Note) Slides and example codes are available:
https://github.com/mint-lab/3dv_tutorial

# An Invitation to 3D Vision: Single-View Geometry

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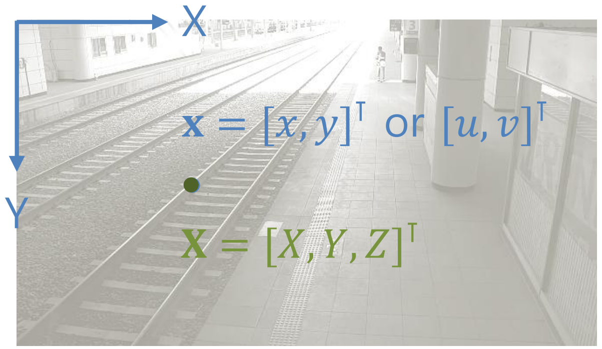
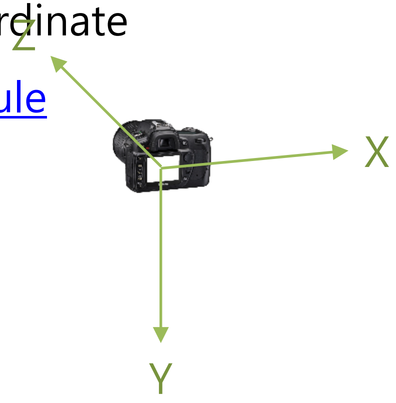
# Getting Started with 2D

- **Image coordinate** (unit: [pixel])




- **Camera coordinate** (unit: [meter])
  - Position: [Focal point]
  - X/Y direction: Image coordinate
  - Z direction: [Right-hand rule]



# Getting Started with 2D

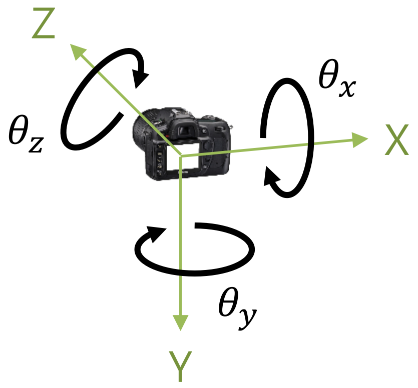
- **2D rotation matrix**
  - Rotational direction: Right-hand rule


$$\mathrm{R}(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

**Properties of a rotation matrix**

- $\mathrm{R}^{-1} = \mathrm{R}^{\mathsf{T}}$ (orthogonal matrix)
- $\det(\mathrm{R}) = 1$

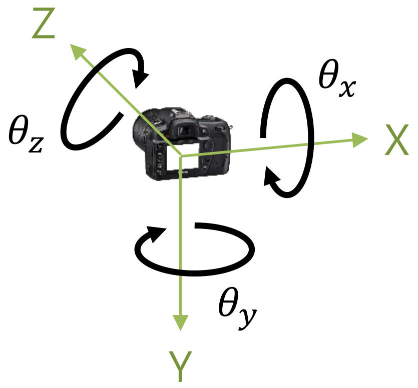
- **3D rotation matrix**
  - Rotational direction: Right-hand rule

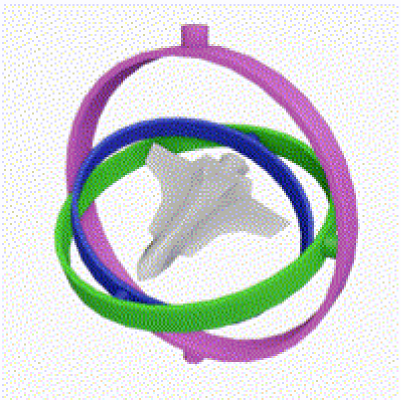


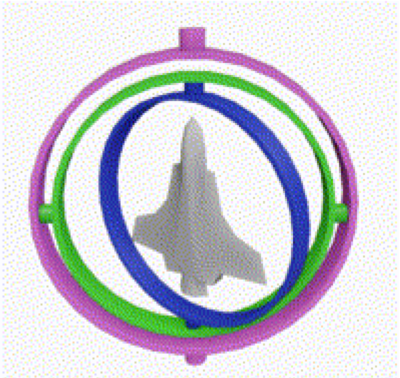

| | Cameras | Vehicles | Airplanes | Telescopes |
|---|---|---|---|---|
| $\theta_x$ | Tilt | Pitch | Attitude | Elevation |
| $\theta_y$ | Pan | Yaw | Heading | Azimuth |
| $\theta_z$ | Roll | Roll | Bank | Horizon |

# Getting Started with 2D

- 3D rotation representation (3 DOF)
  - 3D rotation matrix (9 parameters)
    - Notation: 3x3 matrix
      - e.g. $\mathrm{R} = \mathrm{R}_z(\theta_z)\,\mathrm{R}_y(\theta_y)\,\mathrm{R}_x(\theta_x) = \begin{bmatrix} \cos\theta_z & -\sin\theta_z & 0 \\ \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_y & 0 & \sin\theta_y \\ 0 & 1 & 0 \\ -\sin\theta_y & 0 & \cos\theta_y \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x \\ 0 & \sin\theta_x & \cos\theta_x \end{bmatrix}$
    - Properties: $\mathrm{R}^{-1} = \mathrm{R}^{\intercal}$ (orthogonal matrix), $\det(\mathrm{R}) = 1$
  - Euler angle (3 parameters)
    - Notation: $[\theta_x, \theta_y, \theta_z]$
    - Issues: Not unique, not continuous, Gimbar lock case (loss of DOF)



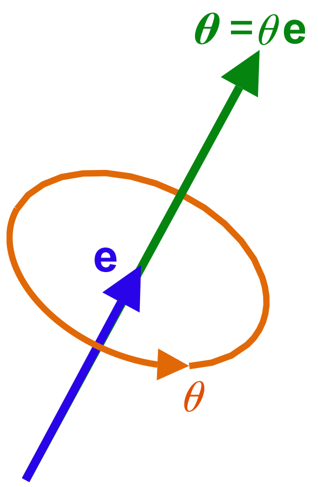





"Gimber lock" case

Image: Wikipedia

# Getting Started with 2D




- 3D rotation representation (3 DOF)
  - Axis-angle representation (3 parameters; a.k.a. *rotation vector*, *Rodrigues notation*)
    - Notation: $\boldsymbol{\theta} = \theta\mathbf{e}$
      - e.g. Axis (unit vector): $\mathbf{e} = [0, 0, 1]$, angle: $\theta = {}^{\pi}/_{2} \rightarrow \boldsymbol{\theta} = [0, 0, {}^{\pi}/_{2}]$
    - Properties: Log map of SO(3), dual ($-\mathbf{e}$ with $-\theta \rightarrow \boldsymbol{\theta}$), reverse angle ($-\boldsymbol{\theta}$)
    - Note) The standard notation in OpenCV with `cv.Rodrigues()` (`R` ↔ `rvec`)
  - (Unit) Quaternion (4 parameters)
    - Notation: $\mathbf{q} = [q_w, q_x, q_y, q_z]$ or $[q_x, q_y, q_z, q_w]$
      - Meaning: $\mathbf{q} = \cos\frac{\theta}{2} + (e_x\mathbf{i} + e_y\mathbf{j} + e_z\mathbf{k})\sin\frac{\theta}{2}$
    - Property: ${q_x}^2 + {q_y}^2 + {q_z}^2 + {q_w}^2 = 1$, dual ($-\mathbf{q}$), reverse angle ($\bar{\mathbf{q}}$; conjugate)

- Note) 3D rotation conversion
  - Python: `scipy.spatial.transform.Rotation`
  - Web apps: NinjaCalc, Glowbuzzer, Andre Gaschler

Image: Wikipedia

# Getting Started with 2D

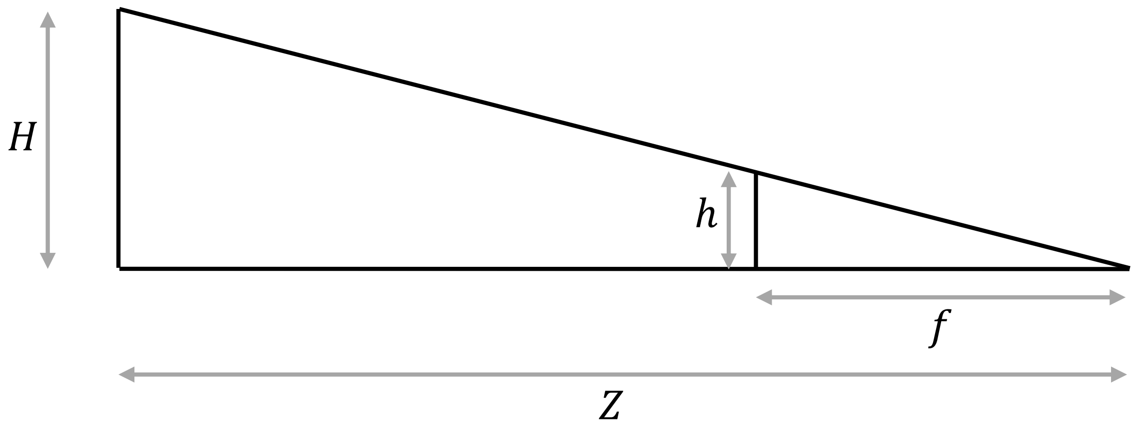
- Example) **3D rotation conversion** [3d_rotation_conversion.py]

```
import numpy as np
from scipy.spatial.transform import Rotation

# The given 3D rotation
euler = (45, 30, 60) # Unit: [deg] in the XYZ-order

# Generate 3D rotation object
robj = Rotation.from_euler('zyx', euler[::-1], degrees=True)

# Print other representations
print('\n## Euler Angle (ZYX)')
print(np.rad2deg (robj.as_euler('zyx'))) # [60, 30, 45] [deg] in the ZYX-order
print('\n## Rotation Matrix')
print(robj.as_matrix())
print('\n## Rotation Vector')
print(robj.as_rotvec())                 # [0.97, 0.05, 1.17]
print('\n## Quaternion (XYZW)')
print(robj.as_quat())                   # [0.44, 0.02, 0.53, 0.72]
```

# Getting Started with 2D

- **Similarity**

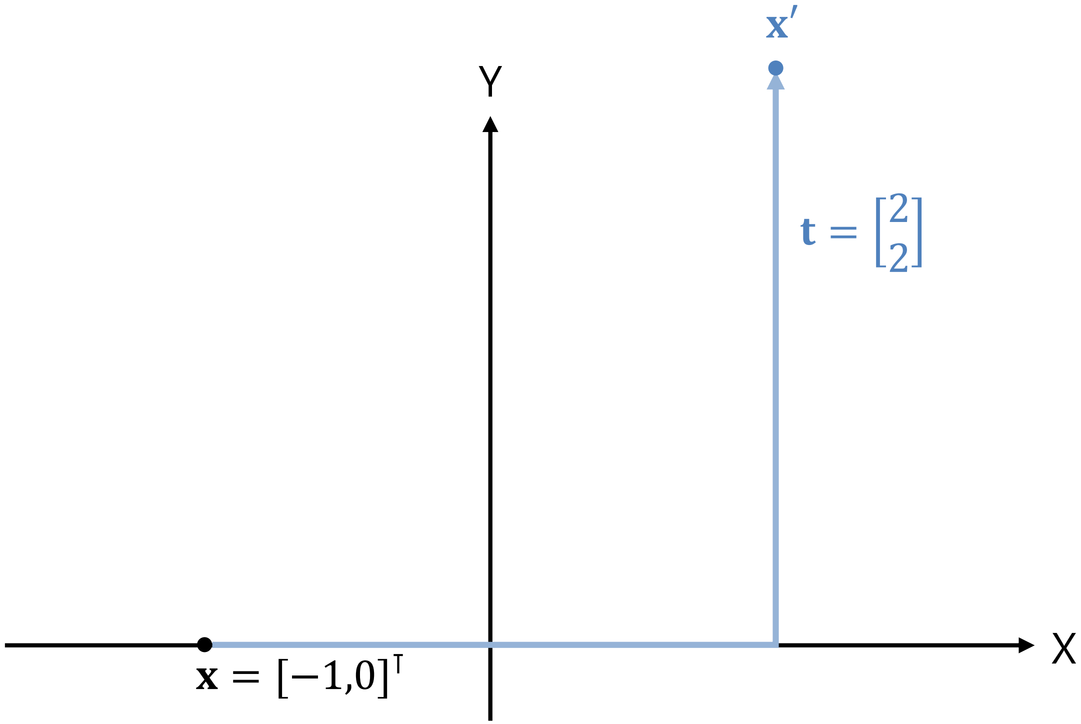
$\frac{h}{H} = \frac{f}{Z}$ or $\frac{h}{f} = \frac{H}{Z}$ $\rightarrow$ $h = f\,\frac{H}{Z}$



# Getting Started with 2D

- **Point translation**

$\mathbf{x}' = \mathbf{x} + \mathbf{t}$



# Getting Started with 2D

- **Coordinate translation**

  Valid for the following point transformation?

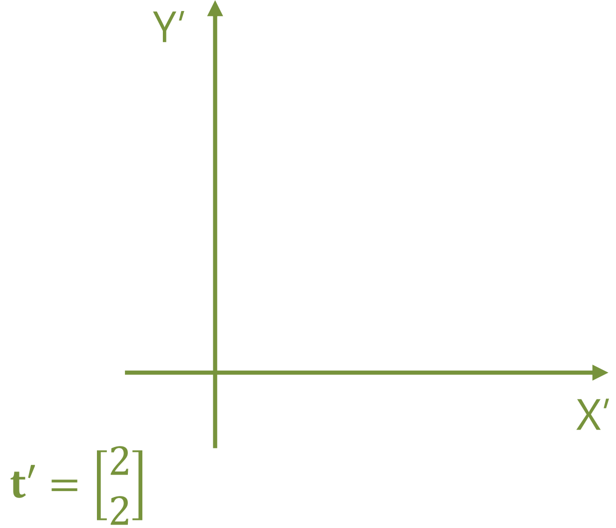
  $\mathbf{x}' = \mathbf{x} + \mathbf{t}'$ ?

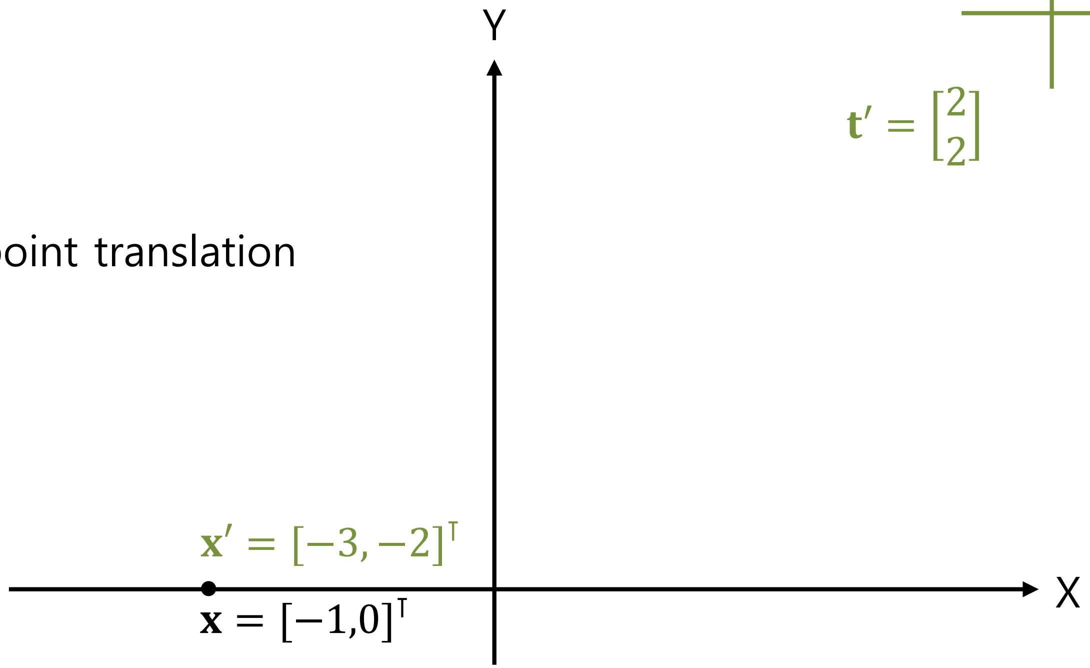
  **No!**

  The **inverse** of point translation

  $\mathbf{x} = \mathbf{x}' + \mathbf{t}'$

  $\mathbf{x}' = \mathbf{x} - \mathbf{t}'$

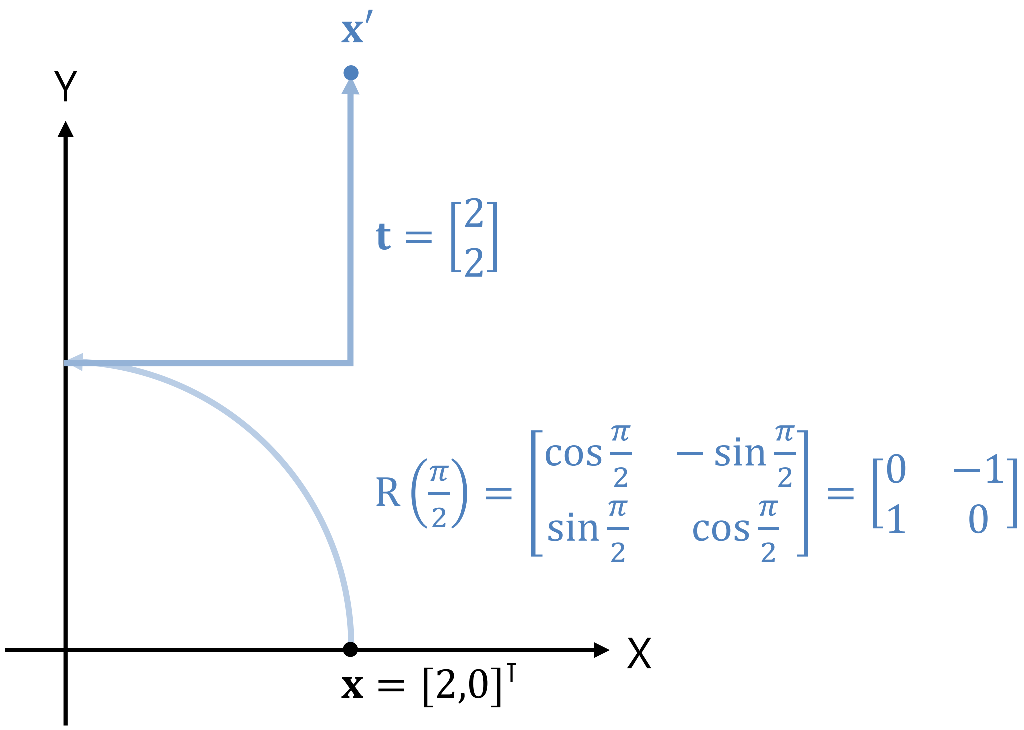


# Getting Started with 2D

- **Point transformation**

$$\mathbf{x}' = \mathrm{R}\mathbf{x} + \mathbf{t} = [\mathrm{R} \mid \mathbf{t}] \begin{bmatrix} \mathbf{x} \\ 1 \end{bmatrix}$$

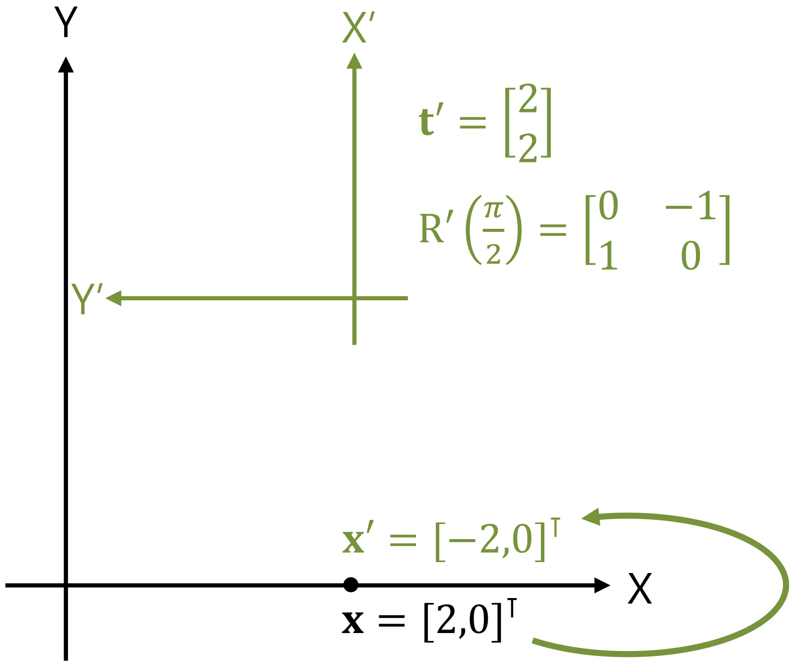


# Getting Started with 2D

- **Coordinate transformation**

  Valid for the following point transformation?

  $\mathbf{x}' = \mathrm{R}'\mathbf{x} + \mathbf{t}'$ ?



# Getting Started with 2D

- **Coordinate transformation**

Valid for the following point transformation?

$\mathbf{x}' = \mathrm{R}'\mathbf{x} + \mathbf{t}'$ ?

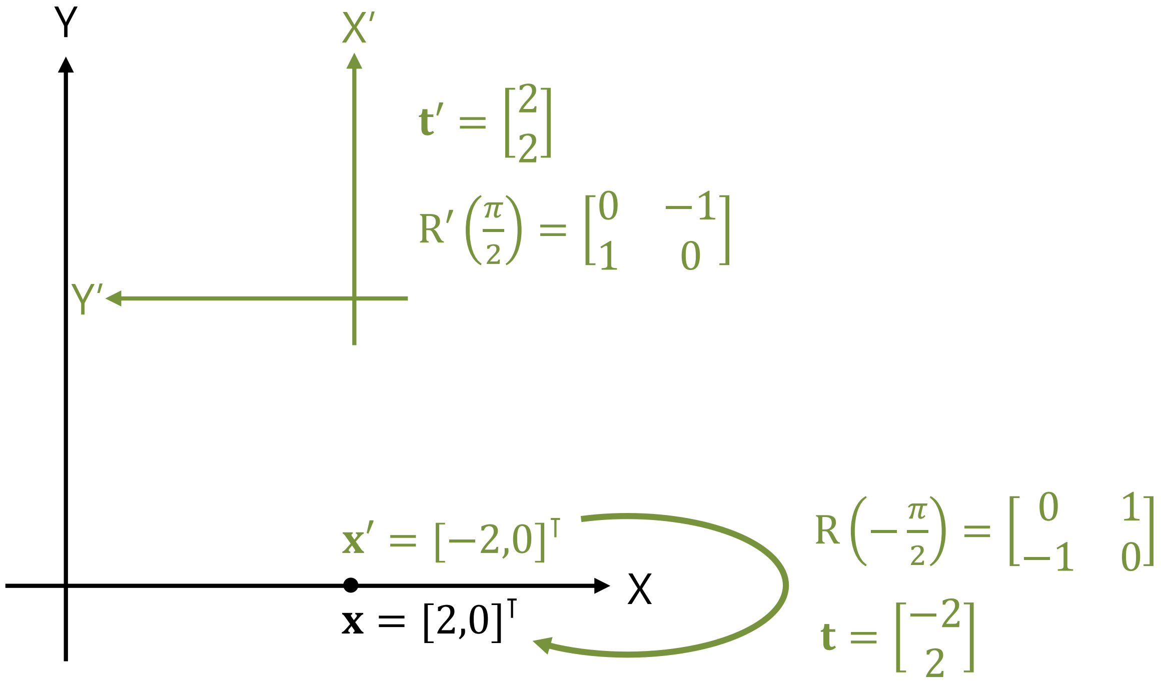
**No!**

The **inverse** of point transformation

$\mathbf{x} = \mathrm{R}'\mathbf{x}' + \mathbf{t}'$

$\downarrow \mathrm{R}'^{\top}(\mathbf{x} - \mathbf{t}') = \mathbf{x}'$

$\mathbf{x}' = \mathrm{R}\mathbf{x} + \mathbf{t}$  ( $\mathrm{R} = \mathrm{R}'^{\top}$ and $\mathbf{t} = -\mathrm{R}'^{\top}\mathbf{t}'$ )



# Table of Contents: Single-view Geometry

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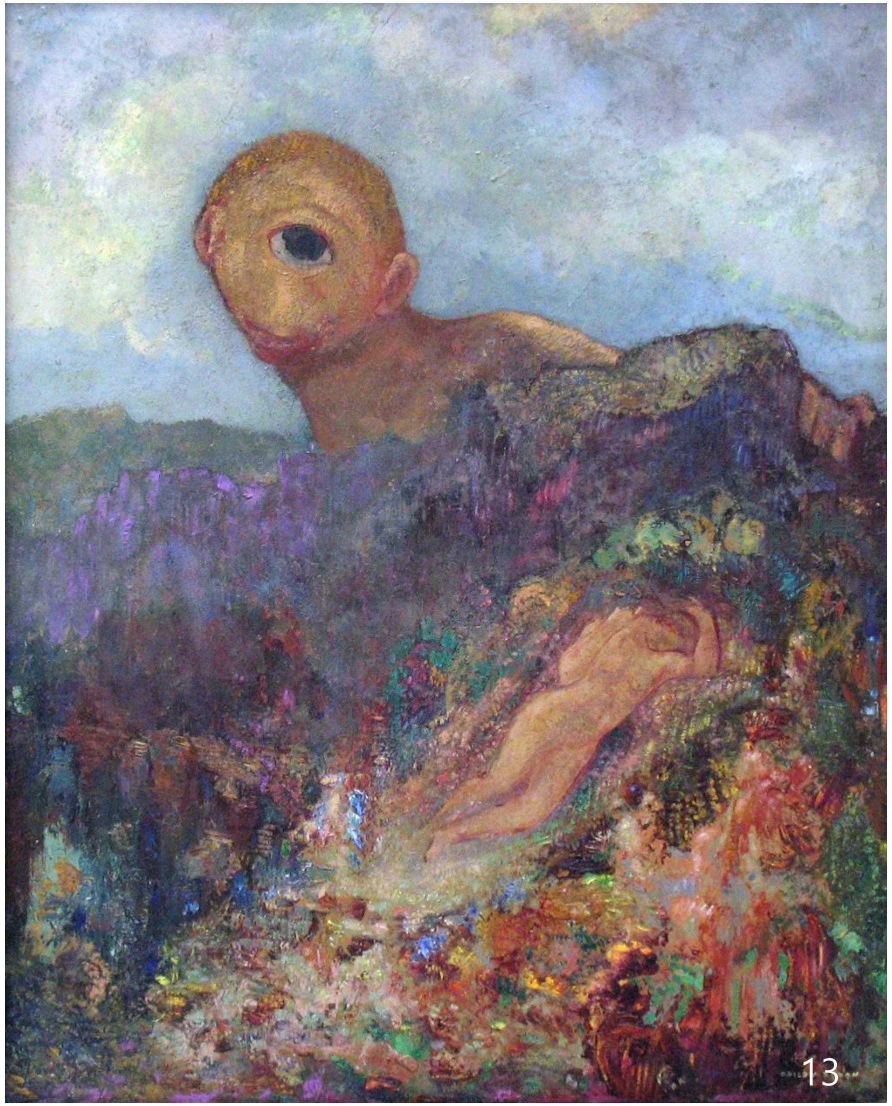
  - Point transformation, coordinate transformation: inverse relationship
- **Camera Projection Models**
  - Pinhole camera model
  - Geometric distortion models
- **Camera Calibration**
- **Absolute Camera Pose Estimation**

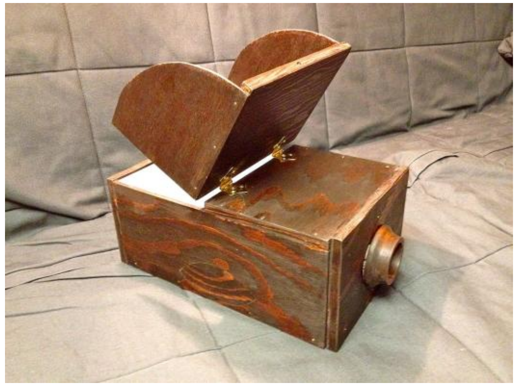


*The Cyclops*, gouache and oil by Odilon Redon

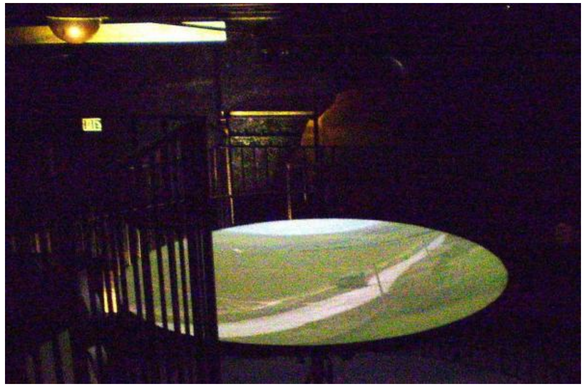
# Pinhole Camera Model

- Pinhole camera model


A large-scale camera obscura at San Francisco, California



A modern-day camera obscura



An Image in a camera obscura at Portslade, England

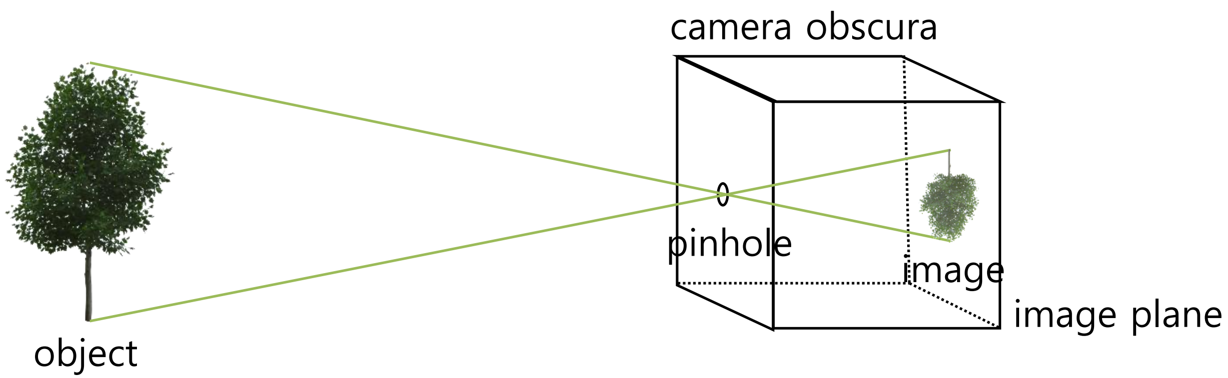
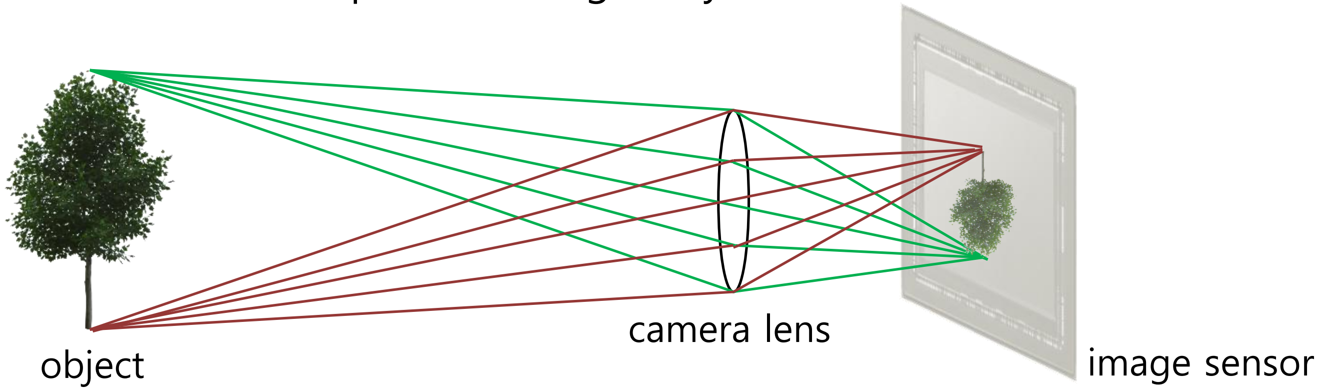



Image: Wikipedia, PNG EGG

# Pinhole Camera Model

- **Real camera with a lens**
  - Q) Why does a camera use a lens? To acquire more light rays




- **Pinhole camera model**

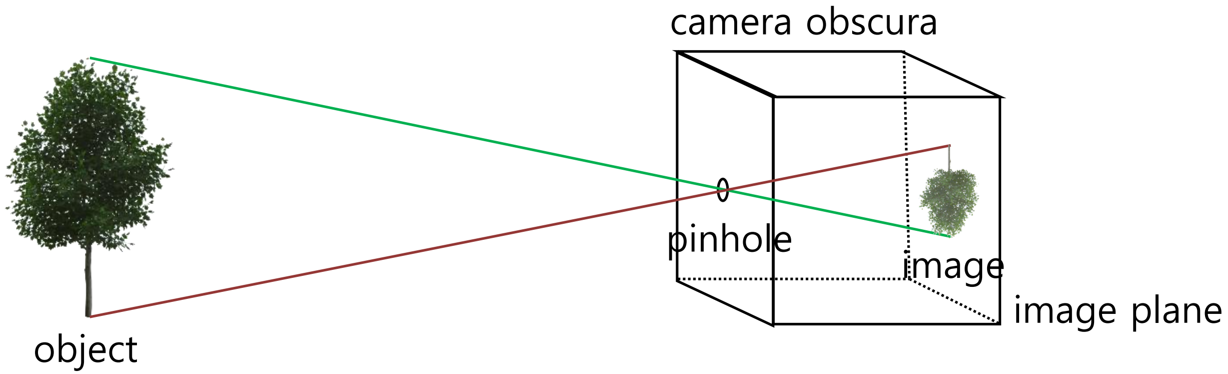
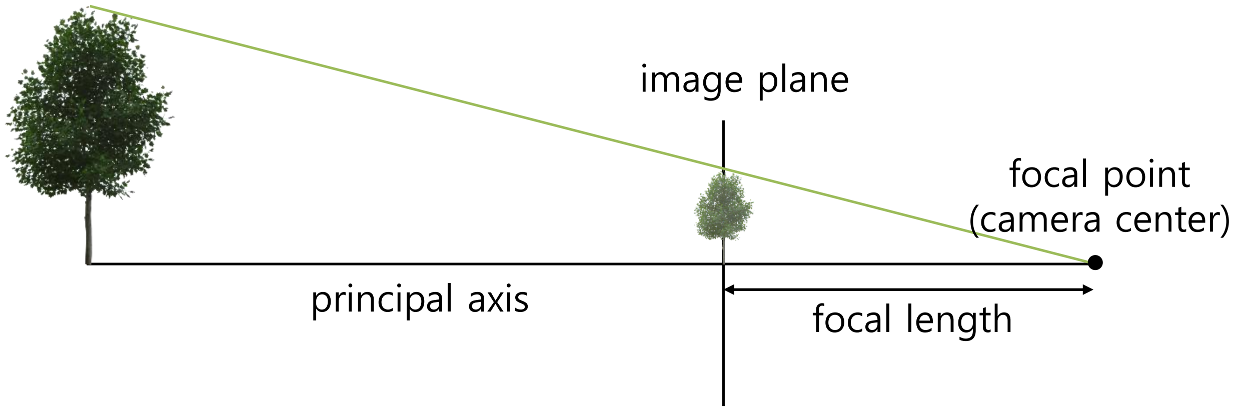
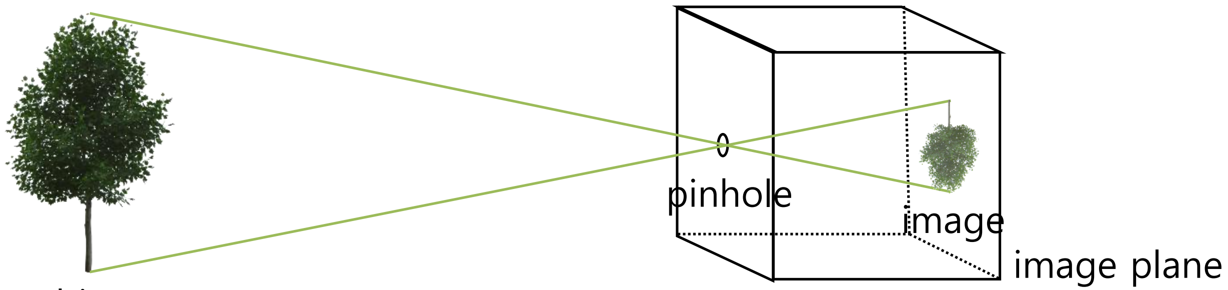



Image: PNG EGG

# Pinhole Camera Model

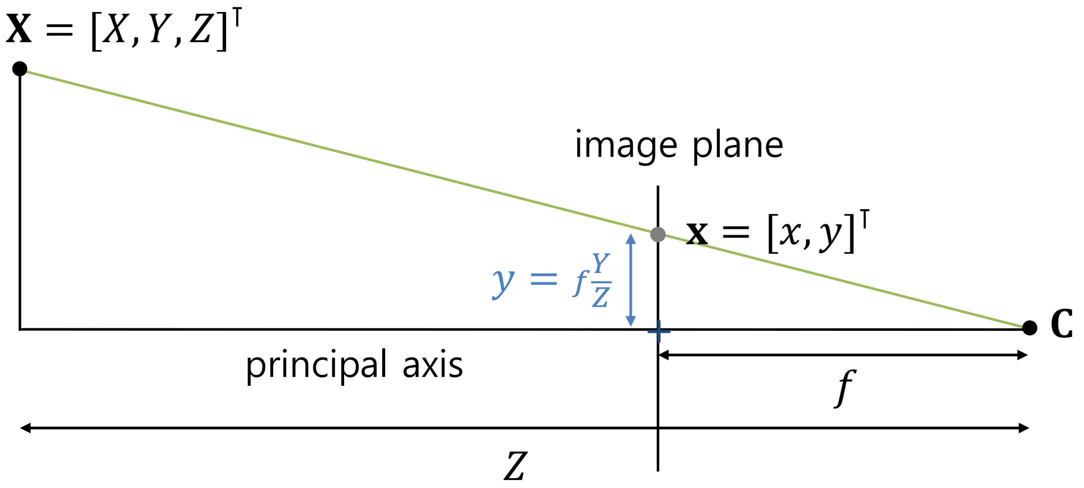
- **Pinhole camera model**
  - In conclusion (without lens distortion), $\mathbf{x} = \mathrm{P}\mathbf{X}$ $(\mathrm{P} = \mathrm{K}[\mathrm{R} \mid \mathbf{t}])$

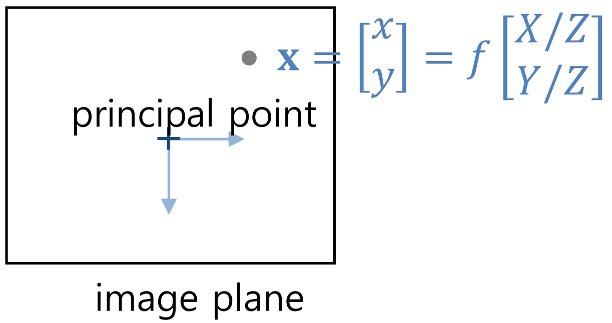


# Pinhole Camera Model

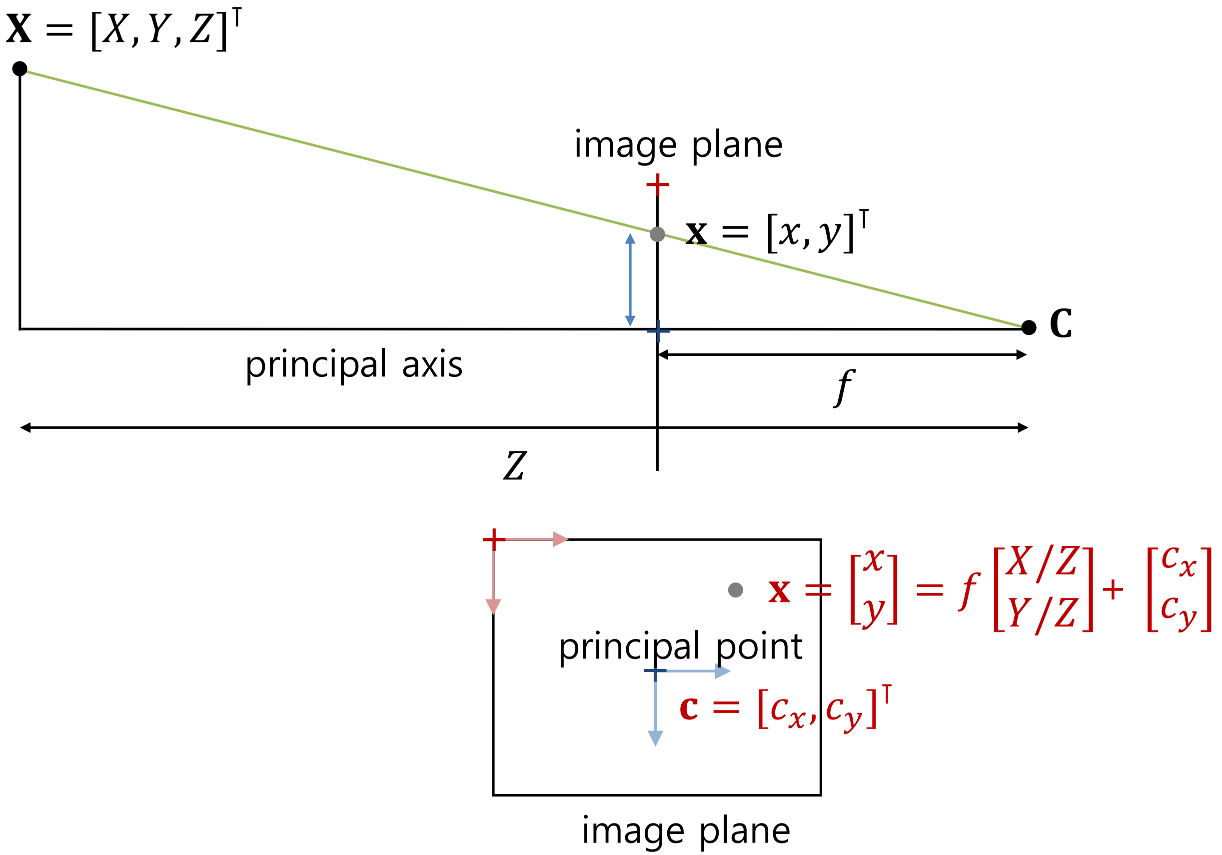
- Pinhole camera model



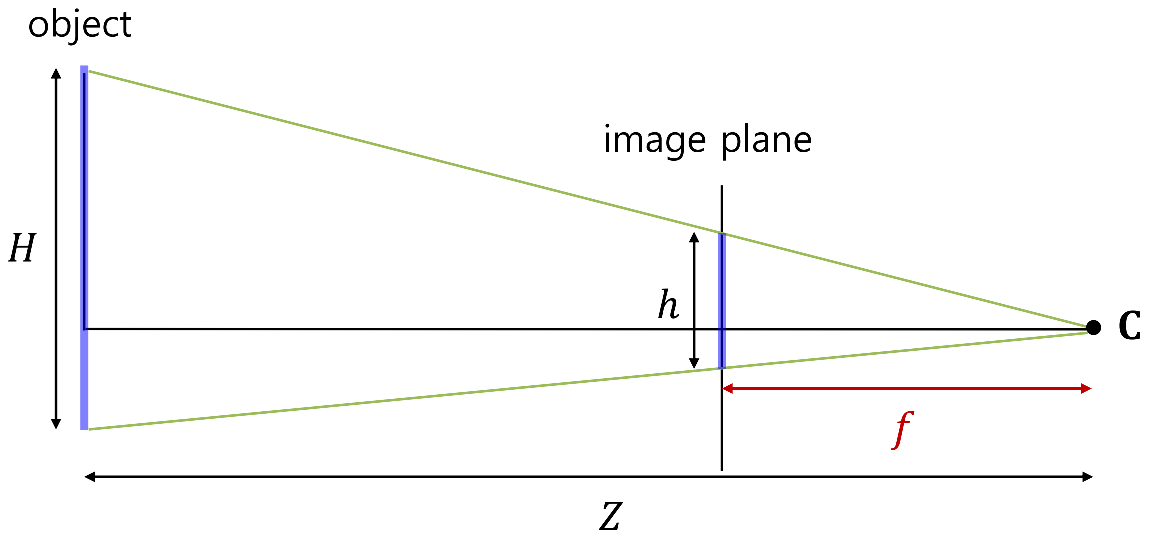
# Pinhole Camera Model

- **Pinhole camera model**



# Pinhole Camera Model

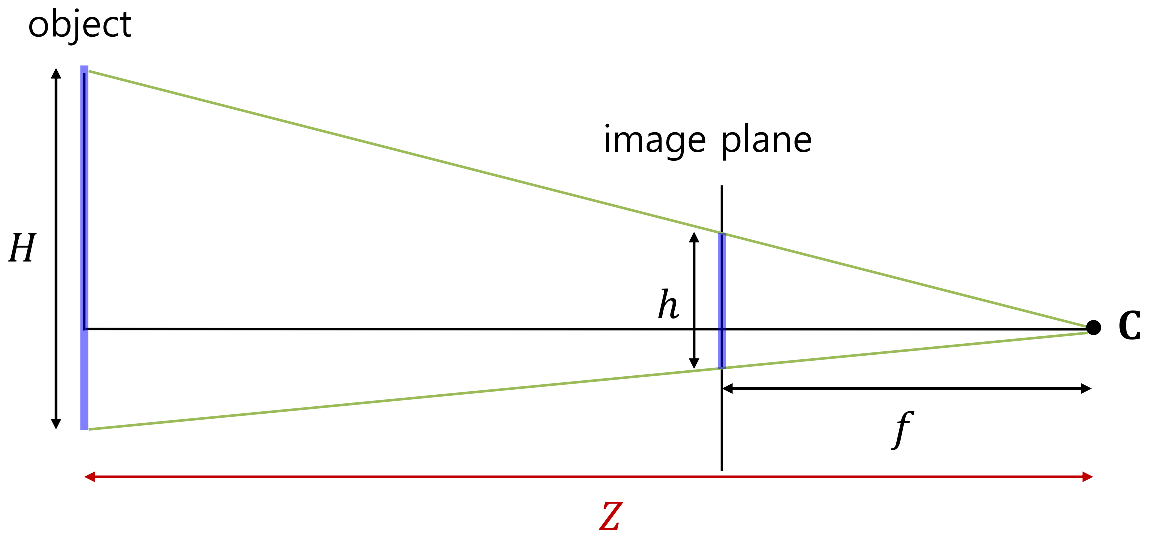
- Example) **Simple camera calibration**
  - Unknown: Focal length ($f$) of the camera (unit: [pixel])
  - Given: The observed object height ($h$) on the image plane (unit: [pixel])
  - Assumptions
    - The object height ($H$) and distance ($Z$) from the camera are known.
    - The object is aligned with the image plane.




$$\therefore f = \frac{Z}{H} h$$

# Pinhole Camera Model

- Example) **Simple depth estimation** (object localization)
  - Unknown: Object distance ($Z$) from the camera (unit: [m])
  - Given: The observed object height ($h$) on the image plane (unit: [pixel])
  - Assumptions
    - The object height ($H$) and focal length ($f$) are known.
    - The object is aligned with the image plane.

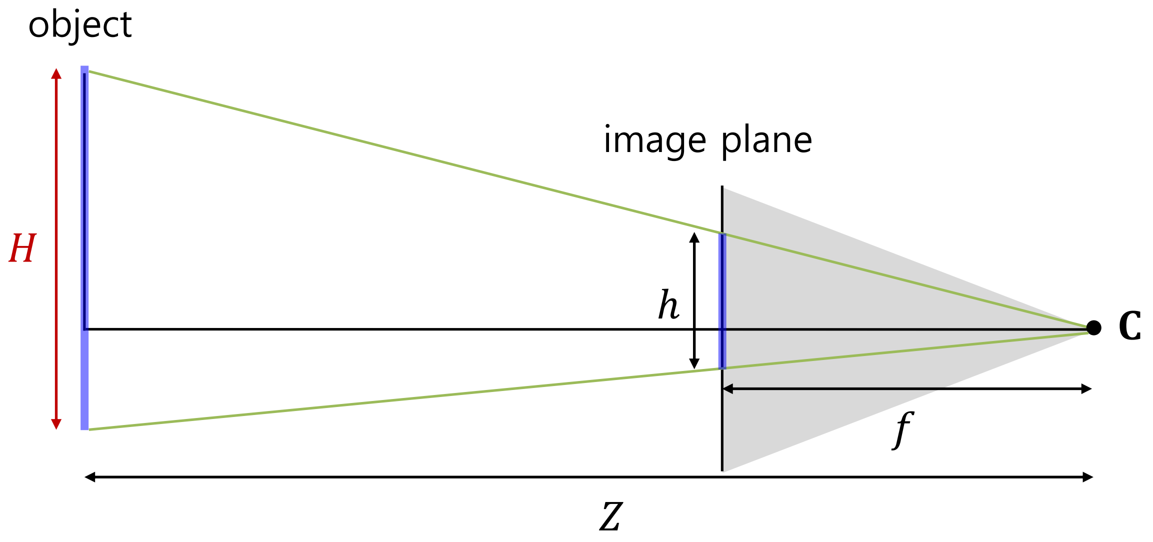



$$\therefore Z = \frac{f}{h} H$$

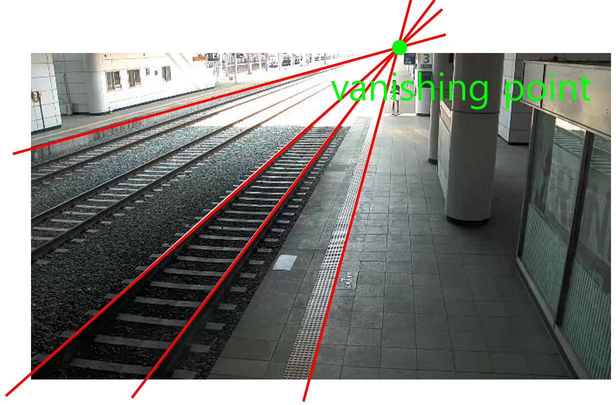
# Pinhole Camera Model

- Example) **Simple object measurement**
  - Unknown: Object height ($H$) (unit: [m])
  - Given: The observed object height ($h$) on the image plane (unit: [pixel])
  - Assumptions
    - The object distance ($Z$) from the camera and focal length ($f$) are known.
    - The object is aligned with the image plane.

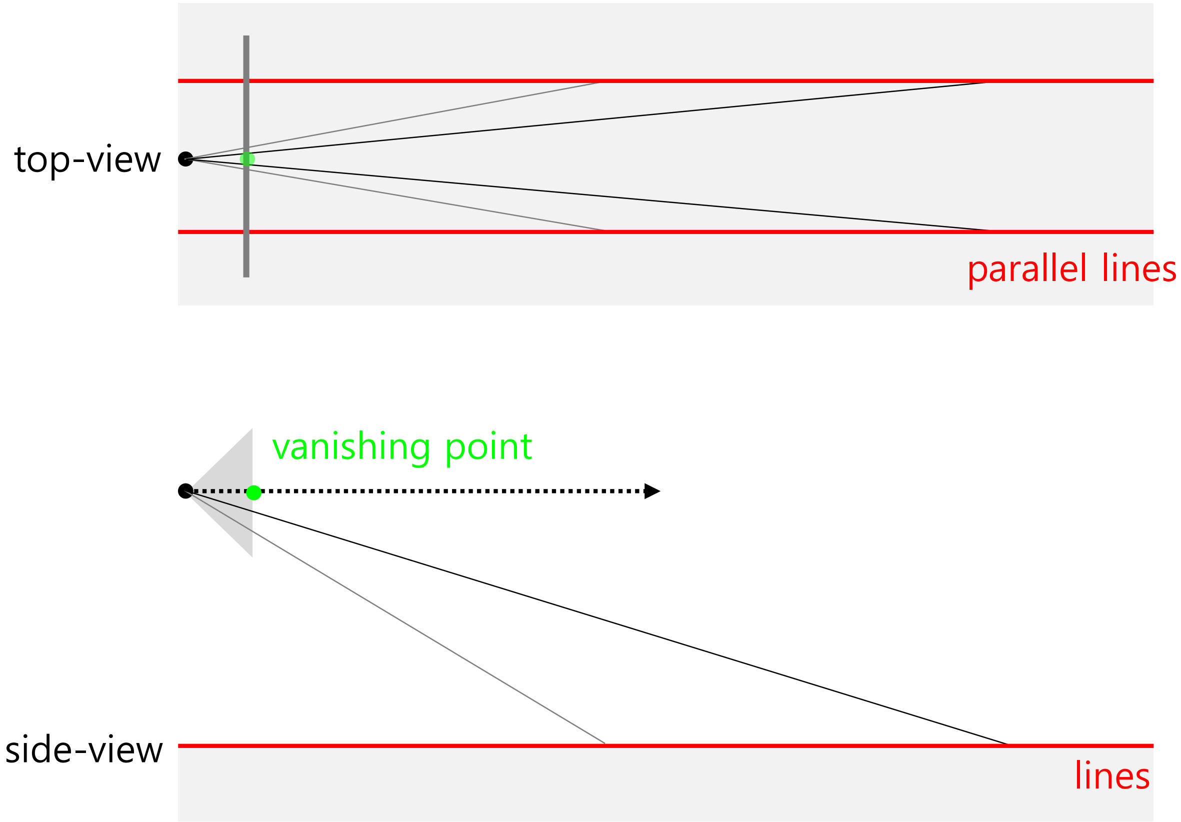
$$\therefore H = \frac{h}{f} Z$$



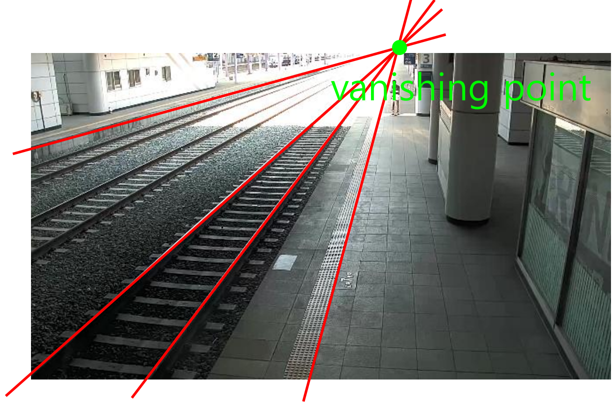
# Pinhole Camera Model




- **Vanishing points**
  - A point on the image plane where mutually parallel lines in 3D spac
    - A vector to the vanishing point is parallel to the lines.
    - A vector to the vanishing point is parallel to the reference plane made by the lines.



# Pinhole Camera Model

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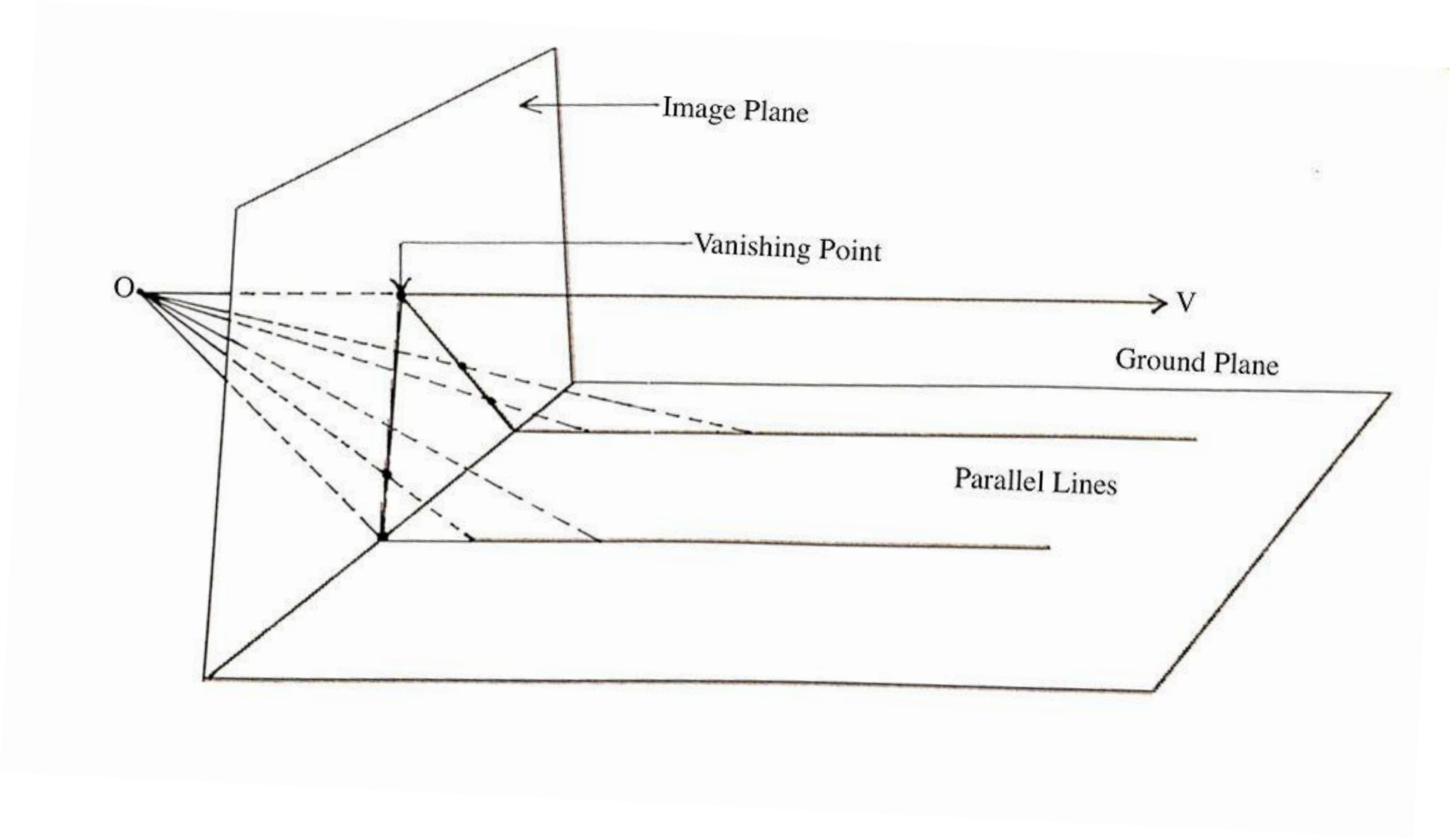
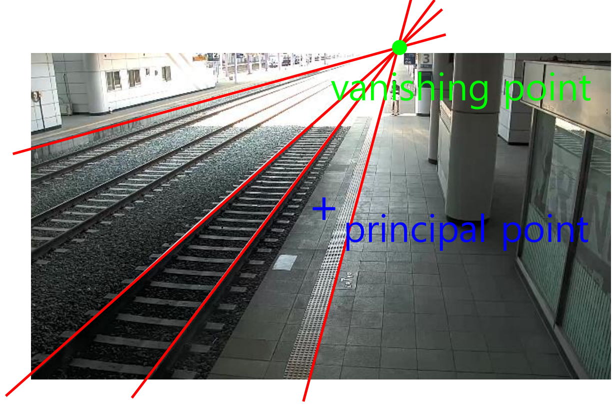


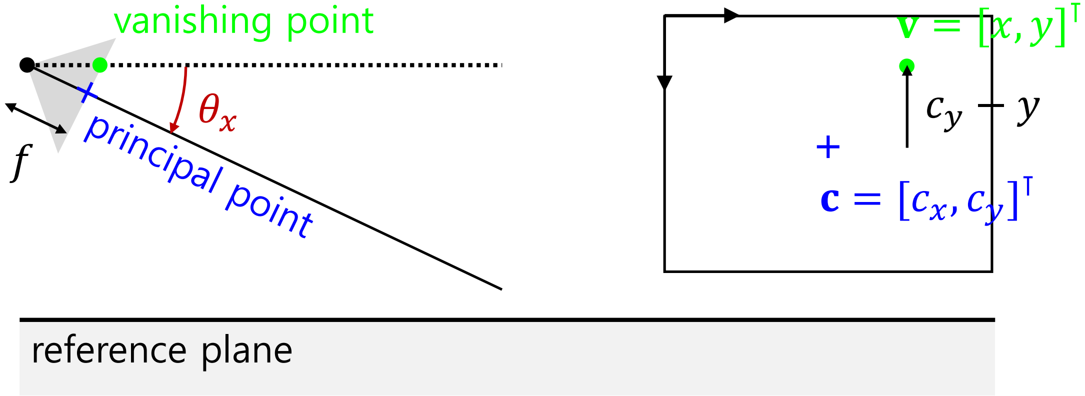

Image: Wikipedia

# Pinhole Camera Model

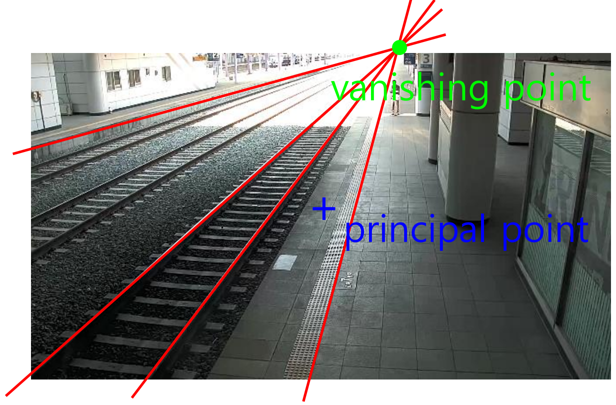



- Example) **Simple camera pose estimation**
  - Unknown: Tilt angle ($\theta_x$) of the camera w.r.t. the reference plane (unit: [rad])
  - Given: A vanishing point $(x, y)$ from the reference plane
  - Assumptions
    - The focal length ($f$) is known.
    - The principal point $(c_x, c_y)$ is known or selected as the center of image.
    - The camera has no roll, $\theta_z = 0$.
  - Note) The tilt angle in this page is defined as the opposite direction of the common notation.

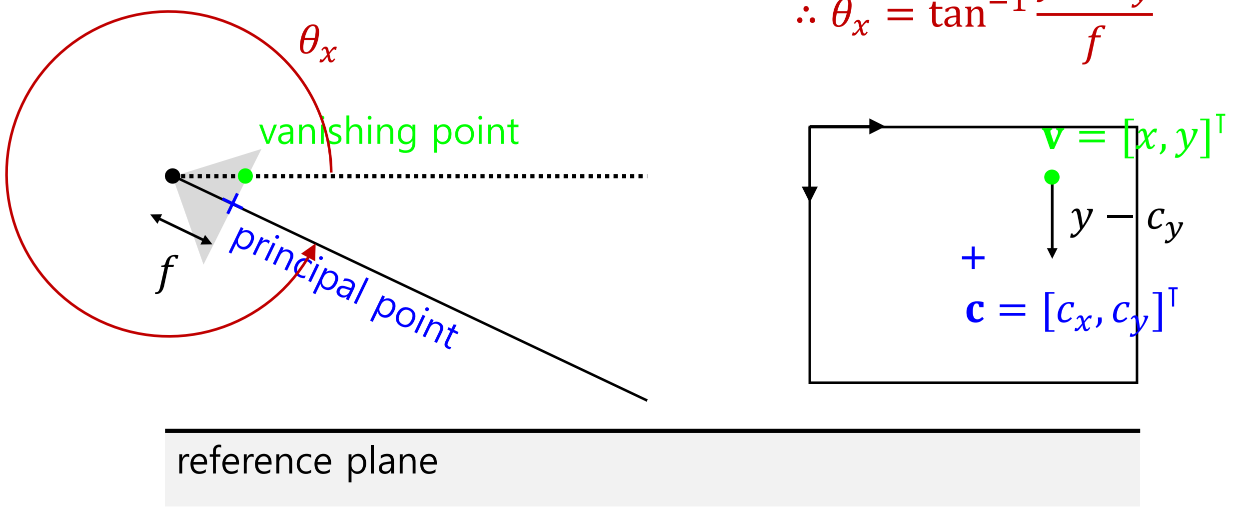
$$\therefore \theta_x = \tan^{-1} \frac{c_y - y}{f}$$



# Pinhole Camera Model

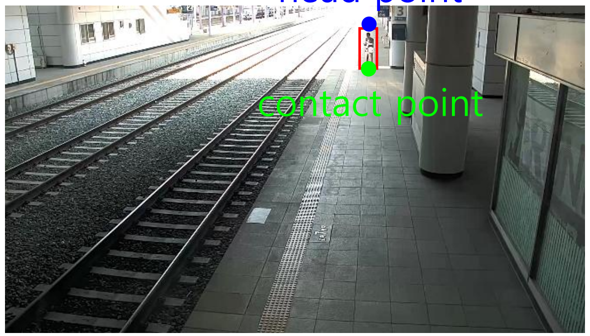



- Example) **Simple camera pose estimation**
  - Unknown: Tilt angle ($\theta_x$) of the camera w.r.t. the reference plane (unit: [rad])
  - Given: A vanishing point $(x, y)$ from the reference plane
  - Assumptions
    - The focal length ($f$) is known.
    - The principal point $(c_x, c_y)$ is known or selected as the center of image.
    - The camera has no roll, $\theta_z = 0$.
  - Note) The pan angle ($\theta_y$) with respect to the rail can be calculated similarly using $x$ instead of $y$.




$$\therefore \theta_x = \tan^{-1} \frac{y - c_y}{f}$$

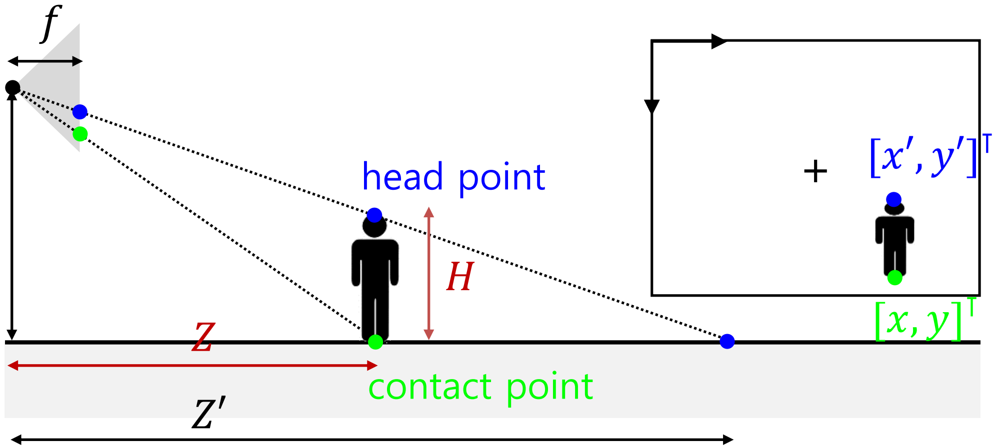
# Pinhole Camera Model



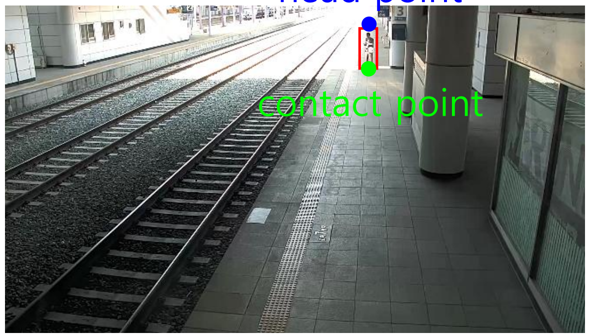

- Example) **Object localization #1**
  - Unknown: Object position and height (unit: [m])
  - Given: The object's contact and head points on the image (unit: [pixel])
  - Assumptions
    - The focal length, principal points, and camera height, are known.
    - The camera is aligned to the reference plane.
    - The object is on the reference plane.

$$\therefore Z = \frac{\bar{z}}{\bar{y}} L \qquad X = \frac{\bar{x}}{\bar{y}} L \qquad H = \left( \frac{\bar{y}}{\bar{z}} - \frac{\bar{y}'}{\bar{z}'} \right) Z$$

$$\begin{bmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{bmatrix} = \begin{bmatrix} x - c_x \\ y - c_y \\ f \end{bmatrix}$$



# Pinhole Camera Model

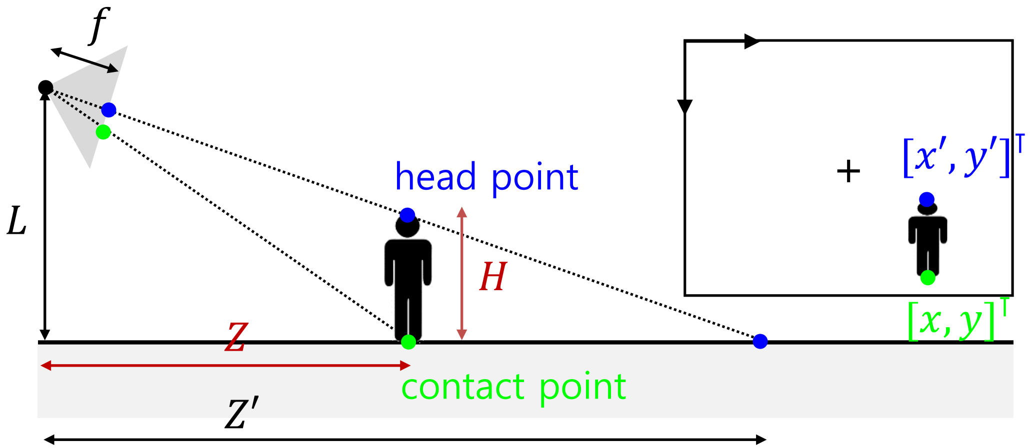



- Example) **Object localization #2**
  - Unknown: Object position and height (unit: [m])
  - Given: The object's contact and head points on the image (unit: [pixel])
  - Assumptions
    - The focal length, principal points, and camera height are known.
    - ~~The camera is aligned to the reference plane.~~ The camera orientation (R) is known.
    - The object is on the reference plane.

$$\therefore Z = \frac{\bar{z}}{\bar{y}} L \qquad X = \frac{\bar{x}}{\bar{y}} L \qquad H = \left( \frac{\bar{y}}{\bar{z}} - \frac{\bar{y}'}{\bar{z}'} \right) Z$$

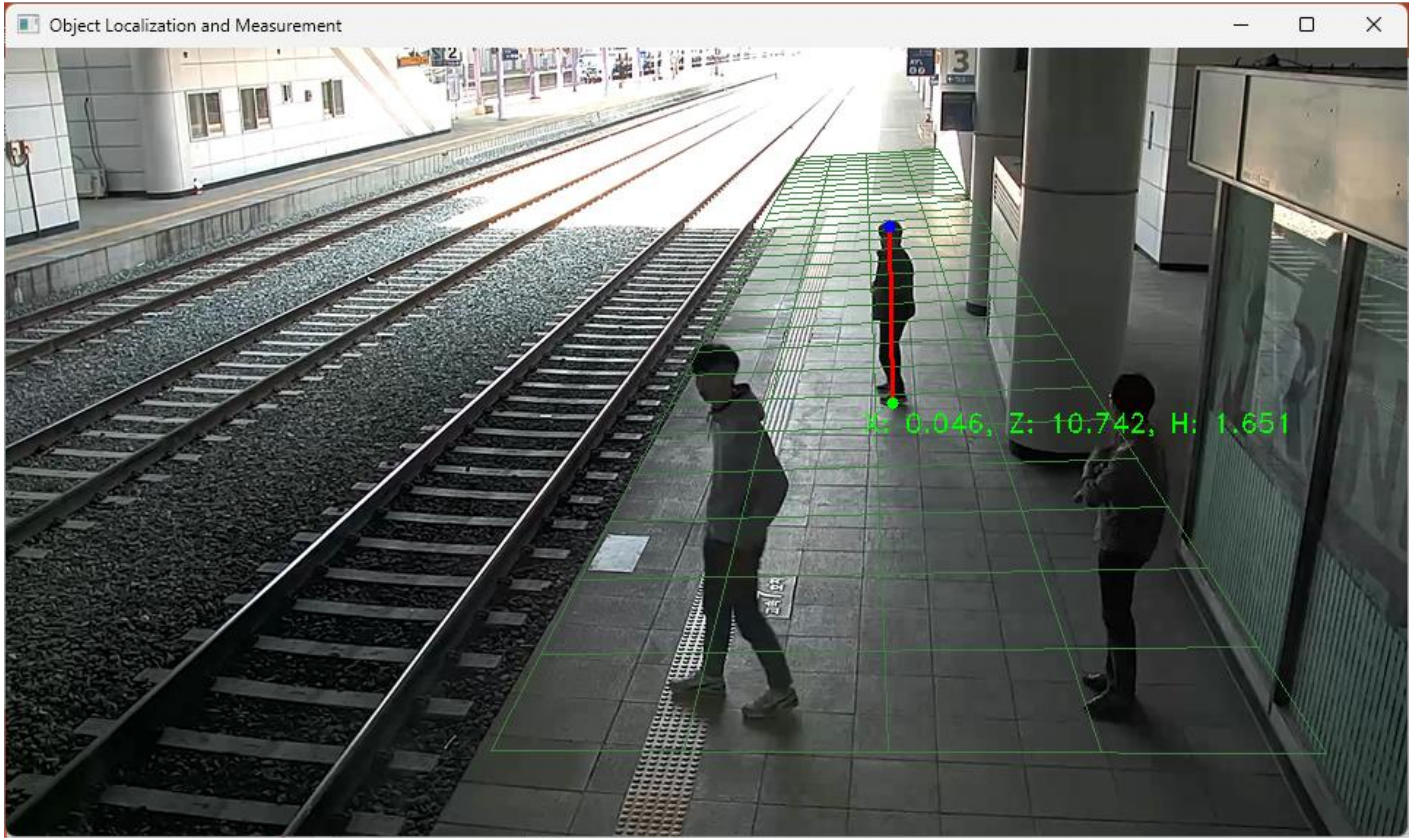
$\because$ same camera center

$$\begin{bmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{bmatrix} = \mathrm{R}^\intercal \begin{bmatrix} x - c_x \\ y - c_y \\ f \end{bmatrix}$$



# Pinhole Camera Model

- Example) **Object localization #2** [object_localization.py]



# Pinhole Camera Model

- Example) **Object localization #2** [object_localization.py]

```python
if __name__ == '__main__':
    ...
    while True:
        img_copy = img.copy()
        if mouse_state['xy_e'][0] > 0 and mouse_state['xy_e'][1] > 0:
            # Calculate object location and height
            c = R.T @ [mouse_state['xy_s'][0] - cx, mouse_state['xy_s'][1] - cy, f]
            h = R.T @ [mouse_state['xy_e'][0] - cx, mouse_state['xy_e'][1] - cy, f]
            if c[1] < 1e-6:
                continue
            X = c[0] / c[1] * L                          # Object location X [m]
            Z = c[2] / c[1] * L                          # Object location Y [m]
            H = (c[1] / c[2] - h[1] / h[2]) * Z          # Object height [m]

            # Draw the head/contact points and location/height
            cv.line(img_copy, mouse_state['xy_s'], mouse_state['xy_e'], (0, 0, 255), 2)
            cv.circle(img_copy, mouse_state['xy_e'], 4, (255, 0, 0), -1) # Head point
            cv.circle(img_copy, mouse_state['xy_s'], 4, (0, 255, 0), -1) # Contact point
            info = f'X: {X:.3f}, Z: {Z:.3f}, H: {H:.3f}'
            cv.putText(img_copy, info, np.array(mouse_state['xy_s']) + (-20, 20), cv.FONT_HERSHEY_DUPLEX, 0.6, (0, 2

        cv.imshow('Object Localization and Measurement', img_copy)
        key = cv.waitKey(10)
        if key == 27: # ESC
            break
    cv.destroyAllWindows()
```

$$\begin{bmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{bmatrix} = \mathrm{R}^\intercal \begin{bmatrix} x - c_x \\ y - c_y \\ f \end{bmatrix}$$

$$X = \frac{\bar{x}}{\bar{y}} L \quad Z = \frac{\bar{z}}{\bar{y}} L \quad H = \left( \frac{\bar{y}}{\bar{z}} - \frac{\bar{y}'}{\bar{z}'} \right) Z$$

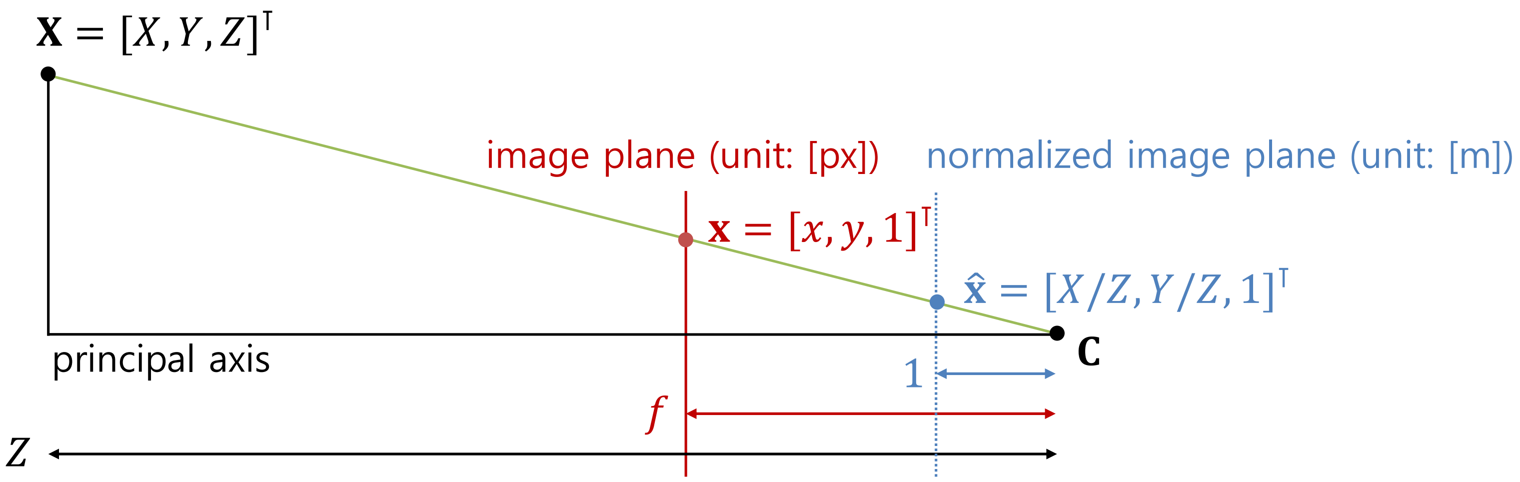
# Pinhole Camera Model

- **Camera matrix** $\mathrm{K}$

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = f \begin{bmatrix} X/Z \\ Y/Z \end{bmatrix} + \begin{bmatrix} c_x \\ c_y \end{bmatrix} \rightarrow \mathbf{x} = \mathrm{K}\hat{\mathbf{x}} \quad \text{where} \quad \mathrm{K} = \begin{bmatrix} f & 0 & c_x \\ 0 & f & c_y \\ 0 & 0 & 1 \end{bmatrix}, \ \mathbf{x} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}, \text{ and } \ \hat{\mathbf{x}} = \begin{bmatrix} X/Z \\ Y/Z \\ 1 \end{bmatrix}$$

Simplified as $\mathrm{K} = \begin{bmatrix} f & 0 & w/2 \\ 0 & f & h/2 \\ 0 & 0 & 1 \end{bmatrix}$ ($w$: image width, $h$: image height)

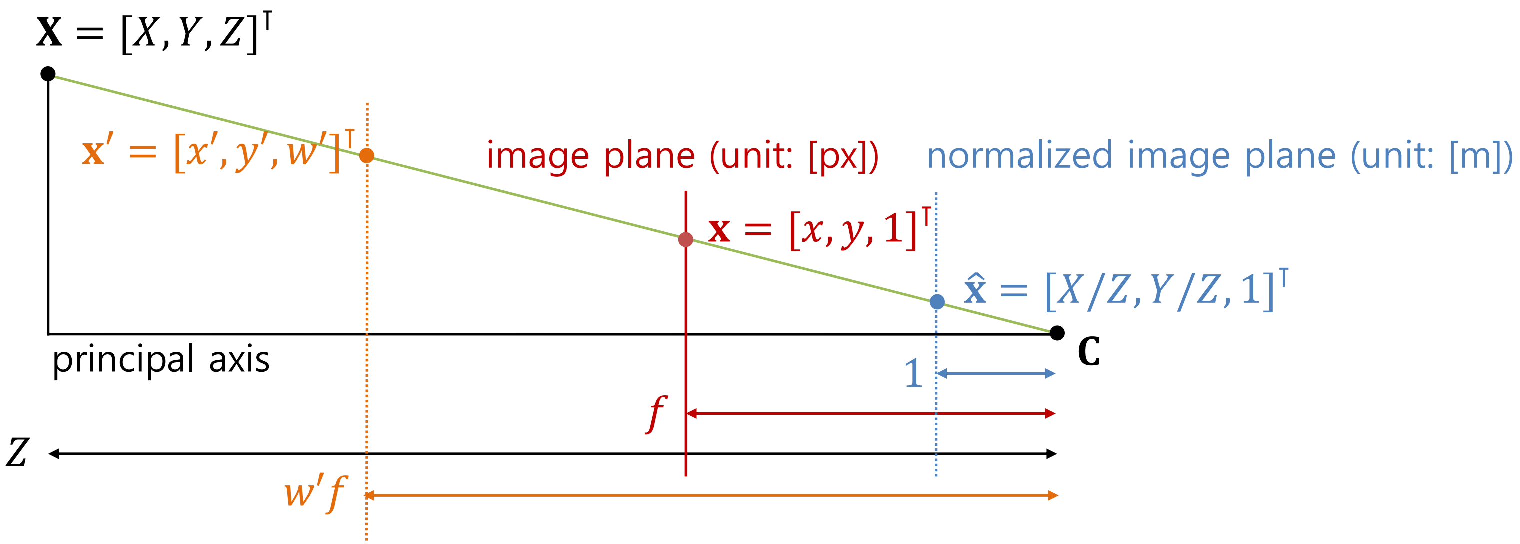
Generalized as $\mathrm{K} = \begin{bmatrix} f_x & s & c_x \\ 0 & f_y & c_y \\ 0 & 0 & 1 \end{bmatrix}$ ($s$: skew parameter)



# Pinhole Camera Model

- **Homogeneous coordinates** (a.k.a. projective coordinates)
  - It describes an $n$-dimensional project space as $n+1$-dimensional coordinate system.
  - It hold a non-conventional equivalence relationship: $(x_1, x_2, \dots, x_{n+1}) \sim (\lambda x_1, \lambda x_2, \dots, \lambda x_{n+1})$ such that $(0 \neq \lambda \in \mathbb{R})$.
    - e.g. (5, 12) is written as (5, 12, 1) which is also equal to (10, 24, 2) or (15, 36, 3) or …
  - On the previous slide, $\mathbf{x} = \mathrm{K}\hat{\mathbf{x}}$ where $\mathrm{K} = \begin{bmatrix} f & 0 & c_x \\ 0 & f & c_y \\ 0 & 0 & 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$, and $\hat{\mathbf{x}} = \begin{bmatrix} X/Z \\ Y/Z \\ 1 \end{bmatrix}$

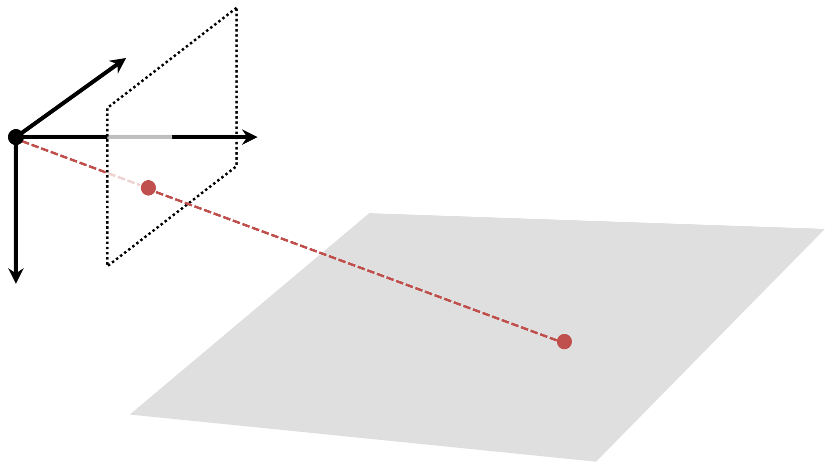
    $\mathbf{x}' = \mathrm{K}\mathbf{X}$ where $\mathrm{K} = \begin{bmatrix} f & 0 & c_x \\ 0 & f & c_y \\ 0 & 0 & 1 \end{bmatrix}$, $\mathbf{x}' = \begin{bmatrix} x' \\ y' \\ w' \end{bmatrix}$, and $\mathbf{X} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$ (Note: $\mathbf{x} = \frac{1}{w'}\mathbf{x}'$)

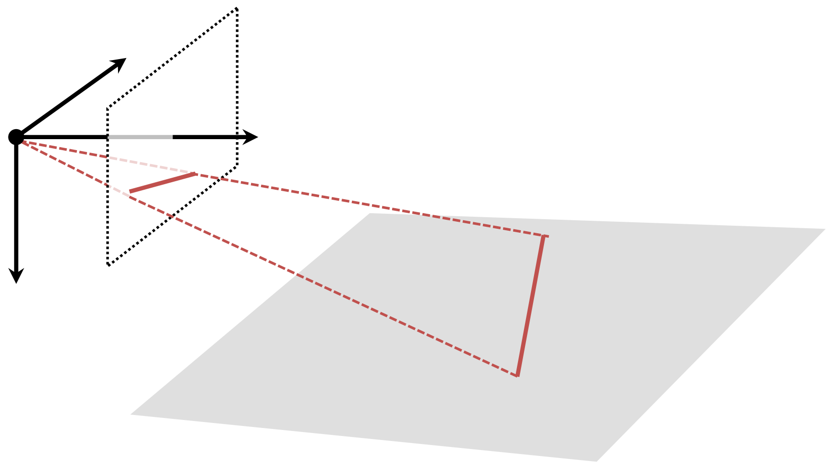


# Pinhole Camera Model

- **Why homogeneous coordinates?**
  - An affine transformation ($\mathbf{y} = \mathrm{A}\mathbf{x} + \mathbf{b}$) is formulated by a single matrix multiplication.
  - A point at infinity (a.k.a. ideal point) is numerically represented by $w = 0$.
  - A point and line ($ax + by + c = 0$) are described beautifully as like $\mathbf{l}^\intercal \mathbf{x} = 0$ or $\mathbf{x}^\intercal \mathbf{l} = 0$ ($\mathbf{l} = [a, b, c]^\intercal$).
    - Intersection of two lines: $\mathbf{x} = \mathbf{l}_1 \times \mathbf{l}_2$
    - A line by two points: $\mathbf{l} = \mathbf{x}_1 \times \mathbf{x}_2$

    } *Duality* of a point and line
  - A light ray (line at the camera center) is observed as a point on the image plane.
    - A plane at the camera center is observed as a line on the image plane.
    - A conic whose peak is at the camera center is observed as a conic section on the image plane.

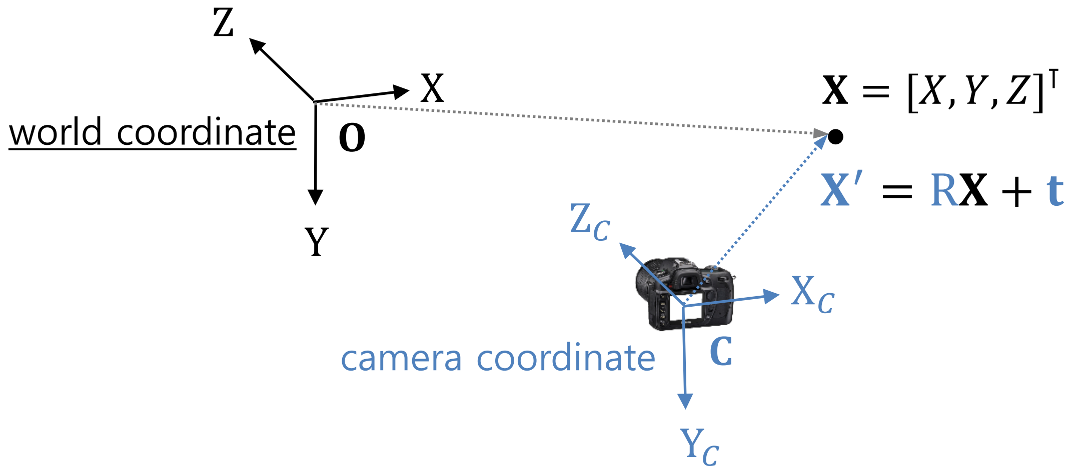




# Pinhole Camera Model

- **Projection matrix** $\mathrm{P}$
  - Generally, a point $\mathbf{X}$ is not based on the camera coordinate so that it needs be transformed to the camera coordinate.

$$\mathbf{X}' = \mathrm{R}\mathbf{X} + \mathbf{t} \;\rightarrow\; \mathbf{X}' = [\mathrm{R} \mid \mathbf{t}] \begin{bmatrix} \mathbf{X} \\ 1 \end{bmatrix}$$




  - The whole camera projection from the world coordinate to the image coordinate:

$$\mathbf{x} = \mathrm{K}\mathbf{X}' = \mathrm{K}(\mathrm{R}\mathbf{X} + \mathbf{t}) = \mathrm{K}\,[\mathrm{R} \mid \mathbf{t}] \begin{bmatrix} \mathbf{X} \\ 1 \end{bmatrix}$$

$\rightarrow$ $\mathbf{x} = \mathrm{P}\mathbf{X}$ where $\mathrm{P} = \mathrm{K}\,[\mathrm{R} \mid \mathbf{t}]$ (3x4 matrix), $\mathbf{x}$ and $\mathbf{X}$ in homogeneous coordinates

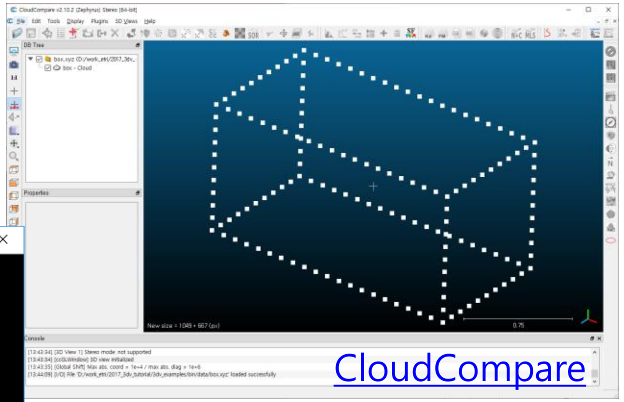
  - Note) The camera pose ($\mathrm{R}^\intercal$ and $-\mathrm{R}^\intercal\mathbf{t}$) can be derived from the inverse of point transformation ($\mathrm{R}$ and $\mathbf{t}$).

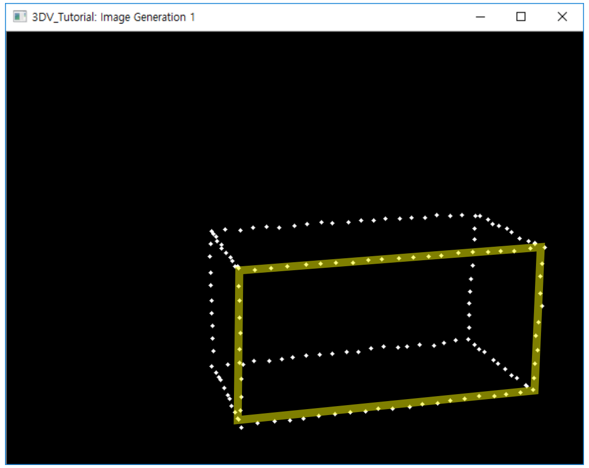
$$\mathrm{T}^{-1} = \begin{bmatrix} \mathrm{R} & \mathbf{t} \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \mathrm{R}^\intercal & -\mathrm{R}^\intercal\mathbf{t} \\ 0 & 1 \end{bmatrix}$$

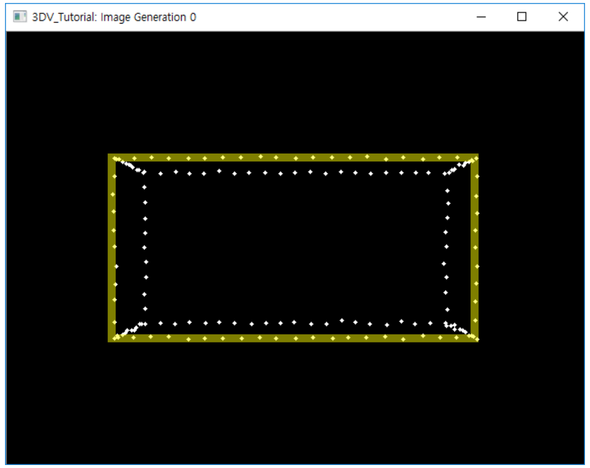
# Pinhole Camera Model

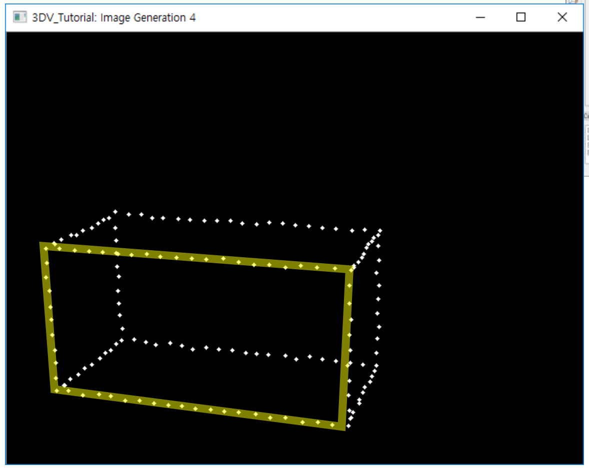
- Example) **Image formation** `[image_formation.py]`

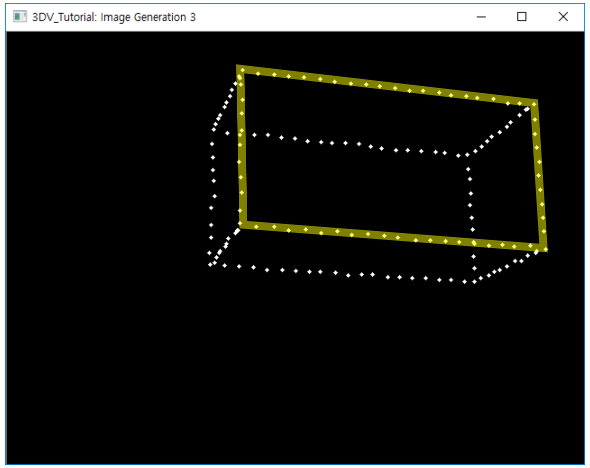
A point cloud: `data/box.xyz`

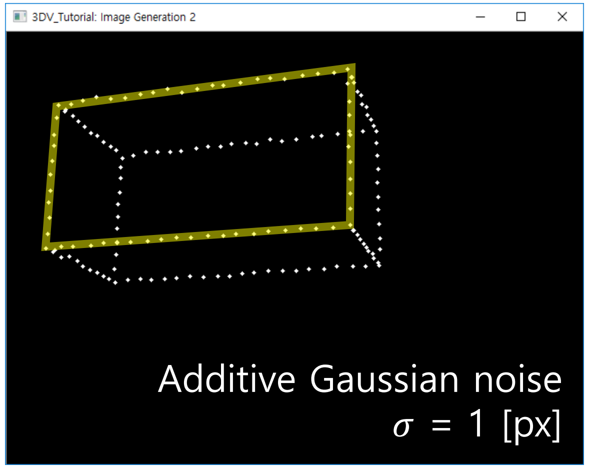


# Pinhole Camera Model

- Example) **Image formation** [image_formation.py]

```python
from scipy.spatial.transform import Rotation

# The given camera configuration: Focal length, principal point, image resolution, position, and orientation
f, cx, cy, noise_std = 1000, 320, 240, 1
img_res = (640, 480)
cam_pos = [[0, 0, 0], [-2, -2, 0], [2, 2, 0], [-2, 2, 0], [2, -2, 0]]           # Unit: [m]
cam_ori = [[0, 0, 0], [-15 , 15, 0], [15, -15, 0], [15, 15, 0], [-15, -15, 0]] # Unit: [deg]

# Load a point cloud in the homogeneous coordinate
X = np.loadtxt('../data/box.xyz') # Size: N x 3

# Generate images for each camera pose
K = np.array([[f, 0, cx], [0, f, cy], [0, 0, 1]])
for i, (pos, ori) in enumerate(zip(cam_pos, cam_ori)):
    # Derive 'R' and 't'
    Rc = Rotation.from_euler('zyx', ori[::-1], degrees=True).as_matrix()
    R = Rc.T
    t = -Rc.T @ pos

    # Project the points (Alternative: `cv.projectPoints()`)
    x = K @ (R @ X.T + t.reshape(-1, 1)) # Size: 3 x N
    x /= x[-1]

    # Add Gaussian noise
    noise = np.random.normal(scale=noise_std, size=(2, len(X)))
    x[0:2,:] += noise
```

$$\mathbf{K} = \begin{bmatrix} f & 0 & c_x \\ 0 & f & c_y \\ 0 & 0 & 1 \end{bmatrix}$$

$\mathbf{R} = \mathrm{R}_c^\intercal$ and $\mathbf{t} = -\mathrm{R}_c^\intercal \mathbf{t}_c$

$$\mathbf{x} = \mathrm{K}(\mathrm{R}\mathbf{X} + \mathbf{t})$$

$$\mathbf{x} = \begin{bmatrix} x/w \\ y/w \\ w/w \end{bmatrix}$$

# Geometric Distortion Models

Q) How to represent such geometric distortion?

- **Geometric distortion models**
  - A camera lens generates geometric distortion, which can be approximated (modeled) as a nonlinear function $f_d$.
  - Geometric distortion models $f_d$ are mostly defined on the normalized image plane.
  - Camera projection with geometric distortion: $\mathbf{x} = \mathrm{proj}(\mathbf{X}; \mathrm{K}, \mathrm{R}, \mathbf{t}, d)$ where $d$ is a set of distortion coefficients.

Note) $\mathbf{x} = \mathrm{K}(\mathrm{R}\mathbf{X} + \mathbf{t})$ without distortion and normalization

$$\underbrace{\mathbf{X} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}}_{\substack{\text{3D point} \\ \text{(the world coordinate)}}} \longrightarrow \underbrace{\underbrace{\mathbf{X}' = \mathrm{R}\mathbf{X} + \mathbf{t}}_{\substack{\text{3D point} \\ \text{(the camera coordinate)}}} \longrightarrow \underbrace{\hat{\mathbf{x}} = \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} = \begin{bmatrix} X'/Z' \\ Y'/Z' \end{bmatrix}}_{\substack{\text{2D point} \\ \text{(the normalized image plane)}}} \longrightarrow \underbrace{\hat{\mathbf{x}}_d = f_d(\hat{\mathbf{x}})}_{\text{geometric distortion}} \longrightarrow \underbrace{\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f_x \hat{x}_d + c_x \\ f_y \hat{y}_d + c_y \end{bmatrix}}_{\text{pinhole camera projection}}}_{\mathbf{x} = \mathrm{proj}(\mathbf{X}; \mathrm{K}, \mathrm{R}, \mathbf{t}, d)}$$

Image: Shawn Becker

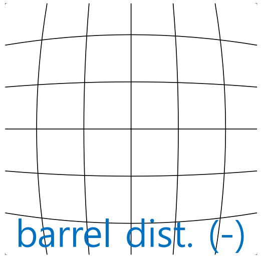
Image: [Camera Calibration Toolbox for MATLAB](), [Wikipedia](), [MathWorks]()

# Geometric Distortion Models

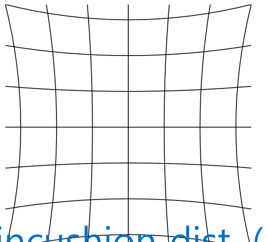
- **Geometric distortion models**
  - **Polynomial distortion model** (a.k.a. Brown-Conrady model; 1919)

$$\begin{bmatrix}\hat{x}_d \\ \hat{y}_d\end{bmatrix} = (1 + k_1 r^2 + k_2 r^4 + \cdots)\begin{bmatrix}\hat{x} \\ \hat{y}\end{bmatrix} + (1 + p_3 r^2 + p_4 r^4 + \cdots)\begin{bmatrix}2p_1\hat{x}\hat{y} + p_2(r^2 + 2\hat{x}^2) \\ 2p_2\hat{x}\hat{y} + p_1(r^2 + 2y^2)\end{bmatrix} \quad \text{where} \quad r^2 = \hat{x}^2 + \hat{y}^2$$

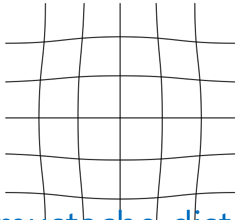
    - OpenCV (default): `cv.projectPoints()` ↔ `cv.undistortPoints()`



barrel dist. (-)

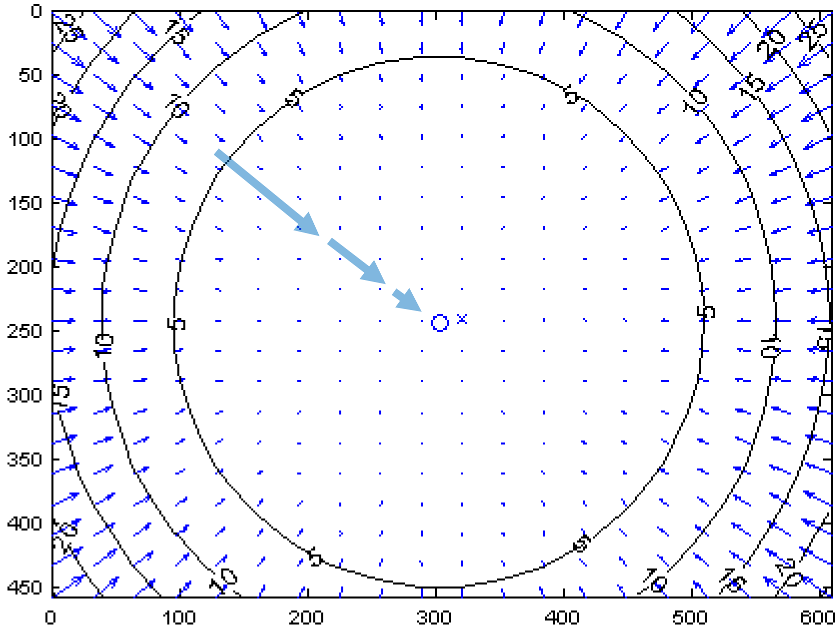


pincushion dist. (+)



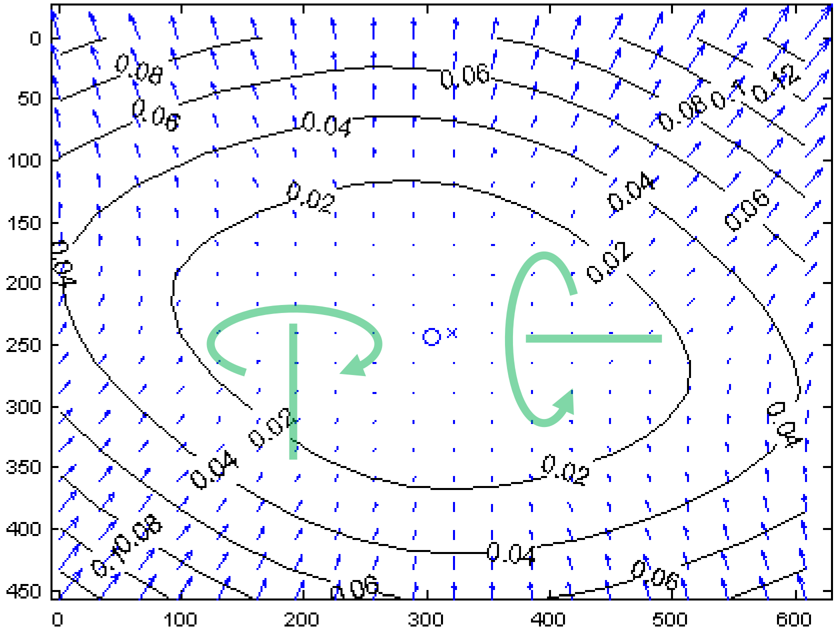
mustache dist.

**Radial distortion**



Pixel error = [0.1174, 0.1159]
Focal Length = (657.303, 657.744) +/- [0.2849, 0.2894]
Principal Point = (302.717, 242.334) +/- [0.5912, 0.5571]
Skew = 0.0004198 +/- 0.0001905
Radial coefficients = (-0.2535, 0.1187, 0) +/- [0.00231, 0.009418, 0]
Tangential coefficients = (-0.0002789, 5.174e-005) +/- [0.0001217, 0.0001208]

**Tangential distortion**



Pixel error = [0.1174, 0.1159]
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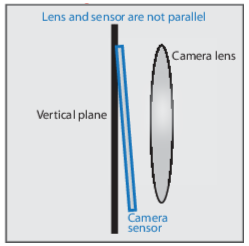
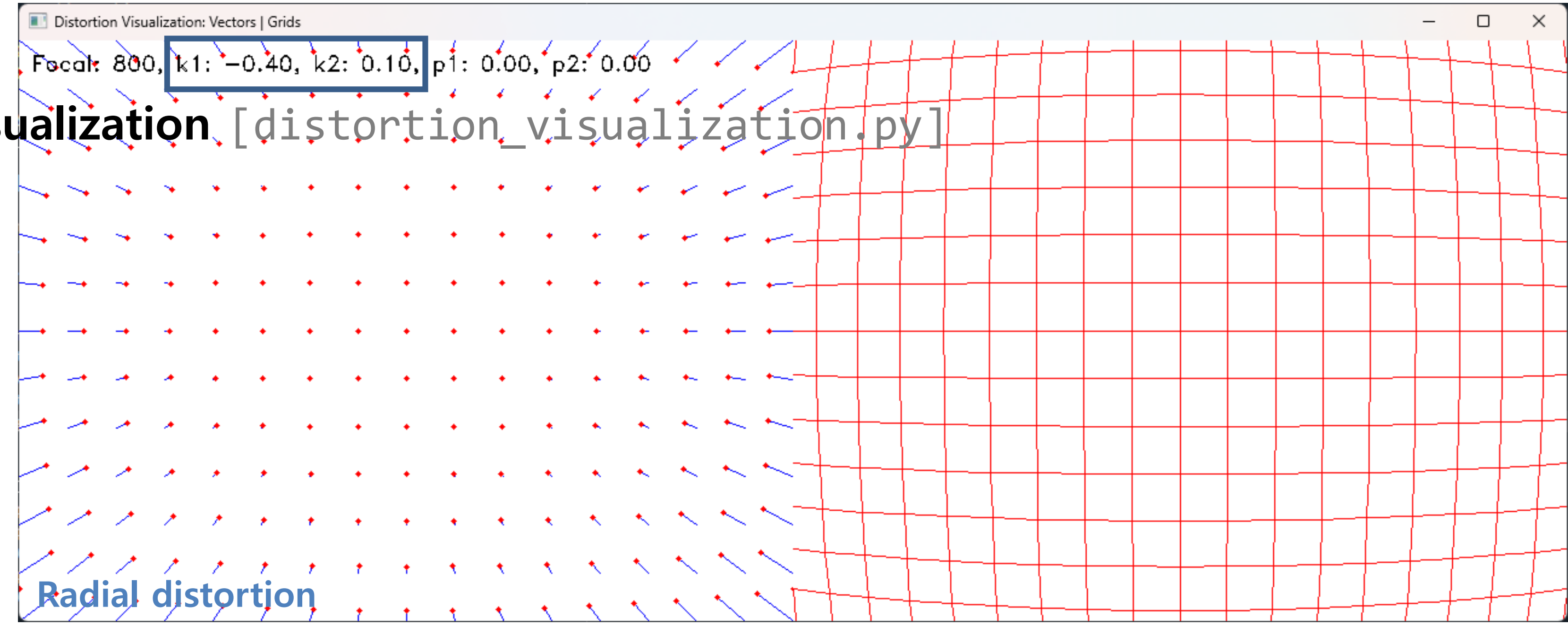
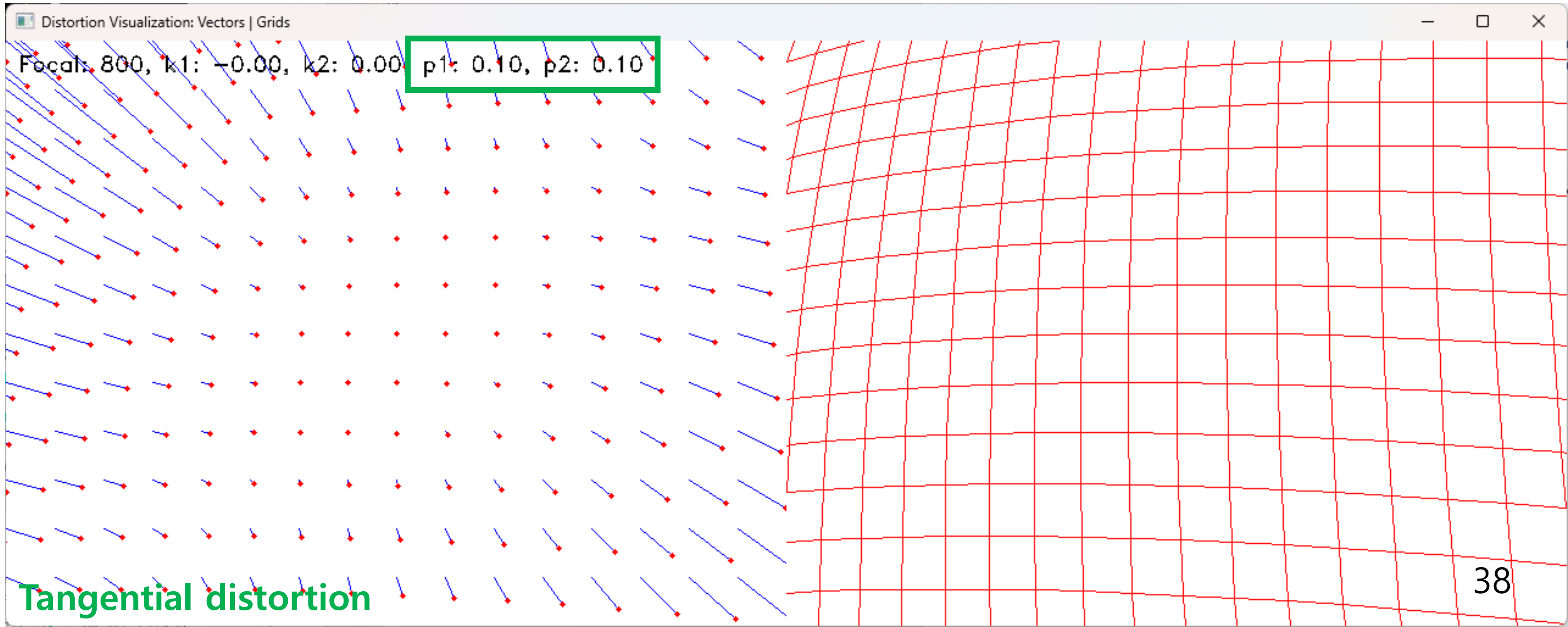



Why?
Lens and sensor are not parallel.
(usually negligible)

# Geometric Distortion Models

- Example) **Geometric distortion visualization** [distortion_visualization.py]



# Geometric Distortion Models

- Example) **Geometric distortion visualization** [distortion_visualization.py]

```python
# The initial camera configuration
img_w, img_h = (640, 480)
K = np.array([[800, 0, 320],
              [0, 800, 240],
              [0, 0, 1.]])
dist_coeff = np.array([-0.2, 0.1, 0, 0])
grid_x, grid_y, grid_z = (-18, 19), (-15, 16), 20

obj_pts = np.array([[x, y, grid_z] for y in range(*grid_y) for x in range(*grid_x)], dtype=np.float32)
while True:
    # Project 3D points with/without distortion
    dist_pts, _ = cv.projectPoints(obj_pts, np.zeros(3), np.zeros(3), K, dist_coeff)
    zero_pts, _ = cv.projectPoints(obj_pts, np.zeros(3), np.zeros(3), K, np.zeros(4))

    # Draw vectors
    img_vector = np.full((img_h, img_w, 3), 255, dtype=np.uint8)
    for zero_pt, dist_pt in zip(zero_pts, dist_pts):
        cv.line(img_vector, np.int32(zero_pt.flatten()), np.int32(dist_pt.flatten()), (255, 0, 0))
    for pt in dist_pts:
        cv.circle(img_vector, np.int32(pt.flatten()), 1, (0, 0, 255), -1)

    # Draw grids
    img_grid = np.full((img_h, img_w, 3), 255, dtype=np.uint8)
    dist_pts = dist_pts.reshape(len(range(*grid_y)), -1, 2)
    for pts in dist_pts:
        cv.polylines(img_grid, [np.int32(pts)], False, (0, 0, 255))
```

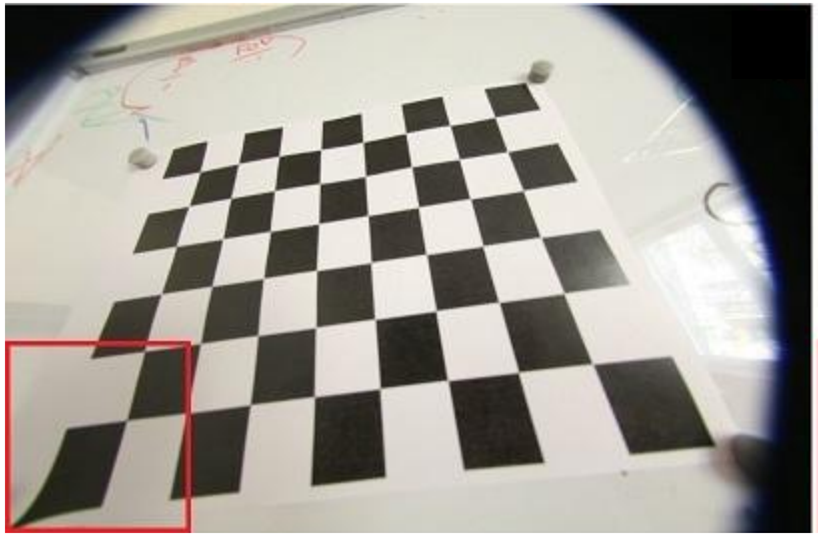
# Camera Projection Model

- **Geometric distortion models**
  - **Fisheye lens model** (a.k.a. Kannala-Brandt model; T-PAMI 2006)

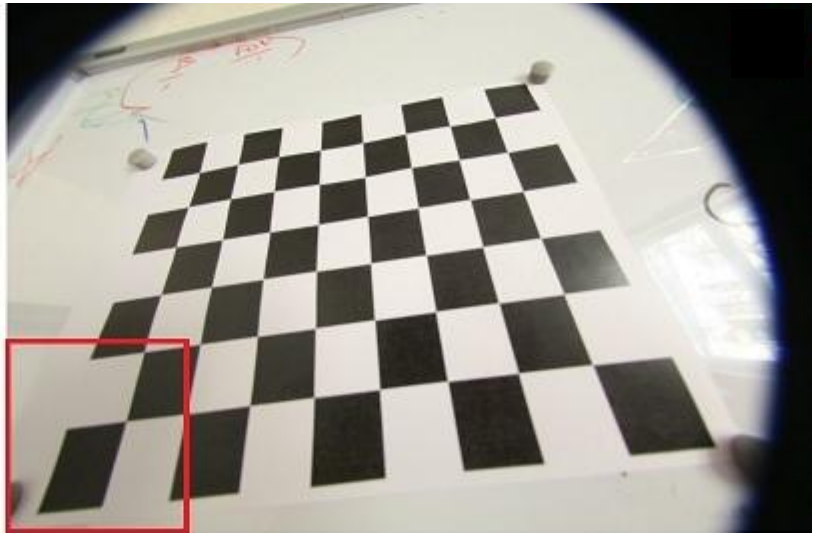
$$\begin{bmatrix} \hat{x}_d \\ \hat{y}_d \end{bmatrix} = (1 + k_1 \theta^2 + k_2 \theta^4 + \cdots) \frac{\theta}{r} \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} \quad \text{where} \quad r^2 = \hat{x}^2 + \hat{y}^2 \quad \text{and} \quad \theta = \tan^{-1} r$$

    - The fisheye lens model can describe strong barrel distortion especially around image boundaries.
    - OpenCV: `cv.fisheye.projectPoints()` ↔ `cv.fisheye.undistortPoints()`

Original fisheye image



Polynomial distortion model



Fisheye lens model

Image: OpenCV

# Geometric Distortion Models

- **Geometric distortion correction**
  - Input: The original image
  - Output: Its rectified image (without geometric distortion)
  - Given: Its camera matrix and distortion coefficients
  - Solutions for the polynomial distortion model
    - OpenCV `cv.undistort()` and `cv.undistortPoints()` (Note: included in `imgproc` module)
      ↔ `cv.projectPoints()` (Note: included in `calib3d` module)

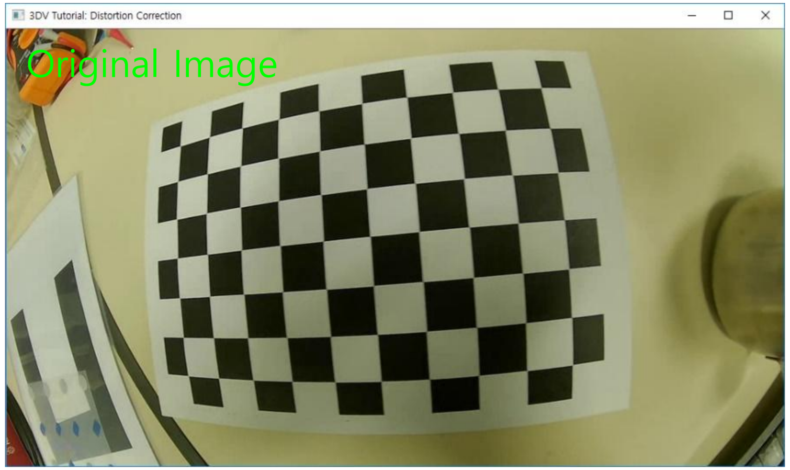
distortion correction →

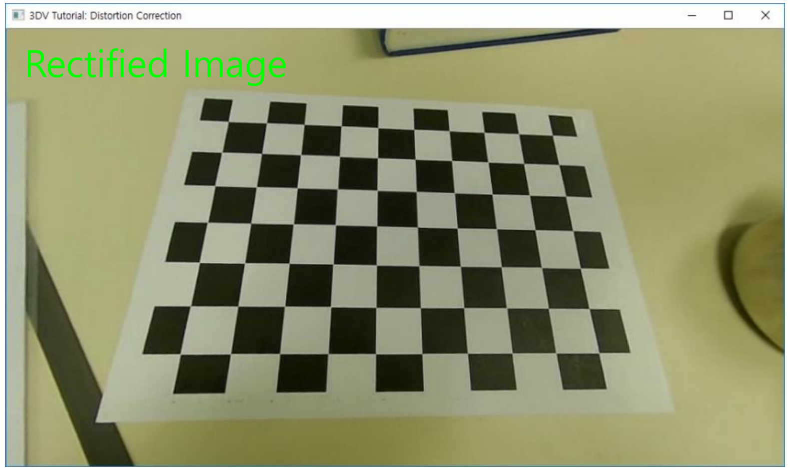
```
K1: 1.105763E-01
K2: 1.886214E-02
K3: 1.473832E-02
P1:-8.448460E-03
P2:-7.356744E-03
```

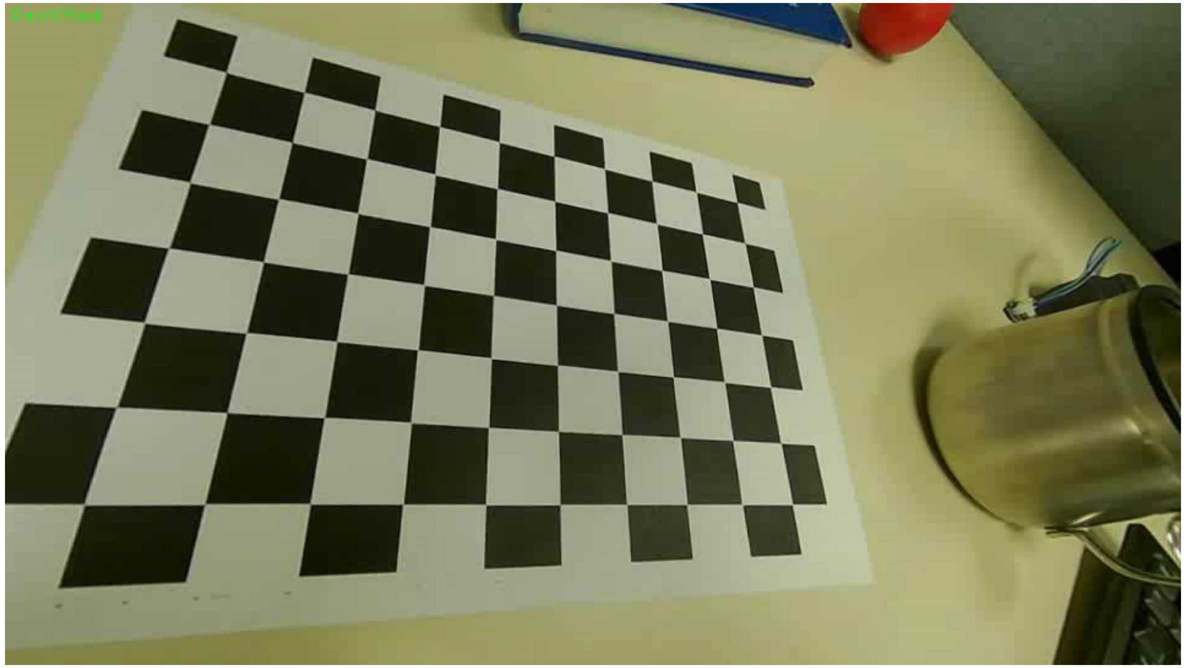
Image: [Shawn Becker]

# Geometric Distortion Models

- Example) **Geometric distortion correction** [distortion_correction.py]



# Geometric Distortion Models

- Example) **Geometric distortion correction** [distortion_correction.py]

```
# The given video and calibration data
video_file = '../data/chessboard.avi'
K = np.array([[432.7390364738057, 0, 476.0614994349778],
              [0, 431.2395555913084, 288.7602152621297],
              [0, 0, 1]]) # Derived from `calibrate_camera.py`
dist_coeff = np.array([-0.2852754904152874, 0.1016466459919075, -0.0004420196146339175, ...])

# Open a video
video = cv.VideoCapture(video_file)

# Run distortion correction
show_rectify = True
map1, map2 = None, None
while True:
    # Read an image from the video
    valid, img = video.read()
    ...

    # Rectify geometric distortion (Alternative: `cv.undistort()`)
    info = "Original"
    if show_rectify:
        if map1 is None or map2 is None:
            map1, map2 = cv.initUndistortRectifyMap(K, dist_coeff, None, None, (img.shape[1], img.shape[0]), cv.CV_3
        img = cv.remap(img, map1, map2, interpolation=cv.INTER_LINEAR)
        info = "Rectified"
    cv.putText(img, info, (10, 25), cv.FONT_HERSHEY_DUPLEX, 0.6, (0, 255, 0))
```

# Camera Projection Model

Simplification *without* consideration of *geometric distortion*

- **Camera parameters**

$$\mathbf{x} = \mathrm{proj}(\mathbf{X}; \mathrm{K}, \mathrm{R}, \mathbf{t}, d)$$

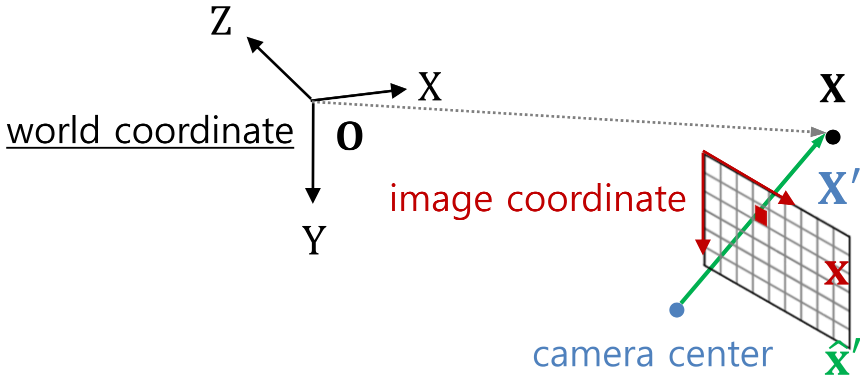
~ projection matrix P

~ $\mathbf{x} = \mathrm{P}\mathbf{X} = \mathrm{K}\,[\mathrm{R} \mid \mathbf{t}]\mathbf{X}$

  - **Intrinsic parameters** ~ camera matrix K
    - e.g. Focal length, principal point, skew, *geometric distortion coefficient*, …
  - **Extrinsic parameters** ~ point transformation R and **t**
    - e.g. Rotation and translation

- A **camera projection model** *maps* a **3D point** (metric) in the 3D world *to* a **2D point** (pixel) on the image.

→ A **camera projection model** *interprets* a **pixel** on the image *as* its physical **ray** in the 3D world.




$\mathbf{X} = [X, Y, Z]^\top$

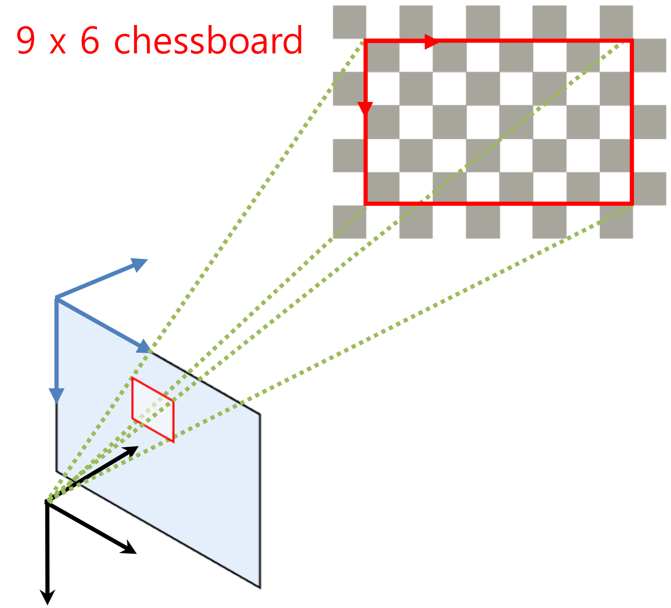
$\mathbf{X}' = \mathrm{R}\mathbf{X} + \mathbf{t}$

$\mathbf{x} = \mathrm{K}\,\mathbf{X}'$

$\hat{\mathbf{x}}' = \mathrm{K}^{-1}\mathbf{x}$

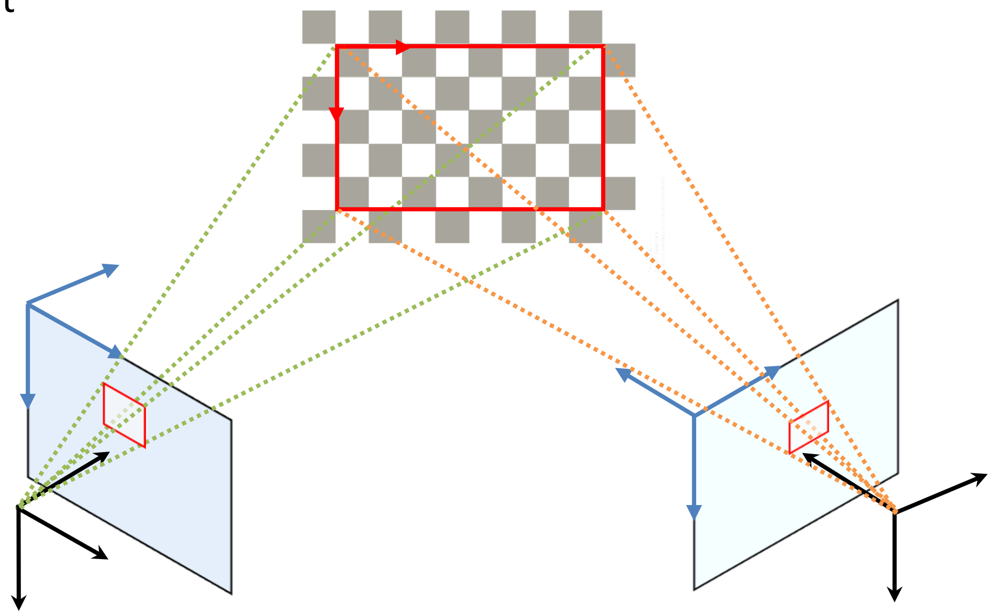
# Camera Calibration

- **Camera calibration**
  - Unknown: Intrinsic + extrinsic parameters (5* + 6 DOF)
    - Note) The number of intrinsic parameters* can vary according to user configuration.
  - Given: 3D points $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ and their projected points $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$
  - Constraints: $n$ x projection $\mathbf{x}_i = \mathrm{K}\,[\,\mathrm{R}\,|\,\mathbf{t}\,]\,\mathbf{X}_i$



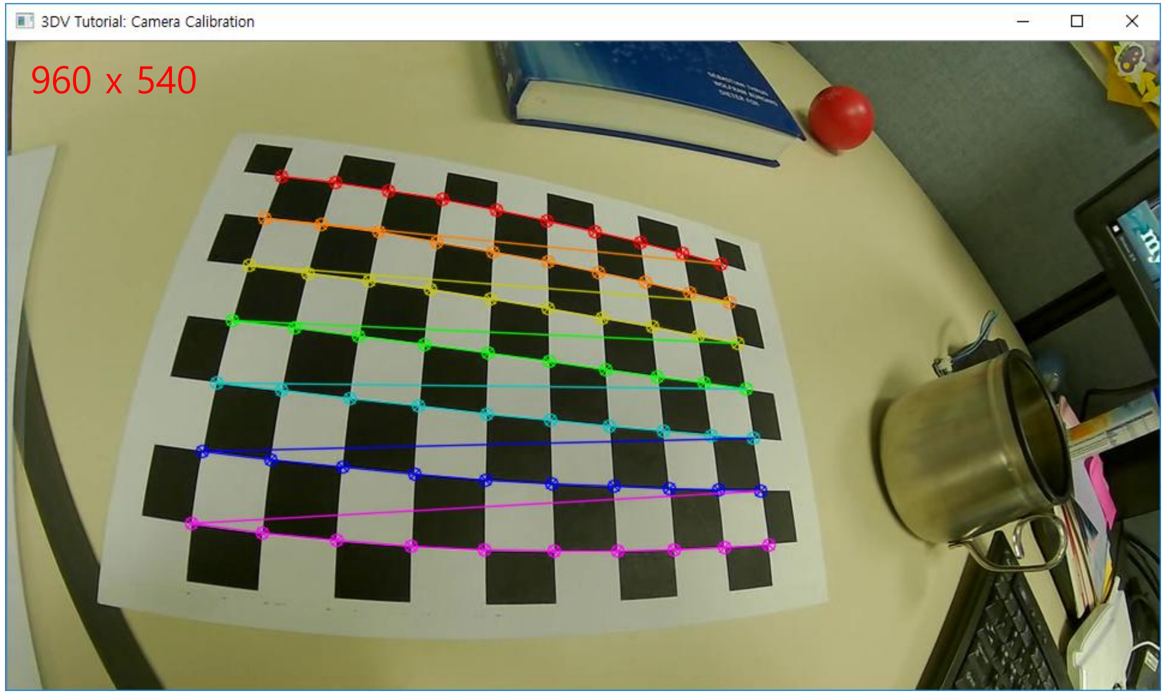
# Camera Calibration

- **Camera calibration**
  - Unknown: Intrinsic + $m$ x extrinsic parameters (5* + $m$ x 6 DOF)
  - Given: 3D points $\mathbf{X}_1$, $\mathbf{X}_2$, ..., $\mathbf{X}_n$ and their projected points, $\mathbf{x}_i^j$, on the $j$th image
    - Note) $m$: the number of images, $n$: the number of 3D points
  - Constraints: $m$ x $n$ x projection $\mathbf{x}_i^j = \mathrm{proj}(\mathbf{X}_i; \mathrm{K}, \mathrm{R}_j, \mathbf{t}_j, d) \sim \mathrm{K}\,[\,\mathrm{R}_j \,|\, \mathbf{t}_j\,]\,\mathbf{X}_i$
  - Solutions [Tools]
    - OpenCV: `cv.calibrateCamera()` and `cv.initCameraMatrix2D()`
    - Camera Calibration Toolbox for MATLAB, Jean-Yves Bouguet
    - DarkCamCalibrator, 다크 프로그래머
  - Note) How to get calibration boards
    - Print out the pattern and stick it on a hard board
    - Calibration Checkerboard Collection, Mark Jones
    - Pattern Generator, calib.io



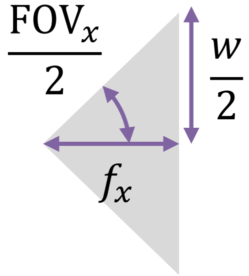
# Camera Calibration

- Example) **Camera calibration** [camera_calibration.py]




Note) **Field-of-view (FOV)** = focal length (w/o distortion)

- Horizontal: $2 \times \tan^{-1} \frac{w/2}{f_x} = 96°$
- Vertical: $2 \times \tan^{-1} \frac{h/2}{f_y} = 64°$




```
## Camera Calibration Results
* The number of applied images = 22
* RMS error = 0.473353
* Camera matrix (K) =
  [432.7390364738057, 0, 476.0614994349778]
  [0, 431.2395555913084, 288.7602152621297]
  [0, 0, 1]
* Distortion coefficient (k1, k2, p1, p2, k3, ...) =
  [-0.2852754904152874, 0.1016466459919075, -0.0004420196146339175, 0.0001149909868437517, -0.01803978785585194]
```

Note) Close to the center of the image, (480, 270)

Note) Close to zero (usually negligible)

# Camera Calibration

- Example) **Camera calibration** [camera_calibration.py]

```
def select_img_from_video(video_file, board_pattern, select_all=False, wait_msec=10):
    # Open a video
    video = cv.VideoCapture(video_file)

    # Select images
    img_select = []
    ...
    return img_select

def calib_camera_from_chessboard(images, board_pattern, board_cellsize, K=None, dist_coeff=None, calib_flags=None):
    # Find 2D corner points from given images
```

$\mathbf{x}_i^j$

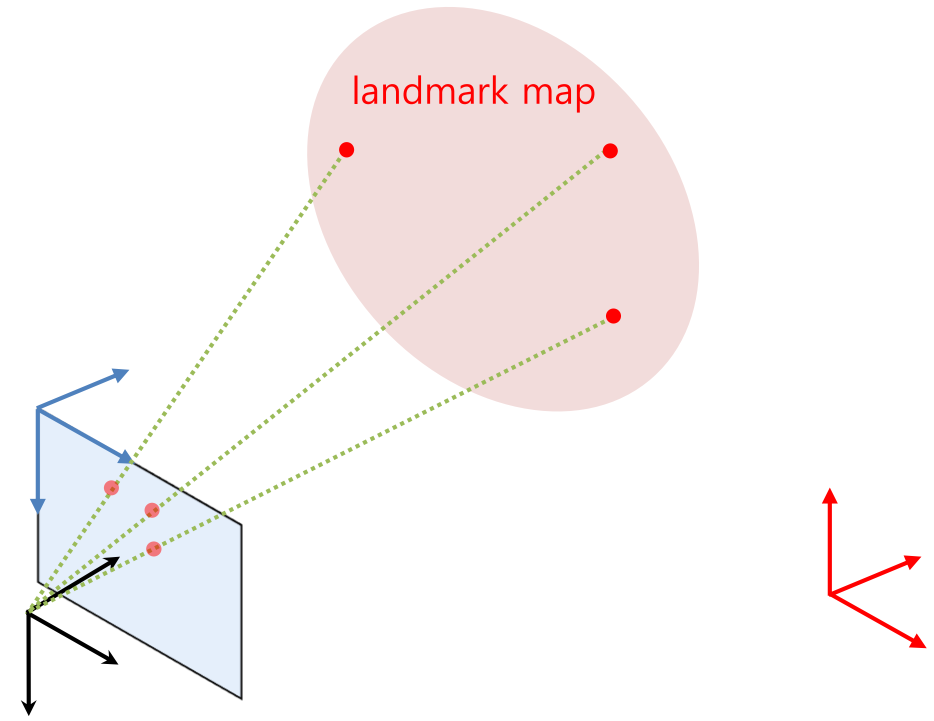
```
    img_points = []
    for img in images:
        gray = cv.cvtColor(img, cv.COLOR_BGR2GRAY)
        complete, pts = cv.findChessboardCorners(gray, board_pattern)
        if complete:
            img_points.append(pts)
    assert len(img_points) > 0, 'There is no set of complete chessboard points!'

    # Prepare 3D points of the chess board
```

$\mathbf{X}_i$

```
    obj_pts = [[c, r, 0] for r in range(board_pattern[1]) for c in range(board_pattern[0])]
    obj_points = [np.array(obj_pts, dtype=np.float32) * board_cellsize] * len(img_points) # Must be `np.float32`

    # Calibrate the camera
    return cv.calibrateCamera(obj_points, img_points, gray.shape[::-1], K, dist_coeff, flags=calib_flags)
```

# Absolute Camera Pose Estimation

- **Absolute camera pose estimation** (perspective-n-point; PnP)
  - Unknown: Camera pose $\mathrm{R}$ and $\mathbf{t}$ (6 DOF)
  - Given: 3D points $\mathbf{X}_1$, $\mathbf{X}_2$, …, $\mathbf{X}_n$, their projected points $\mathbf{x}_1$, $\mathbf{x}_2$, …, $\mathbf{x}_n$, and camera matrix $\mathrm{K}$
  - Constraints: $n$ x projection $\mathbf{x}_i = \mathrm{K}\,[\,\mathrm{R}\,|\,\mathbf{t}\,]\,\mathbf{X}_i$
  - Solutions ($n \geq 3$) → 3-point algorithm
    - OpenCV: `cv.solvePnP()` and `cv.solvePnPRansac()`



# Absolute Camera Pose Estimation

- Example) **Pose estimation (chessboard)** `[pose_estimation_chessboard.py]`

- Prepare a 3D box (AR model)
- Prepare 3D points (chessboard)

↓

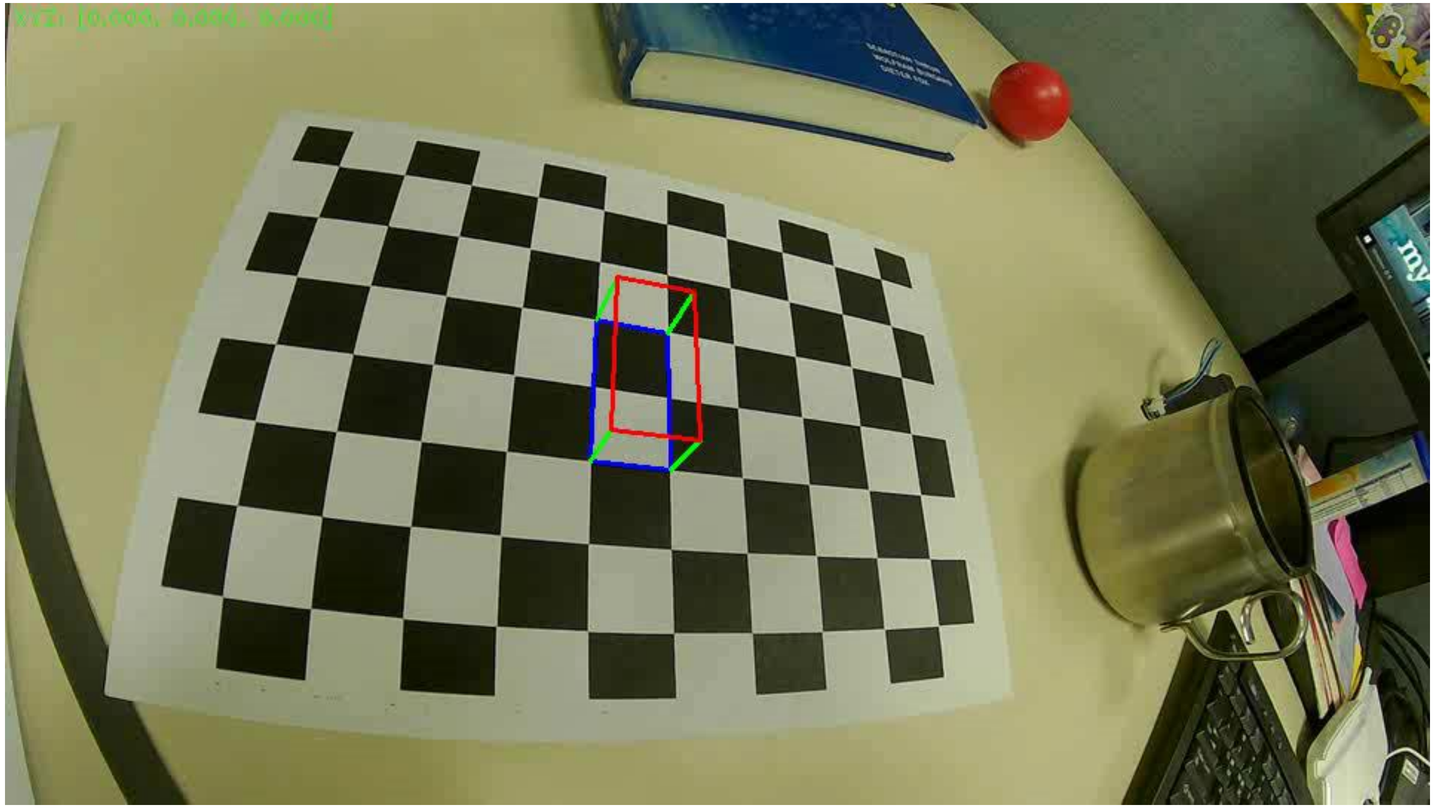
- Grab an image
- Find a chessboard on the image

↓

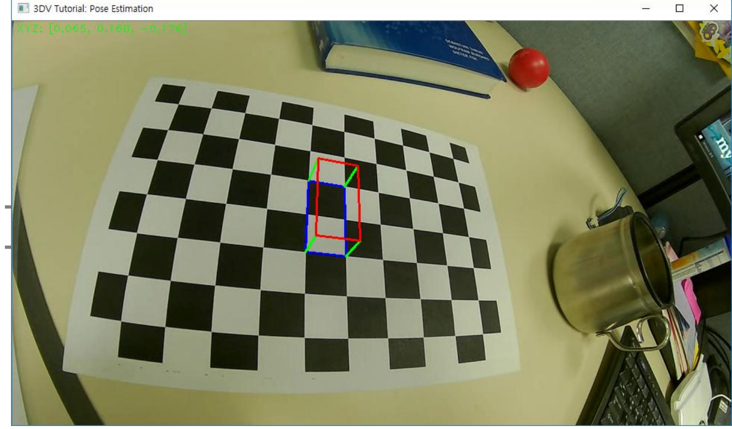
**[Camera pose estimation]**
- Estimate camera pose using `cv.solvePnP()`

↓

**[AR rendering]**
- Draw the 3D box on the image using `cv.projectPoints()`



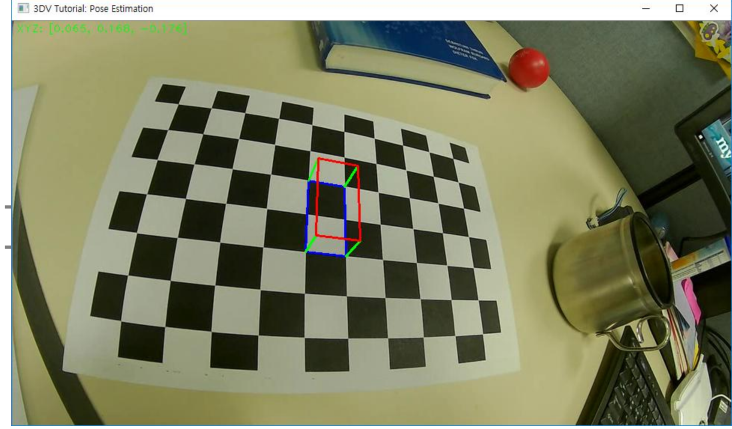
# Absolute Camera Pose Estimation




- Example) **Pose estimation (chessboard)** [pose_estimation_chessboard.py]

```python
# Open a video
video = cv.VideoCapture(video_file)

# Prepare a 3D box for simple AR
box_lower = board_cellsize * np.array([[4, 2,  0], [5, 2,  0], [5, 4,  0], [4, 4,  0]])
box_upper = board_cellsize * np.array([[4, 2, -1], [5, 2, -1], [5, 4, -1], [4, 4, -1]])

# Prepare 3D points on a chessboard
obj_points = board_cellsize * np.array([[c, r, 0] for r in range(board_pattern[1]) for c in range(board_pattern[0])])

# Run pose estimation
while True:
    # Read an image from the video
    valid, img = video.read()
    if not valid:
        break

    # Estimate the camera pose
    complete, img_points = cv.findChessboardCorners(img, board_pattern, board_criteria)
    if complete:
        ret, rvec, tvec = cv.solvePnP(obj_points, img_points, K, dist_coeff)

        # Draw the box on the image
        line_lower, _ = cv.projectPoints(box_lower, rvec, tvec, K, dist_coeff)
        line_upper, _ = cv.projectPoints(box_upper, rvec, tvec, K, dist_coeff)
        cv.polylines(img, [np.int32(line_lower)], True, (255, 0, 0), 2)
```

# Absolute Camera Pose Estimation




- Example) **Pose estimation (chessboard)** [pose_estimation_chessboard.py]

```python
# Open a video
# Prepare a 3D box for simple AR
# Prepare 3D points on a chessboard

# Run pose estimation
while True:
    # Read an image from the video

    # Estimate the camera pose
    complete, img_points = cv.findChessboardCorners(img, board_pattern, board_criteria)
    if complete:
        ret, rvec, tvec = cv.solvePnP(obj_points, img_points, K, dist_coeff)

        # Draw the box on the image
        line_lower, _ = cv.projectPoints(box_lower, rvec, tvec, K, dist_coeff)
        line_upper, _ = cv.projectPoints(box_upper, rvec, tvec, K, dist_coeff)
        cv.polylines(img, [np.int32(line_lower)], True, (255, 0, 0), 2)
        cv.polylines(img, [np.int32(line_upper)], True, (0, 0, 255), 2)
        for b, t in zip(line_lower, line_upper):
            cv.line(img, np.int32(b.flatten()), np.int32(t.flatten()), (0, 255, 0), 2)

        # Print the camera position
        R, _ = cv.Rodrigues(rvec) # Alternative) `scipy.spatial.transform.Rotation`
        p = (-R.T @ tvec).flatten()
        info = f'XYZ: [{p[0]:.3f} {p[1]:.3f} {p[2]:.3f}]'
        cv.putText(img, info, (10, 25), cv.FONT_HERSHEY_DUPLEX, 0.6, (0, 255, 0))
```

# Absolute Camera Pose Estimation

- Example) **Pose estimation (book)** `[pose_estimation_book1.py]`

- Load the cover image
- Extract ORB features
- Build **3D object points** $\mathbf{X}_1, \mathbf{X}_2, \ldots$

Pass tentative matches

Pass correct matches

- Grab an input image
- Extract **ORB features** $\mathbf{x}_1, \mathbf{x}_2, \ldots$
- Match them to the object features

**[Outlier Rejection]**
- Remove wrong matches using `cv::solvePnPRansac()`

**[Camera Pose Estimation]**
- Estimate camera pose using `cv::solvePnP()`

**[Camera Pose Update]**
- Intrinsic: $\mathtt{K}$ (given and fixed)
- Extrinsic: $\mathtt{R}$, $\mathbf{t}$

Draw the box using `cv::projectPoints()`


(due to unknown and changing $\mathtt{K}$; autofocus)

# Absolute Camera Pose Estimation

- Example) **Pose estimation (book) + camera calibration** [pose_estimation_book2.py]

- Load the cover image
- Extract ORB features
- Build **3D object points** $\mathbf{X}_1, \mathbf{X}_2, \ldots$

Pass tentative matches

Pass correct matches

- Grab an input image
- Extract **ORB features** $\mathbf{x}_1, \mathbf{x}_2, \ldots$
- Match them to the object features

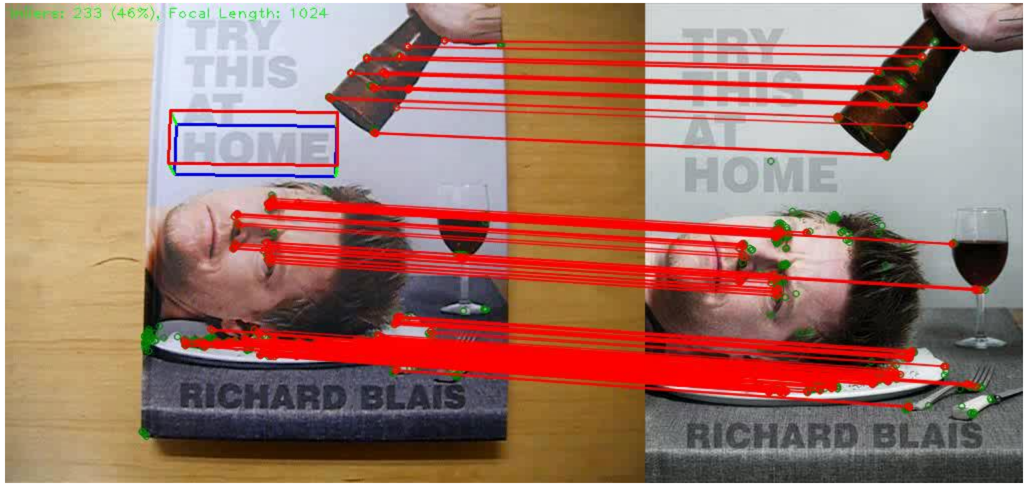
**[Outlier Rejection]**
- Remove wrong matches using `cv::solvePnPRansac()`

**[Camera Pose Estimation]**
- Estimate camera parameters using `cv::calibrateCamera()`

**[Camera Pose Update]**
- Intrinsic: $K$ (initially given)
- Extrinsic: $R$, $\mathbf{t}$

Draw the box using `cv::projectPoints()`

(due to wrong initial $K$)

# Absolute Camera Pose Estimation

- Example) **Pose estimation (book) + camera calibration – initially given** $\mathrm{K}$ `[pose_estimation_book3.py]`

(due to wrong initial $\mathrm{K}$)

- Load the cover image
- Extract ORB features
- Build **3D object points** $\mathbf{X}_1, \mathbf{X}_2, \ldots$

Pass tentative matches

Pass correct matches

- Grab an input image
- Extract **ORB features** $\mathbf{x}_1, \mathbf{x}_2, \ldots$
- Match them to the object features

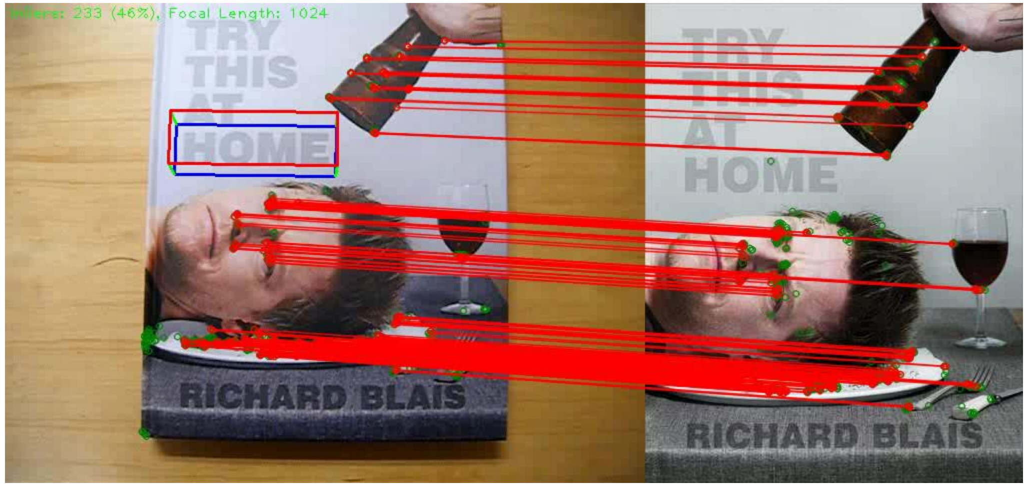
**[Outlier Rejection]**
- Remove wrong matches using `cv::findHomography()`

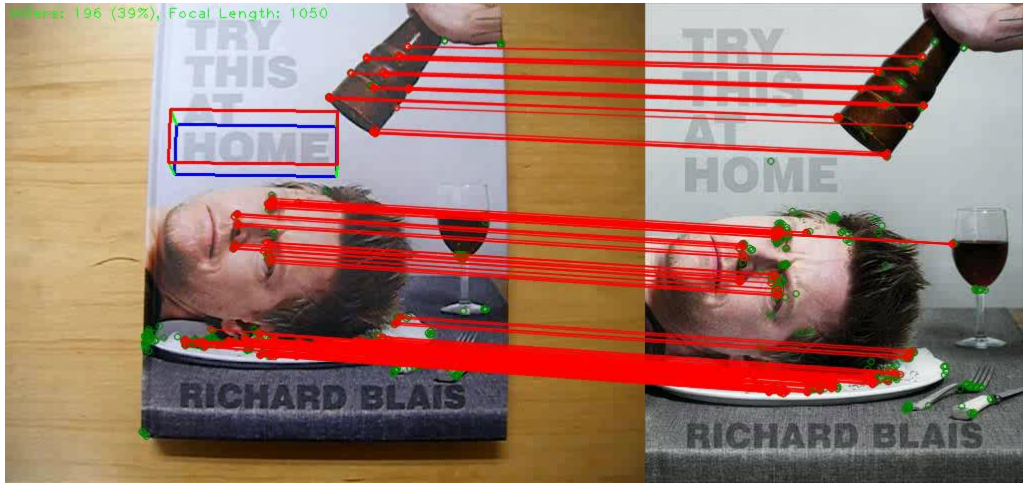
(free from $\mathrm{K}$)

**[Camera Pose Estimation]**
- Estimate camera parameters using `cv::calibrateCamera()`

**[Camera Pose Update]**
- Intrinsic: $\mathrm{K}$ (initially known)
- Extrinsic: $\mathrm{R}$, $\mathbf{t}$

Draw the box using `cv::projectPoints()`



# Summary) Single-view Geometry

- **Camera Projection Models:** $\mathbf{x} = \mathrm{proj}(\mathbf{X}; \mathrm{K}, \mathrm{R}, \mathbf{t}, d)$
  - **Pinhole camera model**: $\mathbf{x} = \mathrm{K}(\mathrm{R}\mathbf{X} + \mathbf{t})$
    - Note) Homogeneous coordinate
    - Example) Object localization / image formation
  - **Geometric distortion models**: $\hat{\mathbf{x}}_d = f_d(\hat{\mathbf{x}})$ on the normalized image plane ($\hat{\mathbf{x}}$; $\hat{z} = 1$)
    - e.g. Polynomial distortion model: Radial distortion and tangential distortion
    - Example) Distortion visualization / distortion correction
- **Camera Calibration**
  - Problem) Finding camera intrinsic parameters ($\mathrm{K}$, distortion coefficients) and extrinsic parameters ($\mathrm{R}$ and $\mathbf{t}$)
  - Example) Camera calibration
- **Absolute Camera Pose Estimation (PnP)**
  - Problem) Finding camera extrinsic parameters ($\mathrm{R}$ and $\mathbf{t}$) $\rightarrow$ camera pose ($\mathrm{R}^\intercal$ and $-\mathrm{R}^\intercal\mathbf{t}$)
  - Example) Pose estimation (chessboard)
  - Example) Pose estimation (book) as three versions
    - Q) What is RANSAC? What is homography?

# Beyond Classic Camera Models and Calibration

- Classic camera models: *Parametric* models
  - Pinhole camera projection + Polynomial lens distortion (a.k.a. Brown-Conrady model)
  - Pinhole camera projection + Fisheye lens distortion (a.k.a. Kannala-Brandt model)
- *Non-parametric* representation
  - Why non-parametric models (more parameters)?

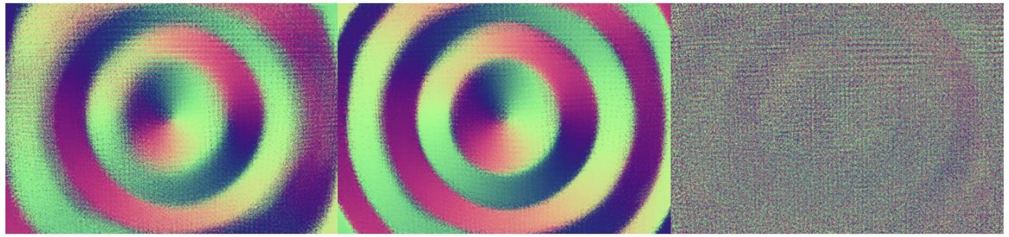


Figure 1. Residual distortion patterns of fitting two parametric camera models (left, center) and a generic model (right) to a mobile phone camera. While the generic model shows mostly random noise, parametric models show strong systematic modeling errors.

How to represent the camera projection?

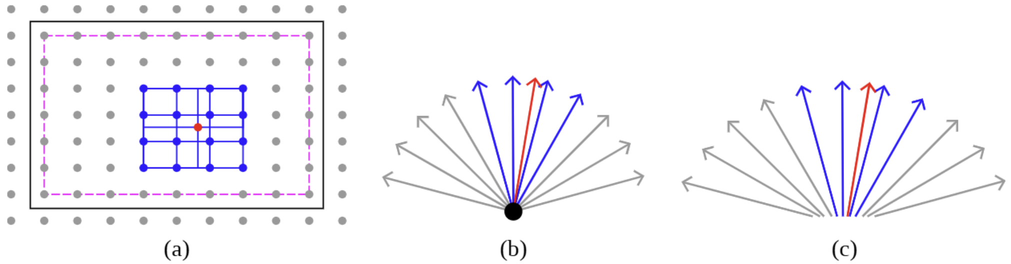



Figure 2. 2D sketch of the two generic camera models considered in this paper. (a) In image space (black rectangle), a grid of control points is defined that is aligned to the calibrated area (dashed pink rectangle) and extends beyond it by one cell. A point (red) is unprojected by B-Spline surface interpolation of the values stored for its surrounding 4x4 points (blue). Interpolation happens among directions (gray and blue arrows) starting from a projection center (black dot) for the central model (b), and among arbitrary lines (gray and blue arrows) for the non-central model (c).

Image: Schops et. al., "Why Having 10,000 Parameters in Your Camera Model is Better Than Twelve", CVPR, 2020 CVF arXiv Github

# Beyond Classic Camera Models and Calibration

- *Neural* representation
  - e.g., Perspective fields [CVPR 2023]

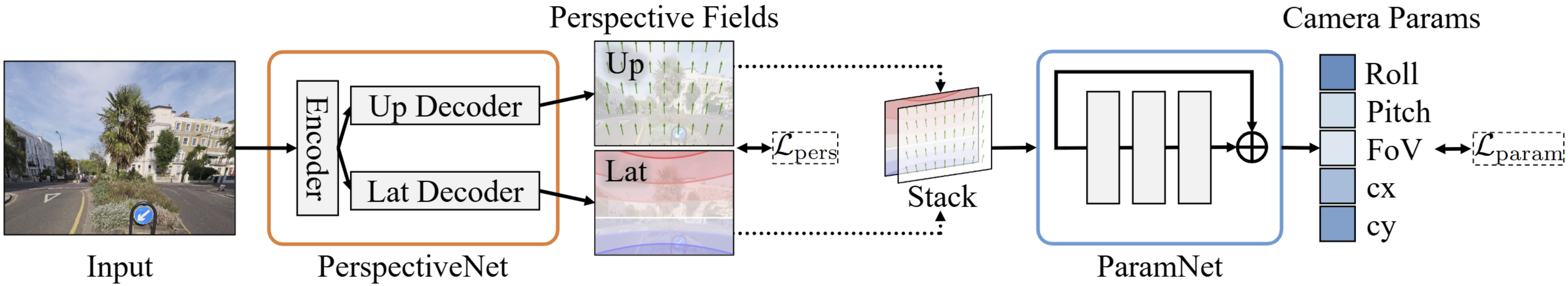



Figure 3. Left: We use a pixel-to-pixel network (PerspectiveNet) to predict Perspective Fields from a single image. Right: When classical camera parameters are needed, we use a ConvNet (ParamNet) to extract this information directly from the Perspective Fields.

Image: Jin et. al., "Perspective Fields for Single Image Camera Calibration", CVPR, 2023 [Homepage](#) [Github](#)