus be associated with a unique pair (m, b) and

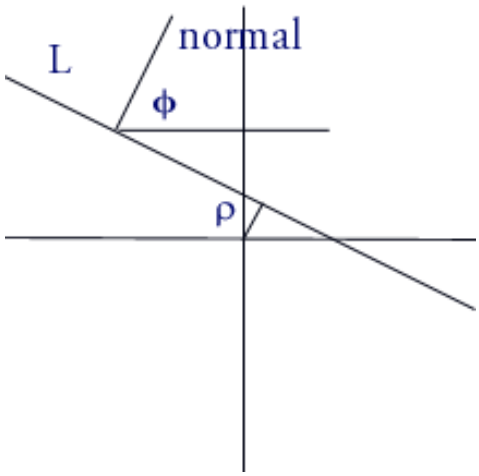
[17] *tomos* means "section" in Greek

vice versa. There's a catch here, however — vertical lines (lines $x =$ constant, infinite slope) are left out of this description.

Another approach, one that allows us to describe *all* lines and that is well suited for the function of L, above, goes as follows. First, a line *through the origin* is determined by its unit normal vector $\mathbf{n}$. Now, $\mathbf{n}$ and $-\mathbf{n}$ determine the same line, so we represent all the (distinct) normal vectors as $(\cos\phi, \sin\phi)$ for an angle ϕ satisfying $0 \leq \phi < \pi$, measured counterclockwise from the x_1-axis. In other words, there is a one-to-one correspondence between the ϕ's with $0 \leq \phi < \pi$ and the collection of all lines through the origin.



A line not through the origin can then be described by its unit normal vector together with the *directed* distance of the line from the origin, a positive number if measured in the direction of $\mathbf{n}$ and a negative number if measured in the direction $-\mathbf{n}$. Call this directed distance ρ. Thus $-\infty < \rho < \infty$.

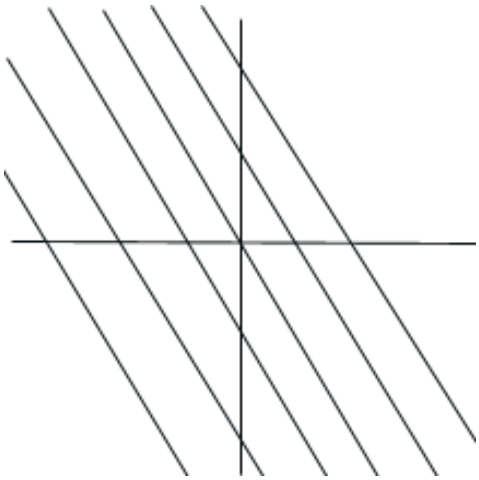


The set of pairs (ρ, ϕ) provides a parameterization for the set of all lines in the plane. Once again:

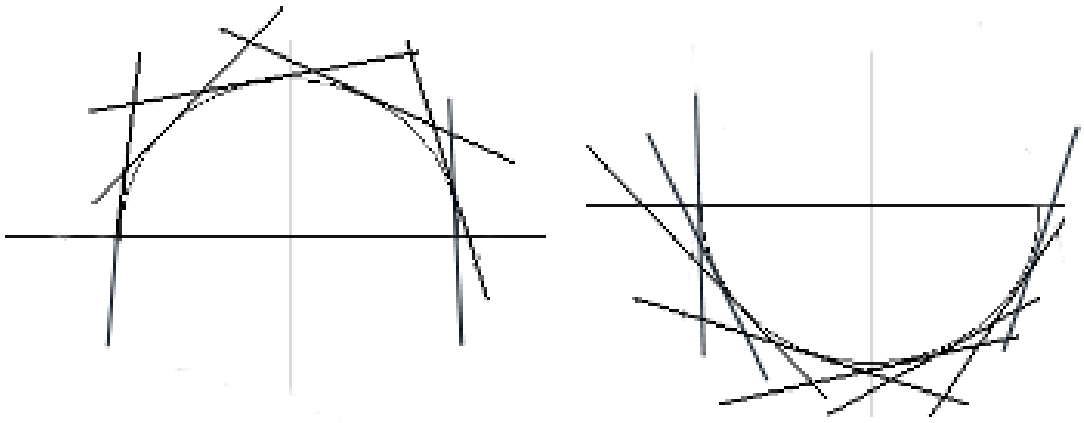
- A pair (ρ, ϕ) means, in this context, the unique line with normal vector $\mathbf{n} = (\cos\phi, \sin\phi)$ which is at a directed distance ρ from the origin, measured in the direction $\mathbf{n}$ if $\rho > 0$ and in the direction $-\mathbf{n}$ if $\rho < 0$.

Anytime you're confronted with a new coordinate system you should ask yourself what the situation is when one of the coordinates is fixed and the other is free to vary. In this case, if ϕ is fixed and ρ varies we

get a family of parallel lines.



For the other case, when ρ is fixed, we have to distinguish some cases. The pairs $(0, \phi)$ correspond to lines through the origin. When ρ is positive and ϕ varies from 0 to π (including 0, excluding π) we get the family of lines tangent to the upper semicircle of radius ρ (including the tangent at $(\rho, 0)$ excluding the tangent at $(-\rho, 0)$). When $\rho < 0$ we get lines tangent to the lower semicircle (including the tangent at $(-|\rho|, 0)$, excluding the tangent at $(|\rho|, 0)$).



Using the coordinates (ρ, ϕ) we therefore have a *transform* of the function $\mu(x_1, x_2)$ to a function $\mathcal{R}\mu(\rho, \phi)$ defined by

$$\mathcal{R}\mu(\rho, \phi) = \int_{L(\rho,\phi)} \mu(x_1, x_2)\, ds\,.$$

This is called the *Radon transform* of μ, introduced by Johann Radon — way back in 1917! The fundamental question of tomography can then be stated as:

- Is there an inversion formula for the Radon transform? That is, from knowledge of the values $\mathcal{R}\mu(\rho, \phi)$ can we recover μ?

We've indicated the dependence of the integral on ρ and ϕ by writing $L(\rho, \phi)$, but we want to use the coordinate description of lines to write the integral in a still more convenient form. Using the dot product, the line determined by (ρ, ϕ) is the set of points (x_1, x_2) with

$$\rho = \mathbf{x} \cdot \mathbf{n} = (x_1, x_2) \cdot (\cos\phi, \sin\phi) = x_1 \cos\phi + x_2 \sin\phi \,.$$

or described via the equation

$$\rho - x_1 \cos\phi - x_2 \sin\phi = 0 \,, \quad -\infty < x_1 < \infty, \; -\infty < x_2 < \infty \,.$$

Now consider the delta function "along the line", that is,

$$\delta(\rho - x_1 \cos\phi - x_2 \sin\phi)$$

as a function of x_1, x_2. This is also called a *line impulse* and it's an example of the greater variety one has in defining different sorts of δ's in two-dimensions. With some interpretation and argument (done in those notes) one can show that integrating a function $f(x_1, x_2)$ against the line impulse associated with a line L results precisely in the line integral of f along L. This is all we'll need here, and with that the Radon transform of $\mu(x_1, x_2)$ can be expressed as

$$\mathcal{R}(\mu)(\rho, \phi) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mu(x_1, x_2) \delta(\rho - x_1 \cos\phi - x_2 \sin\phi) \, dx_1 \, dx_2 \,.$$

This is the form we'll most often work with. One also sees the Radon transform written as

$$\mathcal{R}(\mu)(\rho, \mathbf{n}) = \int_{\mathbf{R}^2} \mu(\mathbf{x}) \delta(\rho - \mathbf{x} \cdot \mathbf{n}) \, d\mathbf{x} \,.$$

This expression suggests generalizations to higher dimensions — interesting, but we won't pursue them.

Projections It's often convenient to work with $\mathcal{R}(\mu)(\rho, \phi)$ by first fixing ϕ and letting ρ vary. Then we're looking at parallel lines passing through the domain of μ, all perpendicular to a particular line making an angle ϕ with the x_1-axis (that line is the common normal to the parallel lines), and we compute the integral of μ along these lines.

This collection of values, $\mathcal{R}(\mu)(\rho, \phi)$ with ϕ fixed, is often referred to as a *projection* of μ, the idea being that the line integrals over parallel lines at a fixed angle are giving some kind of profile, or projection, of μ in that direction.[18] Then varying ϕ gives us a family of projections, and one speaks of the inversion problem as "determining $\mu(x_1, x_2)$ from its projections".

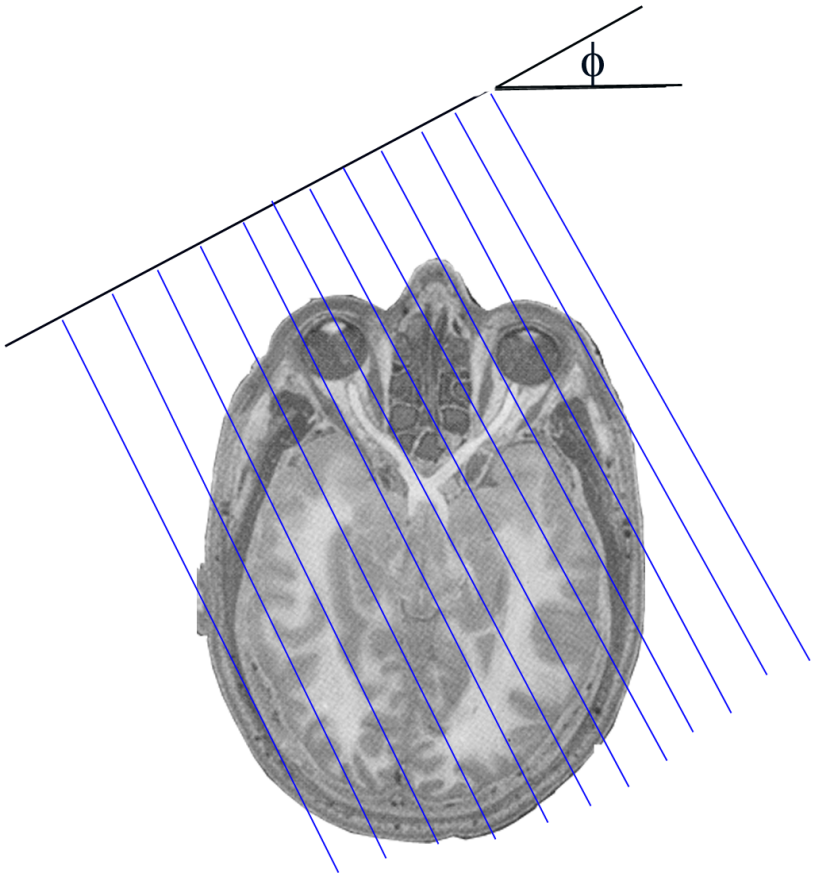
This is especially apt terminology for the medical applications, since that's how a scan is made:

1. Fix an angle and send in a bunch of parallel X-rays at that angle.
2. Change the angle and repeat.

8.9 Getting to Know Your Radon Transform

We want to develop a few properties of the Radon transform, just enough to get some sense of how to work with it. First, a few comments on what kinds of functions $\mu(x_1, x_2)$ one wants to use; it's interesting but we won't make an issue of it.

[18] **Important:** Don't be fooled by the term "projection". You are *not* geometrically projecting the shape of the two-dimensional cross section (that the lines are cutting through). You are looking at the attenuated, parallel X-rays that emerge as we move a source along a line. The line is at some angle relative to a reference axis.



Inspired by honest medical applications, we would *not* want to require that the cross-sectional density $\mu(x_1, x_2)$ be smooth, or even continuous. Jump discontinuities in $\mu(x_1, x_2)$ correspond naturally to a change from bone to muscle, *etc.* Although, mathematically speaking, the lines extend infinitely, in practice the paths are finite. In fact, the easiest thing is just to assume that $\mu(x_1, x_2)$ is zero outside of some region — it's describing the density of a slice of a finite extent body, after all.

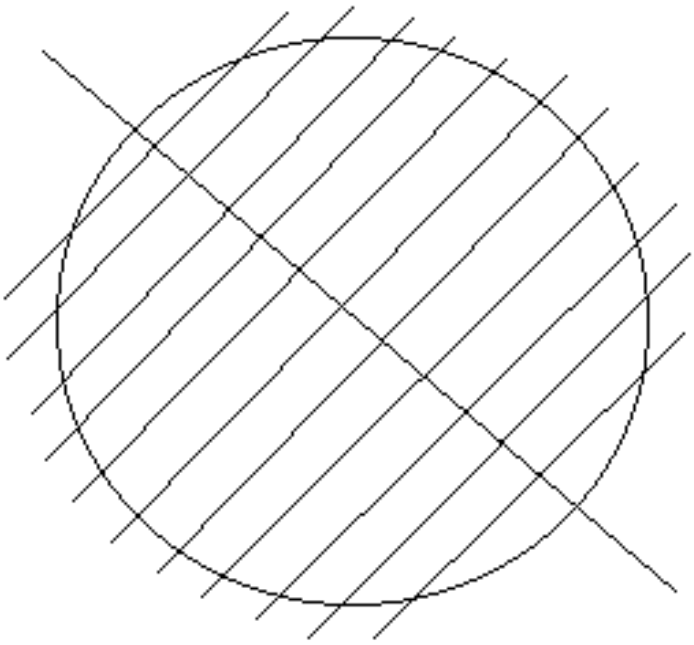
Examples There aren't too many cases where one can compute the Radon transform explicitly. One example is the circ function, expressed in polar coordinates as

$$\mathrm{circ}(r) = \begin{cases} 1 & r \leq 1 \\ 0 & r > 1 \end{cases}$$

We have to integrate the circ function along any line. Think in terms of projections, as defined above. From the circular symmetry, it's clear that the projections are independent of ϕ.

Because of this we can take any convenient value of ϕ, say $\phi = 0$, and find the integrals over the parallel lines in this family. The circ function is 0 outside the unit circle, so we need only to find the integral (of the function 1) over any chord of the unit circle parallel to the x_2-axis. This is easy. If the chord is at a distance ρ from the origin, $|\rho| \leq 1$, then

$$\mathcal{R}(1)(\rho, 0) = \int_{-\sqrt{1-\rho^2}}^{\sqrt{1-\rho^2}} 1 \, dx_2 = 2\sqrt{1-\rho^2}\,.$$



Thus for any (ρ, ϕ),

$$\mathcal{R}\,\mathrm{circ}(\rho,\phi) = \begin{cases} 2\sqrt{1-\rho^2} & |\rho| \le 1 \\ 0 & |\rho| > 1 \end{cases}$$

Gaussians again Another example where we can compute the Radon transform exactly is for a Gaussian:

$$g(x_1, x_2) = e^{-\pi(x_1^2+x_2^2)}\,.$$

Any guesses as to what $\mathcal{R}g$ is? Let's do it.

Using the representation in terms of the line impulse we can write

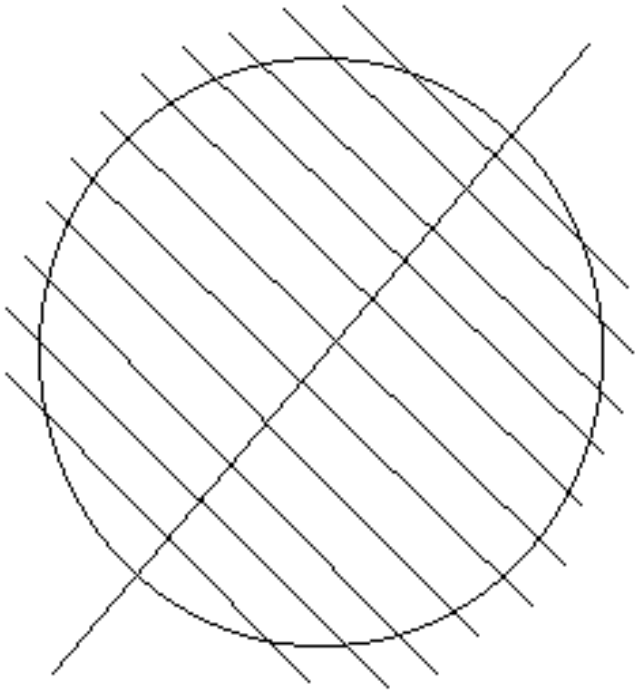
$$\mathcal{R}g(\rho,\phi) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\pi(x_1^2+x_2^2)}\delta(\rho - x_1\cos\phi - x_2\sin\phi)\,dx_1\,dx_2\,.$$

We now make a change of variables in this integral, putting

$$\begin{aligned} u_1 &= x_1\cos\phi + x_2\sin\phi, \\ u_2 &= -x_1\sin\phi + x_2\cos\phi. \end{aligned}$$

This is a rotation of coordinates through an angle ϕ, making the u_1-axis correspond to the x_1-axis. The Jacobian of the transformation is 1, and we also find that

$$u_1^2 + u_2^2 = x_1^2 + x_2^2\,.$$



In the new coordinates the integral becomes:

$$\begin{aligned}
\mathcal{R}g(\rho,\phi) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\pi(u_1^2+u_2^2)}\delta(\rho-u_1)\,du_1du_2 \\
&= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{-\pi u_1^2}\delta(\rho-u_1)\,du_1\right)e^{-\pi u_2^2}\,du_2 \\
&= \int_{-\infty}^{\infty} e^{-\pi\rho^2}e^{-\pi u_2^2}\,du_2 \quad \text{(by the sifting property of } \delta) \\
&= e^{-\pi\rho^2}\int_{-\infty}^{\infty} e^{-\pi u_2^2}\,du_2 \\
&= e^{-\pi\rho^2} \quad \text{(because the Gaussian is normalized to have area 1)}
\end{aligned}$$

Writing this in polar coordinates, $r = x_1^2 + x_2^2$, we have shown that

$$\mathcal{R}(e^{-\pi r^2}) = e^{-\pi\rho^2}\,.$$

How about that.

Linearity, Shifts, and Evenness We need a few general properties of the Radon transform.

Linearity: $\mathcal{R}(\alpha f + \beta g) = \alpha\mathcal{R}(f) + \beta\mathcal{R}(g)$. This holds because integration is a linear function of the integrand.

Shifts: This is a little easier to write (and to derive) in vector form. Let $\mathbf{n} = (\cos\phi, \sin\phi)$. The result is

$$\mathcal{R}(\mu(\mathbf{x} - \mathbf{b})) = (\mathcal{R}\mu)(\rho - \mathbf{b}\cdot\mathbf{n}, \phi)$$

In words: shifting $\mathbf{x}$ by $\mathbf{b}$ has the effect of shifting each projection a distance $\mathbf{b}\cdot\mathbf{n}$ in the ρ-variable.

To derive this we write the definition as

$$\mathcal{R}(\mu(\mathbf{x} - \mathbf{b})) = \int_{\mathbf{R}^2} \mu(\mathbf{x} - \mathbf{b})\delta(\rho - \mathbf{x}\cdot\mathbf{n})\, d\mathbf{x}$$

If $\mathbf{b} = (b_1, b_2)$ then the change of variable $u_1 = x_1 - b_1$ and $u_2 = x_2 - b_2$, or simply $\mathbf{u} = \mathbf{x} - \mathbf{b}$ with $\mathbf{u} = (u_1, u_2)$, converts this integral into

$$\begin{aligned}
\mathcal{R}(\mu(\mathbf{x} - \mathbf{b})) &= \int_{\mathbf{R}^2} \mu(\mathbf{u})\delta(\rho - (\mathbf{u} + \mathbf{b})\cdot\mathbf{n})\, d\mathbf{u} \\
&= \int_{\mathbf{R}^2} \mu(\mathbf{u})\delta(\rho - \mathbf{u}\cdot\mathbf{n} - \mathbf{b}\cdot\mathbf{n}))\, d\mathbf{u} \\
&= (\mathcal{R}\mu)(\rho - \mathbf{b}\cdot\mathbf{n}, \phi)
\end{aligned}$$

Evenness: Finally, the Radon transform always has a certain symmetry — it is always an even function of ρ and ϕ. This means that

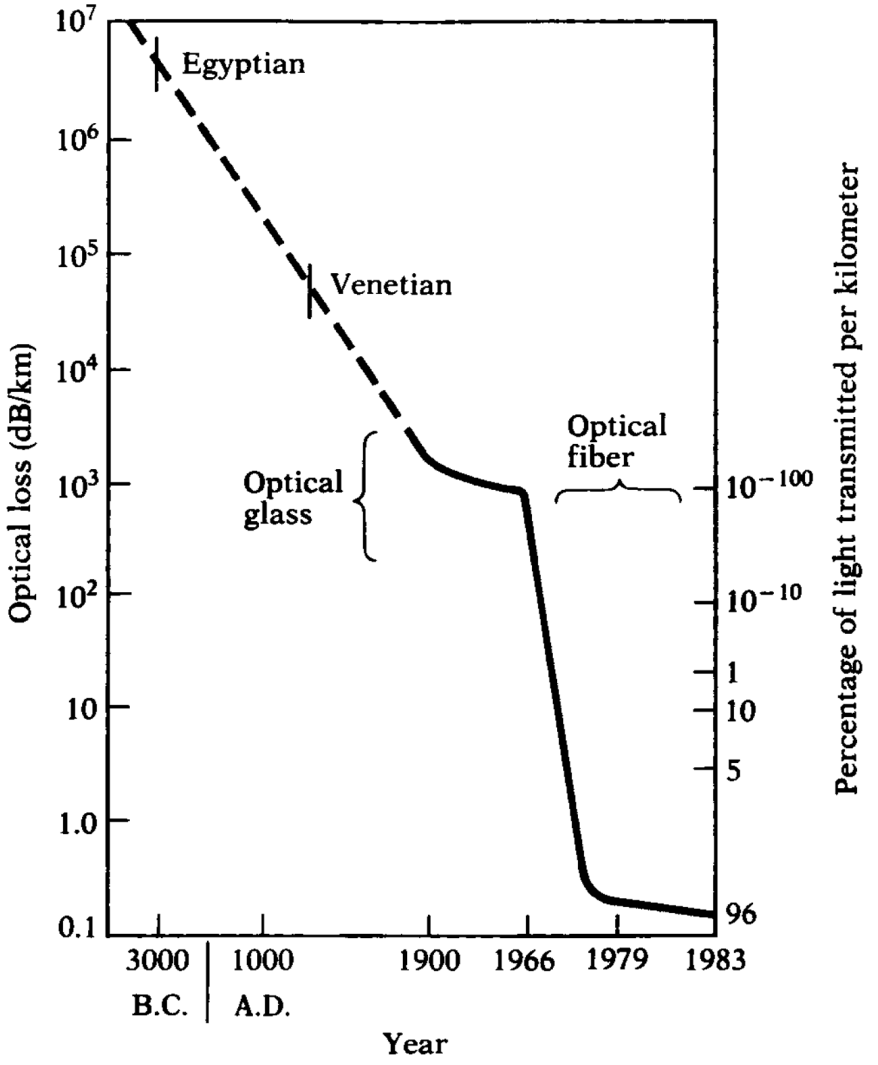
$$\mathcal{R}\mu(-\rho, \phi + \pi) = \mathcal{R}\mu(\rho, \phi)\,.$$

Convince yourself that this makes sense in terms of the projections. The derivation goes:

$$\begin{aligned}
\mathcal{R}\mu(-\rho, \phi + \pi) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu(x_1, x_2)\delta(-\rho - x_1\cos(\phi + \pi) - x_2\sin(\phi + \pi))\, dx_1\, dx_2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu(x_1, x_2)\delta(-\rho - x_1(-\cos\phi) - x_2(-\sin\phi))\, dx_1\, dx_2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu(x_1, x_2)\delta(-\rho + x_1\cos\phi + x_2\sin\phi)\, dx_1\, dx_2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu(x_1, x_2)\delta(\rho - x_1\cos\phi - x_2\sin\phi)\, dx_1\, dx_2 \quad \text{(because } \delta \text{ is even)} \\
&= \mathcal{R}\mu(\rho, \phi)
\end{aligned}$$

8.10 Appendix: Clarity of Glass

Here's a chart showing how the clarity of glass has improved over the ages, with some poetic license in estimating the clarity of the windows of ancient Egypt. Note that on the vertical axis on the left the tick marks are powers of 10 but the units are in decibels — which already involve taking a logarithm! The big jump in clarity going to optical fibers was achieved largely by eliminating water in the glass.



8.11 Medical Imaging: Inverting the Radon Transform

Let's recall the setup for tomography. We have a two-dimensional region (a slice of a body) and a density function $\mu(x_1, x_2)$ defined on the region. The Radon transform of μ is obtained by integrating μ along lines that cut across the region. We write this as

$$\mathcal{R}\mu(\rho, \phi) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mu(x_1, x_2)\delta(\rho - x_1 \cos\phi - x_2 \sin\phi)\, dx_1\, dx_2\,.$$

Here (ρ, ϕ) are coordinates that specify a line; ϕ $(0 \le \phi < \pi)$ is the angle the *normal* to the line makes with the x_1-axis and ρ $(-\infty < \rho < \infty)$ is the directed distance of the line from the origin. $\delta(\rho - x_1 \cos\phi - x_2 \sin\phi)$ is a line impulse, a δ-function along the line whose (Cartesian) equation is $\rho - x_1 \cos\phi - x_2 \sin\phi = 0$.

If we fix ϕ and vary ρ, then $\mathcal{R}\mu(\rho, \phi)$ is a collection of integrals along parallel lines through the region, all making the same angle, $\phi + \pi/2$, with a reference axis, the x_1-axis. This set of values is referred to as a *projection* of μ. Thus one often speaks of the Radon transform as a collection of projections parameterized by an angle ϕ.

In practice $\mu(x_1, x_2)$ is unknown, and what is available are the values $\mathcal{R}\mu(\rho, \phi)$. These values (or rather a constant times the exponential of these values) are what your detector registers when an X-ray reaches

it having gone through the region and having been attenuated according to its encounter with $\mu(x_1, x_2)$. The problem is to reconstruct $\mu(x_1, x_2)$ from these meter readings, in other words to invert the Radon transform.

Among those who use these techniques, $\mu(x_1, x_2)$ is often referred to simply as an *image*. In that terminology the problem is then "to reconstruct the image from its projections".

The Projection-Slice Theorem The inversion problem is solved by a result that relates the *two-dimensional* Fourier transform of μ to a *one-dimensional Fourier transform* of $\mathcal{R}(\mu)$, *taken with respect to* ρ. Once $\mathcal{F}\mu$ is known, μ can be found by Fourier inversion.

The formulation of this relation between the Fourier transforms of an image and its projections is called the *Projection-Slice Theorem*[19] and is the cornerstone of tomography. We'll go through the derivation, but it must be said at once that, for practical applications, all of this has to be implemented *numerically*, i.e., with the DFT (and the FFT). Much of the early work in Computer Assisted Tomography (CAT) was in finding efficient algorithms for doing just this. An important issue are the errors introduced by approximating the transforms, termed *artifacts* when the reconstructed image $\mu(x_1, x_2)$ is drawn on a screen. We won't have time to discuss this aspect of the problem.

Starting with

$$\mathcal{R}\mu(\rho, \phi) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu(x_1, x_2)\delta(\rho - x_1\cos\phi - x_2\sin\phi)\, dx_1\, dx_2\,,$$

what is its Fourier transform with respect to ρ, regarding ϕ as fixed? For lack of a better notation, we write this as $\mathcal{F}_\rho(\mathcal{R}(\mu))$. Calling the frequency variable r — dual to ρ — we then have

$$\begin{aligned}
\mathcal{F}_\rho\mathcal{R}(\mu)(r,\phi) &= \int_{-\infty}^{\infty} e^{-2\pi i r\rho}\mathcal{R}\mu(\rho,\phi)\, d\rho \\
&= \int_{-\infty}^{\infty} e^{-2\pi i r\rho}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu(x_1, x_2)\delta(\rho - x_1\cos\phi - x_2\sin\phi)\, dx_1\, dx_2\, d\rho \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu(x_1, x_2)\left(\int_{-\infty}^{\infty} \delta(\rho - x_1\cos\phi - x_2\sin\phi)e^{-2\pi i r\rho}\, d\rho\right) dx_1\, dx_2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu(x_1, x_2)e^{-2\pi i r(x_1\cos\phi + x_2\sin\phi)}\, dx_1\, dx_2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu(x_1, x_2)e^{-2\pi i (x_1 r\cos\phi + x_2 r\sin\phi)}\, dx_1\, dx_2
\end{aligned}$$

Check out what happened here: By interchanging the order of integration we wind up integrating the line impulse against the complex exponential $e^{-2\pi i r\rho}$. For that integration we can regard $\delta(\rho - x_1\cos\phi - x_2\sin\phi)$ as a shifted δ-function, and the integration with respect to ρ produces $e^{-2\pi i(x_1 r\cos\phi + x_2 r\sin\phi)}$. Now if we let

$$\begin{aligned}
\xi_1 &= r\cos\phi \\
\xi_2 &= r\sin\phi
\end{aligned}$$

the remaining double integral is

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi i(x_1\xi_1 + x_2\xi_2)}\mu(x_1, x_2)\, dx_1\, dx_2 = \int_{\mathbf{R}^2} e^{-2\pi i\mathbf{x}\cdot}\,\mu(\mathbf{x})\, d\mathbf{x}\,.$$

[19] Also called the Central Slice Theorem, or the Center Slice theorem.

This is the two-dimensional Fourier transform of μ.

We have shown

- The Projection-Slice Theorem:

$$\mathcal{F}_\rho \mathcal{R}(\mu)(r,\phi) = \mathcal{F}\mu(\xi_1,\xi_2), \quad \xi_1 = r\cos\phi,\ \xi_2 = r\sin\phi\,.$$

Observe that

$$r^2 = \xi_1^2 + \xi_2^2 \quad \text{and} \quad \tan\phi = \frac{\xi_2}{\xi_1}\,.$$

This means that (r,ϕ) are polar coordinates for the (ξ_1,ξ_2)-frequency plane. As ϕ varies between 0 and π (including 0, excluding π) and r between $-\infty$ and ∞ we get all the points in the plane.

Reconstructing the image That last derivation happened pretty fast. Let's unpack the steps in using the projection-slice theorem to reconstruct an image from its projections.

1. We have a source and a sensor that rotate about some center. The angle of rotation is ϕ, where $0 \le \phi < \pi$.

2. A family of parallel X-rays pass from the source through a (planar) region of unknown, variable density, $\mu(x_1,x_2)$, and are registered by the sensor.

   For each ϕ the readings at the meter thus give a function $g_\phi(\rho)$ (or $g(\rho,\phi)$), where ρ is the (directed) distance that a particular X-ray is from the center of the beam of parallel X-rays.

   Each such function g_ϕ, for different ϕ's, is called a projection.

3. For each ϕ we compute $\mathcal{F}g_\phi(r)$, i.e., the Fourier transform of $g_\phi(\rho)$ with respect to ρ.

4. Since $g_\phi(\rho)$ also depends on ϕ so does its Fourier transform. Thus we have a function of two variables, $G(r,\phi)$, the Fourier transform of $g_\phi(\rho)$. The projection-slice theorem tells us that this is the Fourier transform of μ:

$$\mathcal{F}\mu(\xi_1,\xi_2) = G(r,\phi), \quad \text{where} \quad \xi_1 = r\cos\phi,\ \xi_2 = r\sin\phi\,.$$

   Thus $(\mathcal{F}\mu)(\xi_1,\xi_2)$ is *known*.

5. Now take the inverse two-dimensional Fourier transform to recover μ:

$$\mu(\mathbf{x}) = \int_{\mathbf{R}^2} e^{2\pi i \mathbf{x}\cdot}\, \mathcal{F}\mu(\boldsymbol{\xi})\, d\boldsymbol{\xi}\,.$$

Running the numbers Very briefly, let's go through how one might set up a numerical implementation of the procedure we've just been through. The function that we know is $g(\rho,\phi)$ — that's what the sensor gives us, at least in discrete form. To normalize things we suppose that $g(\rho,\phi)$ is zero for $|\rho| \ge 1$. This means, effectively, that the region we're passing rays through is contained within the circle of radius one — the region is bounded so we can assume that it lies within some disk, so we scale to assume the the region lies within the unit disk.

Suppose we have M equal angles, $\phi_j = j\pi/M$, for $j = 0, \ldots, M-1$. Suppose next that for each angle we send through N X-rays. We're assuming that $-1 \le \rho \le 1$, so the rays are spaced $\Delta\rho = 2/N$ apart and we index them to be

$$\rho_n = \frac{2n}{N}\,, \quad n = -\frac{N}{2}, \ldots, \frac{N}{2} - 1\,.$$

Then our projection data are the MN values

$$g_{nj} = g(\rho_n, \phi_j)\,, \quad j = 0, \ldots, M-1\,,\; n = -\frac{N}{2}, \ldots, \frac{N}{2} - 1\,.$$

The first step in applying the projection slice theorem is to find the one-dimensional Fourier transform of $g(\rho, \phi_j)$ with respect to ρ, which, since the function is zero for $|\rho| \ge 1$, is the integral

$$\mathcal{F}g(r, \phi_j) = \int_{-1}^{1} e^{-2\pi i r\rho} g(\rho, \phi_j)\, d\rho\,.$$

We have to approximate and discretize the integral. One approach to this is very much like the one we took in obtaining the DFT (Chapter 6). First, we're integrating with respect to ρ, and we already have sample points at the $\rho_n = 2n/N$; evaluating g at those points gives exactly $g_{nj} = g(\rho_n, \phi_j)$. We'll use these for a trapezoidal rule approximation.

We also have to discretize in r, the "frequency variable" dual to ρ. According to the sampling theorem, if we want to reconstruct $\mathcal{F}g(r, \phi_j)$ from its samples in r the sampling rate is determined by the extent of $g(\rho, \phi_j)$ in the spatial domain, where the variable ρ is limited to $-1 \le \rho \le 1$. So the sampling rate in r is 2 and the sample points are spaced $1/2$ apart:

$$r_m = \frac{m}{2}\,, \quad m = -\frac{N}{2}, \ldots, \frac{N}{2} - 1\,.$$

The result of the trapezoidal approximation using $\rho_n = 2n/N$ and of discretizing in r using $r_m = m/2$ is

$$\begin{aligned}\mathcal{F}g(r_m, \phi_j) &\approx \frac{2}{N} \sum_{n=-N/2+1}^{N/2} e^{-2\pi i \rho_n r_m} g_{nj} \\ &= \frac{2}{N} \sum_{n=-N/2+1}^{N/2} e^{-2\pi i n m/N} g_{nj}\,.\end{aligned}$$

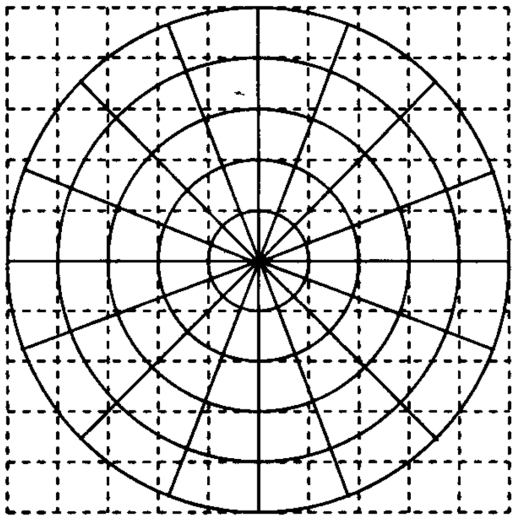
(The 2 in $2/N$ comes in from the form of the trapezoidal rule.) Up to the constant out front, this is a DFT of the sequence (g_{nj}), $n = -N/2+1, \ldots, N/2$. (Here n is varying, while j indexes the projection.) That is,

$$\mathcal{F}g(r_m, \phi_j) \approx \frac{2}{N} \underline{\mathcal{F}}\,(g_{nj})[m]\,.$$

Computing this DFT for each of the M projections ϕ_j $(j = 0, \ldots, M-1)$ gives the data $\mathcal{F}g(r_m, \phi_j)$. Call this

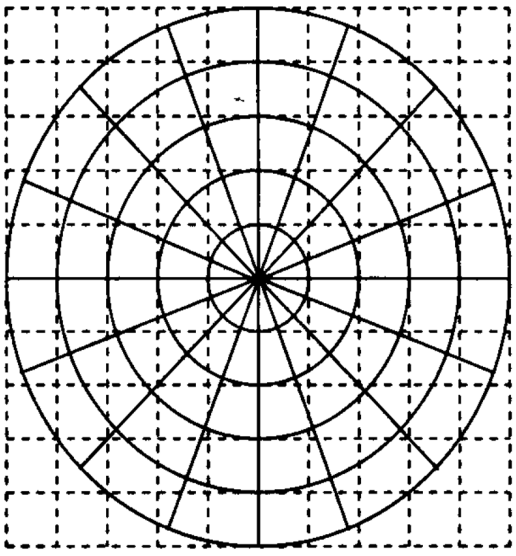
$$G_{mj} = \underline{\mathcal{F}}\,(g_{nj})[m]\,.$$

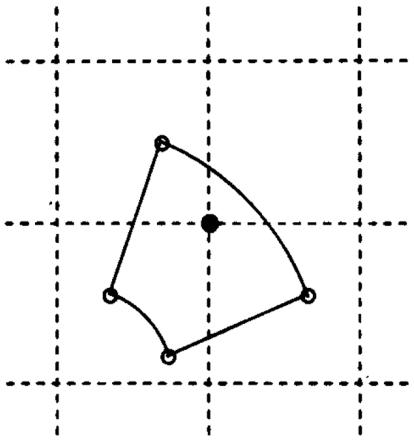
The next step is to take the two-dimensional *inverse* Fourier transform of the data G_{mj}. Now there's an interesting problem that comes up in implementing this efficiently. The G_{mj} are presented as data points based on a *polar coordinate* grid in the frequency domain:



The vertices in this picture are the points (r_m, ϕ_j) and that's where the data points G_{mj} live. However, efficient FFT algorithms depend on the data being presented on a *Cartesian* grid. One way this is often done is to manufacture data at Cartesian grid points by taking a weighted average of the G_{mj} at the polar grid points which are nearest neighbors:

$$G_{\text{Cartesian}} = w_a G_a + w_b G_b + w_c G_c + w_d G_d \,.$$



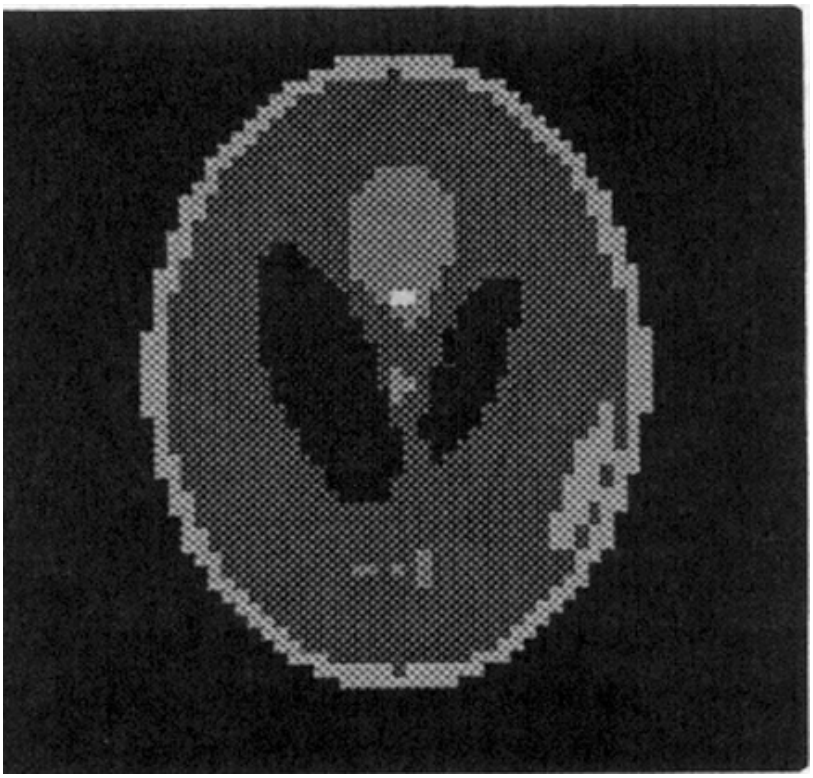


Choosing the weighting factors w_a, w_b, w_c and w_c is part of the art, but *the most significant introductions of error in the whole process come from this step.*

The final picture is then created by

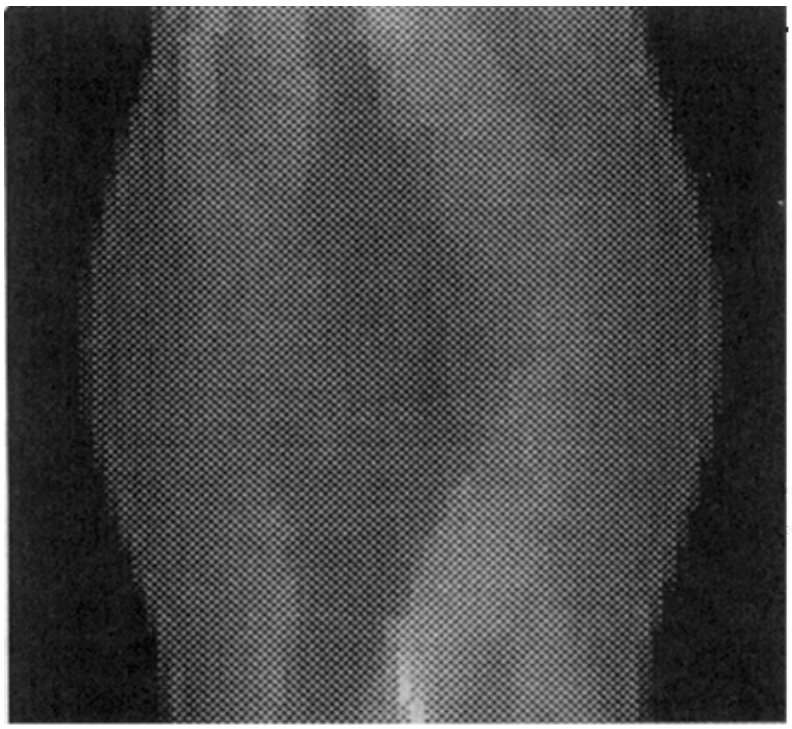
$$\mu(\text{grid points in spatial domain}) = \underline{\mathcal{F}}^{-1}(G_{\text{Cartesian}}) \,.$$

This is your brain. This is your brain on Fourier transforms Here are some pictures of a Fourier reconstruction of a model brain.[20]. The "brain" is modeled by a high density elliptical shell (the skull) with lower density elliptical regions inside.

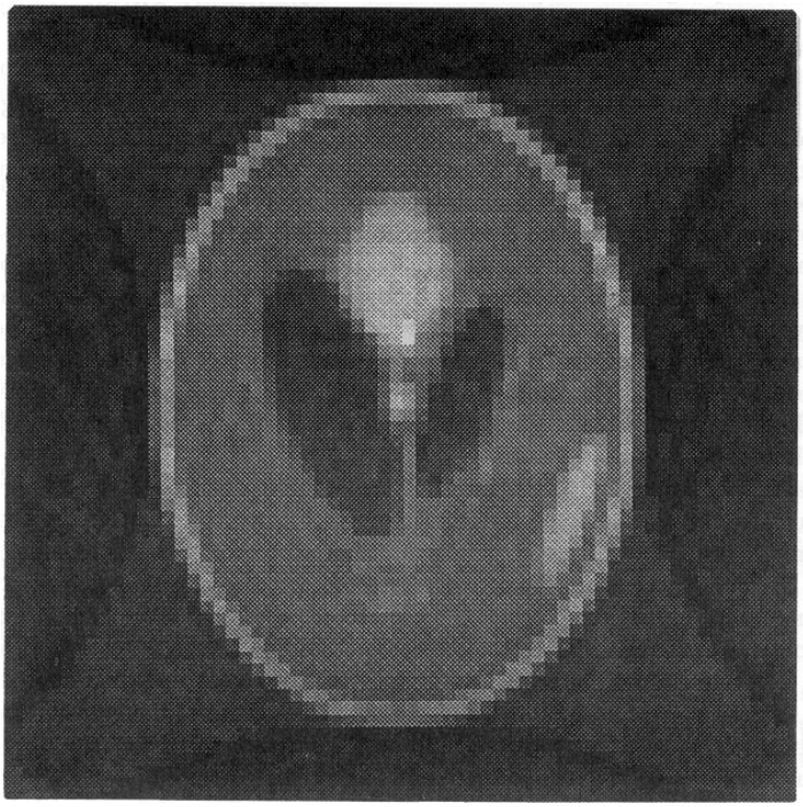


It's possible to compute explicity the Radon transform for lines going through an elliptical region, so the sampling can be carried out based on these formulas. There are 64 projections (64 ϕ_j's) each sampled at 64 points (64 ρ_n's) in the interval $[-1, 1]$. Here's the plot of the values of the projections (the Radon transforms along the lines). As in pictures of the (Fourier) spectrum of images, the values here are represented via shading; white represents large values and black represents small values. The horizontal axis is ρ and the vertical is ϕ.

[20] See the paper: L A. Shepp and B. F. Logan, The Fourier reconstruction of a head section, *IEEE Trans. Nucl. Sci.*, NS-21 (1974) 21–43.



And here is the reconstructed brain.



Appendix A

Mathematical Background

A.1 Complex Numbers

These notes are intended as a summary and review of complex numbers. I'm assuming that the definition, notation, and arithmetic of complex numbers are known to you, but we'll put the basic facts on the record. In the course we'll also use calculus operations involving complex numbers, usually complex valued functions of a real variable. For what we'll do, this will *not* involve the area of mathematics referred to as "Complex Analysis".

For our purposes, the extensions of the formulas of calculus to complex numbers are straightforward and reliable.

> **Declaration of principles** Without apology I will write
>
> $$i = \sqrt{-1}\,.$$
>
> In many areas of science and engineering it's common to use j for $\sqrt{-1}$. If you want to use j in your own work I won't try to talk you out of it. But I'll use i.

Before we plunge into notation and formulas there are two points to keep in mind:

- Using complex numbers *greatly* simplifies the algebra we'll be doing. This isn't the only reason they're used, but it's a good one.
- We'll use complex numbers to represent real quantities — real signals, for example. At this point in your life this should not cause a metaphysical crisis, but if it does my only advice is *to get over it.*

Let's go to work.

Complex numbers, real and imaginary parts, complex conjugates A *complex number* is determined by two real numbers, its *real* and *imaginary* parts. We write

$$z = x + iy$$

where x and y are real and

$$i^2 = -1\,.$$

x the real part and y is the imaginary part, and we write $x = \operatorname{Re} z$, $y = \operatorname{Im} z$. Note: it's y that is the imaginary part of $z = x + iy$, not iy. One says that iy is an *imaginary number* or is *purely imaginary.* One

says that z has positive real part (resp., positive imaginary part) if x (resp., y) is positive. The set of all complex numbers is denoted by $\mathbf{C}$. (The set of all real numbers is denoted by $\mathbf{R}$.)

Elementary operations on complex numbers are defined according to what happens to the real and imaginary parts. For example, if $z = a + ib$ and $w = c + di$ then their sum and product are given by

$$z + w = (a + c) + (b + d)i$$
$$zw = (ac - bd) + i(ad + bc)$$

I'll come back to the formula for the general quotient z/w, but here's a particular little identity that's used often: Since $i \cdot i = i^2 = -1$ we have

$$\frac{1}{i} = -i \quad \text{and} \quad i(-i) = 1\,.$$

The *complex conjugate* of $z = x + iy$ is

$$\bar{z} = x - iy\,.$$

Other notations for the complex conjugate are z^* and sometimes even $z^\dagger$. It's useful to observe that

$$z = \bar{z} \quad \text{if and only if } z \text{ is real, i.e., } y = 0.$$

Note also that

$$\overline{z + w} = \overline{z} + \overline{w}\,, \quad \overline{zw} = \overline{z}\,\overline{w}\,, \quad \overline{\overline{z}} = z\,.$$

We can find expressions for the real and imaginary parts of a complex number using the complex conjugate. If $z = x + iy$ then $\overline{z} = x - iy$ so that in the sum $z + \overline{z}$ the imaginary parts cancel. That is $z + \overline{z} = 2x$, or

$$x = \operatorname{Re} z = \frac{z + \overline{z}}{2}\,.$$

Similarly, in the difference, $z - \bar{z}$, the real parts cancel and $z - \bar{z} = 2iy$, or

$$y = \operatorname{Im} z = \frac{z - \bar{z}}{2i}\,.$$

Don't forget the i in the denominator here.

The formulas $\overline{z + w} = \bar{z} + \bar{w}$ and $\overline{zw} = \bar{z}\bar{w}$ extend to sums and products of more than two complex numbers, and to integrals (being limits of sums), leading to formulas like

$$\overline{\int f(t)g(t)\,dt} = \int \overline{f(t)}\,\overline{g(t)}\,dt \quad \text{(here } dt \text{ is a real quantity.)}$$

This overextended use of the overline notation for complex conjugates shows why it's useful to have alternate notations, such as

$$\left(\int f(t)g(t)\,dt\right)^* = \int f(t)^* g(t)^*\,dt\,.$$

It's best not to mix stars and bars in a single formula, so please be mindful of this. I wrote these formulas for "indefinite integrals" but in our applications it will be definite integrals that come up.

The *magnitude* of $z = x + iy$ is

$$|z| = \sqrt{x^2 + y^2}\,.$$

Multiplying out the real and imaginary parts gives

$$z\bar{z} = (x + iy)(x - iy) = x^2 - i^2y^2 = x^2 + y^2 = |z|^2\,.$$

This formula comes up all the time.

More generally,

$$|z + w|^2 = |z|^2 + 2\,\mathrm{Re}\{z\bar{w}\} + |w|^2 \quad \text{which is also} \quad |z|^2 + 2\,\mathrm{Re}\{\bar{z}w\} + |w|^2\,.$$

To verify this,

$$\begin{aligned} |z + w|^2 &= (z + w)(\bar{z} + \bar{w}) \\ &= z\bar{z} + z\bar{w} + w\bar{z} + w\bar{w} \\ &= |z|^2 + (z\bar{w} + +\overline{z\bar{w}}) + |w|^2 \quad \text{which is also} \quad |z|^2 + (\bar{z}w + \overline{\bar{z}w}) + |w|^2. \end{aligned}$$

The quotient z/w For people who really need to find the real and imaginary parts of a quotient z/w here's how it's done. Write $z = a + bi$ and $w = c + di$. Then

$$\begin{aligned} \frac{z}{w} &= \frac{a + bi}{c + di} \\ &= \frac{a + bi}{c + di}\,\frac{c - di}{c - di} \\ &= \frac{(a + bi)(c - di)}{c^2 + d^2} = \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2}\,. \end{aligned}$$

Thus

$$\mathrm{Re}\,\frac{a + bi}{c + di} = \frac{ac + bd}{c^2 + d^2}\,, \quad \mathrm{Im}\,\frac{a + bi}{c + di} = \frac{bc - ad}{c^2 + d^2}\,.$$

Do *not* memorize this. Remember the "multiply the top and bottom by the conjugate" sort of thing.

Polar form Since a complex number is determined by two real numbers it's natural to associate $z = x+iy$ with the pair $(x, y) \in \mathbf{R}^2$, and hence to identify z with the point in the plane with Cartesian coordinates (x, y). One also then speaks of the "real axis" and the "imaginary axis".

We can also introduce polar coordinates r and θ and relate them to the complex number $z = x+iy$ through the equations

$$r = \sqrt{x^2 + y^2} = |z| \quad \text{and} \quad \theta = \tan^{-1}\frac{y}{x}\,.$$

The angle θ is called the *argument* or the *phase* of the complex number. One sees the notation

$$\theta = \arg z \quad \text{and also} \quad \theta = \angle z\,.$$

Going from polar to Cartesian through $x = r\cos\theta$ and $y = r = \sin\theta$, we have the *polar form* of a complex number:

$$x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)\,.$$

A.2 The Complex Exponential and Euler's Formula

The real workhorse for us will be the complex exponential function. The exponential function e^z for a complex number z is defined, just as in the real case, by the Taylor series:

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{n!}\,.$$

This converges for all $z \in \mathbf{C}$, but we won't check that.

Notice also that

$$\begin{aligned} \overline{e^z} &= \overline{\left(\sum_{n=0}^{\infty} \frac{z^n}{n!}\right)} \quad \text{(a heroic use of the bar notation)} \\ &= \sum_{n=0}^{\infty} \frac{\bar{z}^n}{n!} \\ &= e^{\bar{z}} \end{aligned}$$

Also, e^z satisfies the differential equation $y' = y$ with initial condition $y(0) = 1$ (this is often taken as a definition, even in the complex case). By virtue of this, one can verify the key algebraic properties:

$$e^{z+w} = e^z e^w$$

Here's how this goes. Thinking of w as fixed,

$$\frac{d}{dz} e^{z+w} = e^{z+w}$$

hence e^{z+w} must be a constant multiple of e^z;

$$e^{z+w} = c e^z\,.$$

What is the constant? At $z = 0$ we get

$$e^w = c e^0 = c\,.$$

Done. Using similar arguments one can show the other basic property of exponentiation,

$$(e^z)^r = e^{zr}$$

if r is real. It's actually a tricky business to define $(e^z)^w$ when w is complex (and hence to establsh $(e^z)^w = e^{zw}$). This requires introducing the *complex* logarithm, and special considerations are necessary. We will not go into this.

The most remarkable thing happens when the exponent is purely imaginary. The result is called Euler's formula and reads

$$e^{i\theta} = \cos\theta + i \sin\theta\,.$$

I want to emphasize that the left hand side has only been defined via a series. The exponential function in the real case has nothing to do with the trig functions sine and cosine, and why it should have anything

to do with them in the complex case is a true wonder.[1]

Plugging $\theta = \pi$ into Euler's formula gives $e^{i\pi} = \cos\pi + i\sin\pi = -1$, better written as

$$e^{i\pi} + 1 = 0\,.$$

This is sometimes referred to as the most famous equation in mathematics; it expresses a simple relationship — and why should there be any at all? — between the fundamental numbers e, π, 1, and 0, not to mention i. We'll probably never see this most famous equation again, but now we've seen it once.

Consequences of Euler's formula The polar form $z = r(\cos\theta + i\sin\theta)$ can now be written as

$$z = re^{i\theta}\,,$$

where $r = |z|$ is the magnitude and θ is the phase of the complex number z. Using the arithmetic properties of the exponential function we also have that if $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$ then

$$z_1z_2 = r_1r_2e^{i(\theta_1+\theta_2)}\,.$$

That is, the magnitudes multiply and the arguments (phases) add.

Euler's formula also gives a dead easy way of deriving the addition formulas for the sine and cosine. On the one hand,

$$\begin{aligned} e^{i(\alpha+\beta)} &= e^{i\alpha}e^{i\beta} \\ &= (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) \\ &= (\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta). \end{aligned}$$

On the other hand,

$$e^{i(\alpha+\beta)} = \cos(\alpha+\beta) + i\sin(\alpha+\beta)\,.$$

Equating the real and imaginary parts gives

$$\begin{aligned} \cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \end{aligned}$$

I went through this derivation because it expresses in a simple way an extremely important principle in mathematics and its applications.

---

[1] Euler's formula is usually proved by substituting into and manipulating the Taylor series for $\cos\theta$ and $\sin\theta$. Here's another more elegant way of seeing it. It relies on results for differential equations, but the proofs of those are no more difficult that the proofs of the properties of Taylor series that one needs in the usual approach. Let $f(\theta) = e^{i\theta}$. Then $f(0) = 1$ and $f'(\theta) = ie^{i\theta}$, so that $f'(0) = i$. Moreover

$$f''(\theta) = i^2e^{i\theta} = -e^{i\theta} = -f(\theta)$$

i.e., f satisfies

$$f'' + f = 0, \quad f(0) = 1, f'(0) = i\,.$$

On the other hand if $g(\theta) = \cos\theta + i\sin\theta$ then

$$g''(\theta) = -\cos\theta - i\sin\theta = -g(\theta), \quad \text{or} \quad g'' + g = 0$$

and also

$$g(0) = 1, \quad g'(0) = i\,.$$

Thus f and g satisfy the same differential equation with the same initial conditions, so f and g must be equal. Slick. I prefer using the second order ordinary differential equation here since that's the one naturally associated with the sine and cosine. We could also do the argument with the first order equation $y' = y$. Indeed, if $f(\theta) = e^{i\theta}$ then $f'(\theta) = ie^{i\theta} = if(\theta)$ and $f(0) = 1$. Likewise, if $g(\theta) = \cos\theta + i\sin\theta$ then $g'(\theta) = -\sin\theta + i\cos\theta = i(cos\theta + i\sin\theta) = ig(\theta)$ and $g(0) = 1$. This implies that $f(\theta) = g(\theta)$ for all θ.

> If you can compute the same thing two different ways, chances are you've done something significant.

Take this seriously.[2]

Symmetries of the sine and cosine: even and odd functions Using the identity

$$\overline{e^{i\theta}} = e^{\overline{i\theta}} = e^{-i\theta}$$

we can express the cosine and the sine as the real and imaginary parts, respectively, of $e^{i\theta}$:

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Once again this is a simple observation. Once again there is something more to say.

You are very familiar with the symmetries of the sine and cosine function. That is, $\cos\theta$ is an *even function*, meaning

$$\cos(-\theta) = \cos\theta\,,$$

and $\sin\theta$ is an *odd function*, meaning

$$\sin(-\theta) = -\sin\theta\,.$$

Why is this true? There are many ways of seeing it (Taylor series, differential equations), but here's one you may not have thought of before, and it fits into a general framework of evenness and oddness that we'll find useful when discussing symmetries of the Fourier transform.

If $f(x)$ is *any* function, then the function defined by

$$f_e(x) = \frac{f(x) + f(-x)}{2}$$

is *even*. Check it out:

$$f_e(-x) = \frac{f(-x) + f(-(-x))}{2} = \frac{f(-x) + f(x)}{2} = f_e(x)\,.$$

Similarly, the function defined by

$$f_o(x) = \frac{f(x) - f(-x)}{2}$$

is *odd*. Moreover

$$f_e(x) + f_0(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2} = f(x)$$

The conclusion is that any function can be written as the sum of an even function and an odd function. Or, to put it another way, $f_e(x)$ and $f_o(x)$ are, respectively, the *even and odd parts* of $f(x)$, and the function is the sum of its even and odd parts. We can find some symmetries in a function even if it's not symmetric.

And what are the even and odd parts of the function $e^{i\theta}$? For the even part we have

$$\frac{e^{i\theta} + e^{-i\theta}}{2} = \cos\theta$$

and for the odd part we have

$$\frac{e^{i\theta} - e^{-i\theta}}{2} = i\sin\theta\,.$$

Nice.

---

[2] In some ways this is a maxim for the Fourier transform class. As we shall see, the Fourier transform allows us to view a signal in the time domain and in the frequency domain; two different representations for the same thing. Chances are this is something significant.

A.3 Algebra and Geometry

To wrap up this review I want to say a little more about the complex exponential and its use in representing sinusoids. To set the stage for this we'll consider the mix of algebra and geometry — one of the reasons why complex numbers are often so handy.

We not only think of a complex number $z = x + iy$ as a point in the plane, we also think of it as a vector with tail at the origin and tip at (x, y). In polar form, either written as $re^{i\theta}$ or as $r(\cos\theta + i\sin\theta)$, we recognize $|z| = r$ as the length of the vector and θ as the angle that the vector makes with the x-axis (the real axis). Note that

$$|e^{i\theta}| = |\cos\theta + i\sin\theta| = \sqrt{\cos^2\theta + \sin^2\theta} = 1\,.$$

Many is the time you will use $|e^{i(\text{something real})}| = 1$.

Once we make the identification of a complex number with a vector we have an easy back-and-forth between the algebra of complex numbers and the geometry of vectors. Each point of view can help the other.

Take addition. The sum of $z = a + bi$ and $w = c + di$ is the complex number $z + w = (a + c) + (c + d)i$. Geometrically this is given as the vector sum. If z and w are regarded as vectors from the origin then $z+w$ is the vector from the origin that is the diagonal of the parallelogram determined by z and w.

Similarly, as a vector, $z - w = (a - c) + (b - d)i$ is the vector that goes from the tip of w to the tip of z, i.e., along the other diagonal of the parallelogram determined by z and w. Notice here that we allow for the customary ambiguity in placing vectors; on the one hand we identify the complex number $z - w$ with the vector with tail at the origin and tip at $(a - c, b - d)$. On the other hand we allow ourselves to place the (geometric) vector anywhere in the plane as long as we maintain the same magnitude and direction of the vector.

It's possible to give a geometric interpretation of zw (where, you will recall, the magnitudes multiply and the arguments add) in terms of similar triangles, but we won't need this.

Complex conjugation also has a simple geometric interpretation. If $z = x + iy$ then the complex conjugate $\bar{z} = x - iy$ is the mirror image of z in the x-axis. Think either in terms of reflecting the point (x, y) to the point $(x, -y)$ or reflecting the vector. This gives a natural geometric reason why $z+\bar{z}$ is real — since z and $\bar{z}$ are symmetric about the real axis, the diagonal of the parallelogram determined by z and $\bar{z}$ obviously goes along the real axis. In a similar vein, $-\bar{z} = -(x - iy) = -x + iy$ is the reflection of $z = x + iy$ in the y-axis, and now you can see what $z - \bar{z}$ is purely imaginary.

There are plenty of examples of the interplay between the algebra and geometry of complex numbers, and the identification of complex numbers with points in the plane (Cartesian or polar coordinates) often leads to some simple approaches to problems in analytic geometry. Equations in x and y (or in r and θ) can often be recast as equations in complex numbers, and having access to the arithmetic of complex numbers frequently simplifies calculations.

A.4 Further Applications of Euler's Formula

We've already done some work with Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$, and we agree it's a fine thing to know. For additional applications we'll replace θ by t and think of

$$e^{it} = \cos t + i \sin t$$

as describing a point in the plane that is moving in time. How does it move? Since $|e^{it}| = 1$ for every t, the point moves along the unit circle. In fact, from looking at the real and imaginary parts separately,

$$x = \cos t, \quad y = \sin t$$

we see that e^{it} is a (complex-valued) parametrization of the circle; the circle is traced out exactly once in the *counterclockwise* direction as t goes from 0 to 2π. We can also think of the vector from the origin to z as rotating counterclockwise about the origin, like a (backwards moving) clock hand.

For our efforts I prefer to work with

$$e^{2\pi it} = \cos 2\pi t + i \sin 2\pi t$$

as the "basic" complex exponential. Via its real and imaginary parts, the complex exponential $e^{2\pi it}$ contains the sinusoids $\cos 2\pi t$ and $\sin 2\pi t$, each of frequency 1 Hz. If you like, including the 2π or not is the difference between working with frequency in units of Hz, or cycles per second, and "angular frequency" in units of radians per second. With the 2π, as t goes from 0 to 1 the point $e^{2\pi it}$ traces out the unit circle exactly once (one cycle) in a counterclockwise direction. The units in the exponential $e^{2\pi it}$ are (as they are in $\cos 2\pi t$ and $\sin 2\pi t$)

$$e^{2\pi \text{ radians/cycle} \cdot i \cdot 1 \text{ cycles/sec} \cdot t \text{ sec}}.$$

Without the 2π the units in e^{it} are

$$e^{i \cdot 1 \text{ radians/sec} \cdot t \text{ sec}}.$$

We can always pass easily between the "complex form" of a sinusoid as expressed by a complex exponential, and the real signals as expressed through sines and cosines. But for many, many applications, calculations, prevarications, etc., it is *far* easier to stick with the complex representation. As I said earlier in these notes, if you have philosophical trouble using complex entities to represent real entities the best advice I can give you is to *get over it*.

We can now feel free to change the amplitude, frequency, and to include a phase shift. The general (real) sinusoid is of the form, say, $A\sin(2\pi\nu t + \phi)$; the amplitude is A, the frequency is ν (in Hz) and the phase is ϕ. (We'll take A to be positive for this discussion.) The general complex exponential that includes this information is then

$$Ae^{i(2\pi\nu t + \phi)}.$$

Note that i is multiplies the entire quantity $2\pi\nu t + \phi$. The term *phasor* is often used to refer to the complex exponential $e^{2\pi i\nu t}$.

And what is $Ae^{i(2\pi\nu t+\phi)}$ describing as t varies? The magnitude is $|Ae^{i(2\pi i\nu t+\phi)}| = |A| = A$ so the point is moving along the circle of radius A. Assume for the moment that ν is positive — we'll come back to negative frequencies later. Then the point traces out the circle in the counterclockwise direction at a rate of ν cycles per second — 1 second is ν times around (including the possibility of a fractional number of times around). The phase ϕ determines the starting point on the circle, for at $t = 0$ the point is $Ae^{i\phi}$. In fact, we can write

$$Ae^{i(2\pi\nu t+\phi)} = e^{2\pi i\nu t}\, Ae^{i\phi}$$

and think of this as the (initial) vector $Ae^{i\phi}$ set rotating at a frequency ν Hz through multiplication by the time-varying phasor $e^{2\pi i\nu t}$.

What happens when ν is negative? That simply reverses the direction of motion around the circle from counterclockwise to clockwise. The catch phrase is just so: positive frequencies means counterclockwise rotation and negative frequencies means clockwise rotation. Now, we can write a cosine, say, as

$$\cos 2\pi\nu t = \frac{e^{2\pi i\nu t} + e^{-2\pi i\nu t}}{2}$$

and one sees this formula interpreted through statements like "a cosine is the sum of phasors of positive and negative frequency", or similar phrases. The fact that a cosine is made up of a positive and negative frequency, so to speak, is important for some analytical considerations, particularly having to do with the Fourier transform (and we'll see this phenomenon more generally), but I don't think there's a geometric interpretation of negative frequencies without appealing to the complex exponentials that go with real sines and cosines —"negative frequency" is clockwise rotation of a phasor, period.

Sums of sinusoids As a brief, final application of these ideas we'll consider the sum of two sinusoids of the same frequency.[3] In real terms, the question is what one can say about the superposition of two signals

$$A_1 \sin(2\pi\nu t + \phi_1) + A_2 \sin(2\pi\nu t + \phi_2)\,.$$

Here the frequency is the same for both signals but the amplitudes and phases may be different.

If you answer too quickly you might say that a phase shift between the two terms is what leads to beats. Wrong. Perhaps physical considerations (up to you) can lead you to conclude that the frequency of the sum is again ν. That's right, but it's not so obvious looking at the graphs of the individual sinusoids and trying to imagine what the sum looks like, e.g., (see graph below):

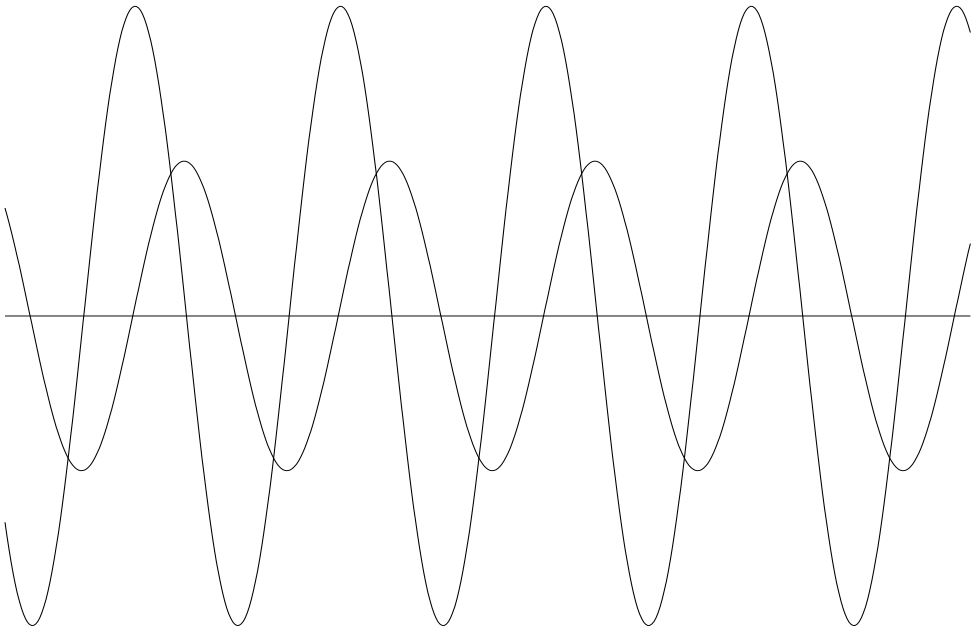


Figure A.1: Two sinusoids of the same frequency. What does their sum look like?

[3] The idea for this example comes from *A Digital Signal Processing Primer* by K. Stieglitz

An algebraic analysis based on the addition formulas for the sine and cosine does not look too promising either. But it's easy to see what happens if we use complex exponentials.

Consider

$$A_1 e^{i(2\pi\nu t+\phi_1)} + A_2 e^{i(2\pi\nu t+\phi_2)}$$

whose imaginary part is the sum of sinusoids, above. Before messing with the algebra, think geometrically in terms of rotating vectors. At $t = 0$ we have the two vectors from the origin to the starting points, $z_0 = A_1 e^{i\phi_1}$ and $w_0 = A_2 e^{i\phi_2}$. Their sum $z_0 + w_0$ is the starting point (or starting vector) for the sum of the two motions. But how do those two starting vectors move? They rotate together at the same rate, the motion of each described by $e^{2\pi i\nu t} z_0$ and $e^{2\pi i\nu t} w_0$, respectively. Thus their sum also rotates at that rate — think of the whole parallelogram (vector sum) rotating rigidly about the vertex at the origin. Now mess with the algebra and arrive at the same result:

$$A_1 e^{i(2\pi\nu t+\phi_1)} + A_2 e^{i(2\pi\nu t+\phi_2)} = e^{2\pi i\nu t}(A_1 e^{i\phi_1} + A_2 e^{i\phi_2})\,.$$

And what is the situation if the two exponentials are "completely out of phase"?

Of course, the simple algebraic manipulation of factoring out the common exponential does not work if the frequencies of the two terms are different. If the frequencies of the two terms are different . . . now *that* gets interesting.

Appendix B

Some References

Two books that have been used often as texts for 261 are:

R. M. Gray and J. W. Goodman *Fourier Transforms*, Kluwer, 1995

R. N. Bracewell, *The Fourier Transform and its Applications*, McGraw Hill, 1986

Gray and Goodman is the main reference for this version of the course and is at the bookstore as a 'recommended' book. The feature of Gray & Goodman that makes it different from most other books is the parallel treatment of the continuous and discrete cases throughout. Though we won't follow that approach *per se* it makes good parallel reading to what we'll do. Bracewell, now out in its third edition, is also highly recommended. Both books are on reserve in Terman library along with several others listed below.

Some other references (among *many*) are:

J. F. James, *A Student's Guide to Fourier Transforms*, Cambridge, 1995

This is a good, short book (130 pages), paralleling Bracewell to some extent, with about 70% devoted to various applications. The topics and examples are interesting and relevant. There are, however, some pretty obscure mathematical arguments, and some errors, too.

Jack D. Gaskill, *Linear Systems, Fourier Transforms, and Optics*, Wiley, 1978

This is sometimes used as a text for 261. The applications are drawn primarily from optics (nothing wrong with that) but the topics and treatment mesh very well with the course overall. Clearly written.

A. Papoulis, *The Fourier Transform and its Applications*, McGraw Hill, 1960

Same title as Bracewell's book, but a more formal mathematical treatment. Papoulis has written a whole slew of EE books. Two others that are relevant to the topics in this class are:

A. Papoulis, *Systems and Transforms With Applications in Optics*, Krieger Publishing Company, 1981

A. Papoulis, *Probability, Random Variables, and Stochastic Processes*, McGraw Hill, 1991

This last one has very general forms of the sampling theorem, including reconstruction by random sampling. Read this and be one of the few people on earth to know these results.

P. J. Nahim, *The Science of Radio*, 2nd edition, Springer, 2001

This is an interesting and entertaining book on the history and practice of radio. Of relevance to our course are treatments of the Fourier analysis of radio signals, from sparks to AM. The author's intention is to start from scratch and take a 'top down' approach.

Some references for the discrete Fourier transform and the fast Fourier transform algorithm are:

E. O. Brigham, *The Fast Fourier Transform*, Prentice Hall, 1974

This is a standard reference and I included it because of that; I think it's kind of clunky, however.

W. Briggs and V. Henson, *The DFT: An Owner's Manual for the Discrete Fourier Transform*, SIAM, 1995

I really like the treatment in this book; the topics, the examples, the problems are all well chosen.

A highly respected, advanced book on the FFT algorithm is

C. van Loam, *Computational Frameworks for the Fast Fourier Transform*, SIAM 1992

Books more often found on mathematician's shelves include:

H. Dym and H. P. McKean, *Fourier Series and Integrals*, Academic Press, 1972

This is a very well written, straightforward mathematical treatment of Fourier series and Fourier transforms. It includes a brief development of the theory of integration needed for the mathematical details (the L^2 and L^1 theory). Breezy style, but sophisticated.

T. W. Körner, *Fourier Analysis*, Cambridge, 1988

This is a good, long book (580 pages) full of the lore of Fourier analysis for mathematicians. It's written with a light touch with lots of illuminating comments.

R. Strichartz, *A Guide to Distribution Theory and Fourier Transforms*, CRC Press, 1994

This is an accessible introduction to distributions (generalized functions) and their applications, at the advanced undergraduate, beginning graduate level of mathematics. It's a good way to see how distributions and Fourier transforms have become fundamental in studying partial differential equations (at least for proving theorems, if not for computing solutions).

A. Terras, *Harmonic Analysis on Symmetric Spaces and Applications, I, II*, Springer Verlag, 1988

If you want to see how the Fourier transform is generalized to the setting of Lie groups, and why it's such a big deal in number theory, these books are an excellent source. Let me know if you believe the applications.

Index

n-dim Fourier series, 359, 361
    coefficients, 361
n-dim Fourier transform, 333
    convolution theorem, 351
    definition, 334
    general stretch theorem, 348
    inverse, 334
    linearity, 345
    notation, 333
    polar coordinates, 354
    properties, 345
    rotation theorem, 349
    shift and stretch theorem, 350
    shift theorem, 345
    spectrum, 338
    stretch theorem, 347
n-dim Fourier transform of
    delta function, 353
    radial function, 354
    separable functions, 341
n-dim Schwarz functions, 352
n-dim complex exponential, 360
n-dim complex exponentials, 335
n-dim convolution, 351
n-dim functions
    circ, 355
    delta, 352
    Gaussian, 344
    jinc, 356
    parallelogram rect, 350
    rect, 342
n-dim periodic functions, 359
n-dim stretch theorem derivation, 357
n-dim tempered distributions, 352

AC component, 9
Ahlfors Lars, 148
alias, 239
aliasing, 238, 245
analytic signal, 316
antenna radiation patterns, 201
artifacts, 395
attenuation coefficient, 385
averaging and least squares, 320

BachJ.S., 9
bandlimited function on $\mathbf{R}^2$, 380
bandwidth, 13, 224
basis
    dual, 376
    natural, 28
    orthonormal, 28
basis of $\mathbf{C}^N$
    complex exponentials, 265
    shifted delta, 263
bed of nails, 370
bell-shaped curve, 99
Bessel function of the first kind
    first-order, 356
    zero-order, 355
bit reversal, 289
    via permutation matrices, 289
Bracewell, 189, 201
Brownian motion, 362
butterfly diagram, 287
buzz, 47

Cantor G., 2
CAT, *see* Computer Assisted Tomography
Cauchy-Schwarz inequality, 33
    and matched filters, 310
causal function or signal, 312
causality, 311
    and linear systems, 311
    and linear time invariant systems, 311
    and physical realizability, 312
Central Limit Theorem, 116, 118, 129
Central Slice Theorem, *see* Projection-Slice Theorem
change of variables formula
    for multiple integrals, 357
circ function, 355
circularly (radially) symmetric function, 354
clarity of glass, 394
Clark Latimer, 45
compact support, 225
complex exponentials
    n-dim, 335
    eigenfunctions of LTI systems, 307
    operator decomposition, 308
complex inner product, 30, 36
Computational Frameworks for the Fast Fourier Transform, 285
Computer Assisted Tomography, 395
continuity of Fourier transform, 136
continuous Fourier transform
    of sampled signal, 251

convergence
  Fourier series, 50
  of power integral, 367
  pointwise, 53
  pointwise vs. uniform, 58
  uniform, 53
convergence rate
  Fourier series, 53
convolution, 43, 91
  and probability density functions, 125
  as a smoothing operation, 116, 196
  as smoothing and averaging, 96
  definition, 93
  derivative theorem, 196
  Dirichlet kernel, 60
  discrete, 273
  in frequency domain, 94
  interpretation, 95
  properties, 97
  theorem, 93
  visualization, 95
convolution integral, 43
Cooley J., 249
crystal lattice dual, 378
crystals, 378
cumulative density function (cdf), 121
cumulative probability, 121
cycle, 5

DC component, 9
  in discrete case, 257
deconvolution, 98
delta
  n-dim, 352
  along the line, 389
  approximating function sequences, 166
  as a tempered distribution, 163
  convolution property, 193
  derivative, 178
  discrete, *see* discrete delta
  Fourier transform, 171
  function, 98, 153
  function origins, 156
  Kronecker, 28
  product with a function, 185
  scaling, 191
  sifting property, 193
derivative of
  delta function, 178
  signum (sign) function, 177
  unit ramp function, 177
  unit step function, 176
DFT, *see* discrete Fourier transform
differential calculus, 175
differential equations
  solution via Fourier transform, 106
differential operator, 106
diffraction, 199
  by a single slit, 204
  by two point sources, 206
diffraction gratings, 209
diffusion, 39
  submarine telegraphy, 45, 109
digital filters, 319
  analysis in the frequency domain, 323
  computation in the frequency domain, 325
digital watermarking, 339
Dirac comb, 210
Dirac P., 139, 156
Dirichlet kernel, 49, 59
discrete delta, 263
  properties, 274
discrete filters
  band-pass, 327
  low-pass, 325
discrete Fourier transform, 249, 252
  alternative definition, 262, 275
  convolution, 273
  duality relations, 268
  general properties, 259, 271
  matrix form, 255
  modulation theorem, 272
  notation, 252
  of reversed signals, 266
  Parseval's identity, 271
  periodicity, 260
  shift theorem, 272
  vector form, 255
discrete Fourier transform of
  discrete rect, 330
  shifted deltas, 264
  vector complex exponential, 265
discrete linear systems examples
  matrix multiplication, 296
discrete signals
  periodic assumption, 261
  periodicity, 269
disrete convolution
  and random variable probability, 126
distribution
  normal (or Gaussian), 118, 124
  uniform, 123
distributions, 135, 139, 153
  as limits, 153
  as linear functionals, 157
  convolution, 193, 195
  derivative, 176
  duality relations, 183
  evenness and oddness, 184
  other classes, 164
  physical analogy, 165
  product with a function, 185
  reversal, 182
  tempered, 139

distributions and Fourier transform
convolution theorem, 192
derivative theorem, 187
shift theorem, 189
stretch theorem, 191
dual (reciprocal) basis, 376
dual lattice, 375

electron density distribution, 210
electrostatic capacity per unit length, 45
energy spectrum, 13, 73
Euler, 99
existence of Fourier transform, 135
exponential decay
one-sided, 77
two-sided, 88

far field diffraction, 203
Faraday Michael, 45
fast Fourier transform, 277
algorithm, 277
and Gauss, 258
butterfly diagram, 287
calculation, 280
computational complexity, 286
description, 281
divide and conquer approach, 284
sorting indices, 287
via matrix factorization, 285
Feynman, 199
FFT, *see* fast Fourier transform
filtering, 102
filters, 102
bandpass, 104
discrete, 325
highpass, 105
lowpass, 103
notch, 106
finite sampling theorem, 232
formula
integration by parts, 51
Fourier, 39
analysis, 1
coefficients, 11
finite series, 12
infinite series, 12
Fourier coefficients
as Fourier transform, 72
size, 50
Fourier inversion theorem, 72
Fourier optics, 199
Fourier pair, 74
Fourier reconstruction
model brain, 399
Fourier transform, 65
continuity, 136
definition, 71
duality, 79
existance, 135
magnitude and L^1 norm, 138, 141
motivation, 68
of a Fourier series, 220
of reversed signals, 81
polar coordinates, 354
Fourier transform of
$1/x$, 188
delta, 171
shah function, 214
shifted delta, 173
signum, 188
sine and cosine, 174
unit step, 188
Fourier transform properties
derivative formula, 106, 142
duality, 79
even and odd symmetry, 82
general shift and stretch theorem, 87
linearity, 83
shift theorem, 83
stretch (similarity) theorem, 84
Fraunhofer approximation (diffraction), 201
Fraunhofer diffraction, 202
frequencies
positive and negative, 256, 262
frequency domain
polar coordinate grid, 398
frequency response, 307
Friedrich and Kniping, 209
function
absolutely integrable, 36
compact support, 164
global properties, 53
local properties, 53
orthogonal projection, 38
rapidly decreasing, 140
scaling, 78
Schwartz class, 143
smoothness and decay, 140
square integrable, 36
functional analysis, 293
fundamental solution
heat flow on a circle, 43

Gauss, 99, 249
calculating orbit of an asteroid, 258
Gaussian function, 99, 117
general form, 100
heat kernel for infinite rod, 108
its Fourier transform, 100
periodization, 115
Gaussian integral evaluation, 101
generalized Fourier transform, 179
generalized functions, 139
generating function, 365

Gibbs J. W., 57
Gibbs phenomenon, 57
    square wave, 62
Goodman J. W., 199
Gray and Goodman, 189
Green's function
    for infinite rod heat flow, 109
    heat flow on a circle, 43

Hankel transform, 355
harmonic oscillator, 7
harmonics, 7, 21
    amplitude and phase, 9
    energy, 13
heat, 39
    flow, 39
heat equation, 40
    infinite rod, 107, 156
heat flow
    on a circle, 41, 113
    on an infinite rod, 107
    spot on earth, 43
heat kernel
    for infinite rod heat flow, 109
Heaviside, 156
Heaviside function, 176
Heaviside Oliver, 46, 139, 176
Heisenberg uncertainty principle, 132
Heisenberg Werner, 47
Heisenberg's inequality, 132
Helmholtz equation, 201
high-pass filter, 198
Hilbert transform, 178, 312
    and analytic signals, 316
    as an LTI system, 315
    as an operator, 313
    Cauchy principal value integral, 314
    inverse, 314
    of sinc, 318
histogram, 119
Huygens Christiaan, 200
Huygens' principle, 202

identity
    infinite sum of Gaussians, 114
IDFT, *see* inverse discrete Fourier transform
improper integral, 180
impulse function, 98, 153
impulse response, 102, 198, 300
    heat flow on a circle, 43
impulse train, 49
independence, 124
independent periodicity, 359
inner product
    complex-valued functions in L^2, 30
    geometric formula, 35
    in n-dim, 360
    real-valued functions in L^2, 29
    vectors in $\mathbf{R}^n$, 27
instantaneous frequency, 318
integer lattice, 361, 370
    self-dual, 375
integral
    convergence, 149
integration, 146
    contour and the residue theorem, 148
    positive functions, 146
interpolation
    general form, 228
    Lagrange, 229
inverse n-dim Fourier transform
    definition, 334
inverse discrete Fourier transform, 268
    matrix form, 269
inverse Fourier transform
    definition, 72
    motivation, 72
inverse Hilbert transform, 314

Jacobi theta function, 114
Jacobi's identity, 114
jinc function, 356
Joy of Convolution, 97

Körner T., 148
Kac Mark, 119
kernel
    Dirichlet, 49, 59
Kronecker delta, 28

Lagrange, 229
Lagrange interpolation polynomial, 230
Laplacian, 201
lattice, 217, 370
    and shah, 373
    area, 372
    dual, 375
    fundamental parallelogram, 372
    general, 372
    reciprocal, 375
    unit cell, 372
lattice sampling formula, 382
Law of squares, 46
least squares curve fitting, 258
Lebesgue, 23
    integral, 138
Lebesgue dominated convergence theorem, 149
Lebesgue integral, 148
Lectures on Physics, 199
light
    linearly polarized, 200
    monochromatic, 200
    point source, 207
    waves, 201
limits of distributions, 166

line
  (ρ, ϕ) parametrization, 387
  (m, b) parametrization, 387
  parametric description, 387
line impulse, 389
linear change of variables, 348
linear combination, 294
linear system
  additivity, 294
  homogeneity, 294
linear systems, 293
  and causality, 311
  and convolution, 297
  and translation, 298
  composition or cascade, 299
  impulse response, 300
  kernel, 296
  superposition theorem, 300
  via integration against a kernel, 296
linear systems examples
  integration, 296
  multiplication, 295
  periodization, 298
  sampling, 295
  switching, 295
linear time invariant systems
  and causality, 311
  and Fourier transform, 307
  definition, 302
  superposition theorem, 303
  transfer function, 307
Lippman G., 118
Lord Kelvin, 45
Lorenz profile curve, 78
Los Alamos, 47
LTIS, *see* linear time invariant systems

Markov process, 362
matched filter theorem, 310
matched filters, 309
matrix
  circulant, 298
  Hermitian, 270
  orthogonal, 270, 349
  rotation, 349
  symmetric, 270
  Toeplitz, 298
  unitary, 270
Maxwell's theory of electromagnetism, 156
measure, 148
  theory, 148
medical imaging, 384
  numerical computations, 397
  reconstruction, 396
Michael Frayn's play Copenhagen, 47
Michelson and Stratton's device, 57
minimum sampling rate, 227
module, 372
musical
  pitch, 8
  tuning, 8
musical tone, 47

narrowband signal, 317
Newton, 200
Newton's law of cooling, 40
nonrecursive filter, 321
norm
  L^1, 138
  L^2 (square), 23
normal approximation or distribution, 118
notch filter, 199
Nyquist frequency, 227
Nyquist Harry, 227

one-sided exponential decay, 77
operators, 153
ordinary differential equations
  solution, 107
orthogonality, 26
orthonormal basis, 28

Pólya G., 364
Parseval's theorem, 24
  for Fourier transforms, 73
period
  fundamental, 4
periodic distributions, 218
  and Fourier series, 217
periodic functions, 2, 4
  summation of, 5
periodicity
  and integer lattice, 371
  independent, 359
  spatial, 2
  temporal, 2
periodizing sinc functions, 235
permutation matrix, 282
perpendicularity, 26
plane wave field, 202
Poisson summation formula, 215
  n-dim case, 383
polar coordinates Fourier transform, 354
power integral convergence, 367
power spectrum, 13, 73
principal value distribution, 180
principle of superposition, 294
principle value integrals, 137
probability, 120, 121
  generating function, 365
probability density function (pdf), 119
probability distribution, 119
Projection-Slice Theorem, 395
projections, 390
Pythagorean theorem, 26

quadrature function, 315
quantum mechanics
  inequalities, 132
  momentum, 134
  observable, 134
  particle moving in one dimension, 134
  position of particle, 134

radial function, 354
Radon Johann, 388
Radon transform, 387, 388
  evenness, 393
  linearity, 393
  properties, 393
  shift, 393
Radon transform of
  circ, 391
  Gaussian, 392
random variable
  continuous, 119
  discrete, 119
  independent and identically distributed (iid), 128
  mean (or average), 121
  standrad deviation, 121
  variance, 121
random variables, 118
random vector, 363
random walk, 362
  recurrent, 365
  transient, 365
randomw walk
  theorem, 364
rapidly decreasing functions, 140, 142
  dual space, 158
  Fourier inversion, 144
  Fourier transform, 143
  Parseval identity, 145
ratio of sine functions, 236
Rayleigh's identity, 24, 32
reciprocal lattice, 375
reciprocal or dual lattice , 217
reciprocal relationship
  spatial and frequency domain, 336
reciprocity
  time and frequency domain, 15
rect function, 65
recursive filter, 321
resistance per unit length, 45
reversed signal, 81
  discrete case, 266
  Fourier transform, 81
Riemann integral, 25
Riemann-Lebesgue lemma, 139, 150
Riesz-Fischer theorem, 24
Roentgen William, 209
running average, 321

sampling, 209
  endpoint problem, 243
  for bandlimited periodic signal, 230
  in frequency domain, 251
  sines and bandlimited signals, 222
  with shah function, 213
sampling and interpolation, 222
  bandlimited signals, 225
sampling on a lattice, 380
sampling theorem, 226
Savage Sam, 118
sawtooth signal, 54
scaling operator, 191
Schwartz kernel, *see* impulse response
Schwartz Laurent, 139, 143
self-dual lattice, 375
separable functions, 341
separation of variables method
  in partial differential equations, 341
Serber Robert, 47
shah
  distribution, 210
  Fourier series, 219
  Fourier transform, 214
  function, 210
  scaling identity, 214
Shannon C., 140, 227
shift (delay) operator, 189
shifted delta
  Fourier transform, 173
sifting property of delta, 193
signal
  bandlimited, 13, 71, 73, 224
  bandwidth, 13, 224
  reversal, 81
signal conversion
  continuous to discrete, 249
signal-to-noise ratio, 310
signum (sign) function, 177
similarity, 84
sinc function, 69
  as a convolution identity, 225
  orthonormal basis, 228
sinusoid, 7
Smith Julius, 20
smooth windows, 150
SNR, *see* signal-to-noise ratio
Sommerfeld, 199
sorting algorithm
  merge and sort, 277
spatial variable, 333
spectral power density, 73
spectrum, 13, 73
  analyzer, 13
  continuous, 71
  musical instruments, 20
  unbounded, 224
square integrable functions, 21

square sampling formula, 381
Stokes G., 45
superposition, 294
    principle, 294
superposition theorem
    for discrete linear systems, 302
    for discrete linear time invariant systems, 304
    for linear systems, 300
    for linear time invariant systems, 303
support of a function, 164, 224
system, 293

Tekalp A. M., 217
telegraph equation, 46
temperature, 40
tempered distribution
    Fourier inversion, 171
    Fourier transform, 169
    inverse Fourier transform, 170
tempered distributions, 153, 157
    continuity, 158
    defined by function, 159
    dual space, 158
    linearity, 158
    regularization, 197
tensor products, 341
test functions
    class, 139
thermal capacity, 40
thermal resistance, 40
Thomson William (Lord Kelvin), 45, 99, 109
timelimited vs. bandlimited, 225, 233
Toeplitz Otto, 298
tomography, 386, 395
top-hat function, 66
transfer function, 102
transfer function, 198
transfer function of linear time invariant system, 307
trapezoidal rule approximation, 397
triangle function, 75
triangle inequality, 35
Tukey John, 26, 249
tuning
    equal tempered scale, 8
    natural, 8
Two Slits experiments, 200
two-dimensional shah, 370
two-sided exponential decay, 88

unit ramp function, 177
unit step function, 176

Van Loan Charles, 285
vector, 26
    inner (dot) product, 27
    norm, 26
    projection, 28
vector complex exponential, 253
vector complex exponentials
    eigenvectors, 309
    geometric series, 328
    orthogonality, 264
von Laue Max, 209
von Neumann John, 34

Wandell Brian, 4
wave equation, 201
wavefront, 201
Whitehouse Edward, 46
Whittaker E., 227
windowing, 145

X-ray diffraction, 378
X-rays, 209

Young Thomas, 200
Young's experiment, 205

zero padding, 290
zero phase, 336