Collage of Two-Dimensional Words

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Abstract

We consider a new operation on one-dimensional (resp. two-dimensional) word languages, obtained by piling up, one on top of the other, words of a given recognizable language (resp. two-dimensional recognizable language) on a previously empty one-dimensional (resp. two-dimensional) array. The resulting language is the set of words "seen from above": a position in the array is labeled by the topmost letter. We show that in the one-dimensional case, the language is always recognizable. This is no longer true in the two-dimensional case which is shown by a counter-example, and we investigate in which particular cases the result may still hold.

Key words: Regular languages, picture languages

1 Introduction

The present paper deals with the notion of recognizable collection of pictures, a picture being a matrix whose entries (pixels) are taken in a finite alphabet (colors). The reader unfamiliar with the formal definition might find it suggestive to think of the set of chessboards of arbitrary dimension or of the set of

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squares with, say, their north-west to south-east diagonal marked with some particular color, as typical examples.

Assume we are given a collection of strips of wallpapers of different textures in such a way that it forms a recognizable collection. Assume further that starting from an empty frame we can paste these strips one at a time, in any arbitrary way, with possible overlapping but without rotation. At each position, the visible pixel is that belonging to the last pasted strip. This is reminiscent of the so-called painter's algorithm achieving face elimination in computer graphics where the objects nearest to the observer are painted last. Our result says that if we start with a recognizable collection of strips reduced to one column (resp. to one row), then all possible collages form again a recognizable collection. This property is obtained by studying the particular case of one-dimensional pictures, i.e., words, and by extending to two-dimensional pictures via row- (or column-) Kleene concatenation. Furthermore, we show that this closure property no longer holds when this hypothesis fails; using a counting argument, we show that there exists a finite language consisting of two strips whose collage is not recognizable. There exist general simple conditions guaranteeing recognizability of the collage in terms of the parameters of the collage, such as the maximum number of levels of strips. In the case where the alphabet is unary, yielding thus binary pictures with a color for the background and a color for the foreground, the collage is recognizable whatever the collection of strips (it may even be non-recursive).

As far as we know, the operation of collage as we mean it here is new. In [7, Proposition 5.1.], the author considers the operation consisting of tiling a picture with non-overlapping strips and shows a closure property for recognizable pictures. Concerning one-dimensional pictures, the notion of quasiperiodicity, which is remotely connected to our notion of collage was introduced in [1]. In our terminology it is a collage of a unidimensional picture with a unique strip as explained above and where the overlapping occurrences of the strip are obliged to match. A final word of caution though: the term collage was coined in [5] as a means of defining pictures via recursive geometric functions in the spirit of fractals, [2]. We use it here in a different meaning which we think appropriate for its kinship with the art movement in painting of the first decades of the twentieth century.

2 The unidimensional case

Given a finite alphabet Σ, we denote by Σ^* the free monoid of *words* or *strings* over Σ, and by ϵ the empty string. The product or concatenate of two words u and v is simply denoted by uv. For a string $w \in \Sigma^*$, we denote by $|w|$ its length and by $w[i]$ the i-th symbol of w, $i = 1, \ldots, |w|$. A string $z \in \Sigma^*$ is a *subword* or *factor* of w if there exist two strings $u, v \in \Sigma^*$ such that $w = uzv$ and we write $z = w[i \dots j]$ where $|u| = i - 1$ and $|uz| = j$. If $t \in \Sigma^*$ has same length as z, the substitution of t for z in w results in the word utv which we write $w \to utv$. We say that u is *placed at position* i. The notations $\xrightarrow{r}$ for the r-th iterate and $\xrightarrow{*}$ for the reflexive and transitive closure of $\to$ are used with their standard meaning. Given a subset $W \subseteq \Sigma^*$ of *patches*, the operation of collage consists of producing words in $(\Sigma \cup \{\square\})^*$ ($\square$ is a new symbol not in Σ) by starting with a word of the form $\square^n$ and then repeatedly replacing random factors of the current word with elements of W. A word thus obtained is called a *collage* of W. Formally $\mathtt{C}^0(W) = \square^*$ and for all $k \geq 0$

$$\mathtt{C}^{k+1}(W) = \{w' \mid \exists w \in \mathtt{C}^k(W), w \to w'\}.$$

The set of collages of W is the union $\mathtt{Collage}(W) = \bigcup_{k \geq 0} \mathtt{C}^k(W)$. We say position $0 < j \leq n$ of $w \in \mathtt{C}^k(W)$ is *covered by an occurrence* $u \in W$ whenever there exists an integer $\ell < k$ and two words w' and w'' such that

$$\square^n \xrightarrow{\ell} w' \to w'' \xrightarrow{k-\ell-1} w$$

holds where for some $w_1', w_2', w_3' \in \Sigma^*$ we have $w' = w_1' w_2' w_3'$, $w'' = w_1' u w_3'$ and $|w_1'| < j \leq |w_1'| + |u|$. An occurrence u of W placed on the interval $1, \dots, n$ is *obscured* by some occurrence v placed at some later time, whenever the subintervals corresponding to the two occurrences intersect.

Example 1 Consider $n = 11$ and assume the words *aba*, *bbbbc*, *ca* and *abaabcab*, belong to the subset W and are placed respectively at the positions 2, 4, 10, and 1 in that order. The resulting word is at the top of the following table.

a	b	a	a	b	c	a	b	$\square$	c	a
a	b	a	a	b	c	a	b			
									c	a
			b	b	b	b	c			
	a	b	a							
1	2	3	4	5	6	7	8	9	10	11

Position 4 is covered by the occurrences *aba*, *bbbbc*, *ca* and *abaabcab*. Position 9 is covered by no occurrence and position 2 is covered by the occurrences *aba* and *abaabcab*. ■

Said differently, the collage is the word obtained when reading "from above" the rectangular array. It is convenient to define the structure obtained by packing the words *aba*, *bbbbc* and *abaabcab*. This is achieved by removing all spaces between vertically aligned letters. In the previous example, each occurrence of a letter of a word of W would "fall" in its slot as long as it does not hit another letter or the floor of the structure (the indices in the following examples are just meant for clarifying further explanations and refer to the ordering of the word to which each letter belongs while processing the collage). This leads to a sequence of columns of varying height (possibly height 0), as follows

			a_4							
	b_4	a_4	b_2	b_4	c_4	a_4	b_4			
a_4	a_1	b_1	a_1	b_2	b_2	b_2	c_2		c_3	a_3

Table 1
The pile resulting from example 1

The next two definitions provide a more formal approach (it might prove useful to have the previous example in mind).

Definition 1 *Let W be a collection of words, let n be an integer and let P be a finite sequence of pairs $(x, u) \in \mathbb{N} \times W$ called a* stack. *The* pile $\texttt{Pile}_P^n$ *defined by these data is an array of n words in Σ^* which is defined by induction on the length of the sequence P as follows.*

If $P = \emptyset$ then $\texttt{Pile}_P^n$ is an array of n empty words. Otherwise let P' be the sequence P deprived of its last pair (x, u). Let ℓ be the length of u. For all $1 \leq i \leq n$ we define

$$\texttt{Pile}_P^n[i] = \begin{cases} \texttt{Pile}_{P'}^n[i]u[i - x + 1] & \textit{if } x \leq i \leq x + \ell - 1 \\ \texttt{Pile}_{P'}^n[i] & \textit{otherwise} \end{cases}$$

The height *of the pile is the maximum of the lengths of its entries.* ■

In the running example, the pile is the array with 11 elements and its height is 3 ($a_4, a_1b_4, b_1a_4, a_1b_2a_4, b_2b_4, b_2c_4, b_2a_4, c_2b_4, \epsilon, c_3, a_3$). Now we state a precise definition for what is meant by "seen from above".

Definition 2 *Given a pile $p = \texttt{Pile}_P^n$, its associated* collage *is the word w of length n, denoted by $\texttt{Collage}_P^n$, where for $i = 1, \ldots, n$ the following holds*

$$w[i] = \begin{cases} \square & \textit{if } p[i] = \epsilon \\ a \in \Sigma & \textit{if } p[i] = ua, u \in \Sigma^*, a \in \Sigma \end{cases}$$

■

In other words, the collage is obtained by selecting the topmost letter of each column and by taking $\square$ when the column is empty.

If the resulting word w does not contain the symbol $\square$, i.e., if $w \in \Sigma^*$, it is said to be *completely covered* by W. We extend the operation of collage to subsets by setting for all $W \subseteq \Sigma^*$

$$\texttt{Collage}(W) = \{w \in (\Sigma \cup \Box)^* \mid w = \texttt{Collage}^n_P, n \in \mathbb{N}, P \in (\mathbb{N} \times W)^*\} \quad (1)$$

Observation There are other natural definitions of collage of a language. Indeed, we may suppress the condition that the occurrences of W are contained in the interval $1, \ldots, n$ by allowing to clip them to the interval. Another possibility is not to fix the length of the resulting word a priori, i.e., to achieve the collage along the infinite integer line and consider the smallest interval which contains all occurrences pasted. As far as recognizable languages are concerned, the closure property is equally valid in these three cases. Observe however, that the closure property no longer holds for context-free languages. Indeed, we leave it to the reader to verify that the collage of the context-free language $= \{ca^n b^m d \in \{a, b, c, d\}^* \mid n > m\}$ is not context-free.

Theorem 1 *If $W \subseteq \Sigma^*$ is recognizable then so is $\texttt{Collage}(W)$.*

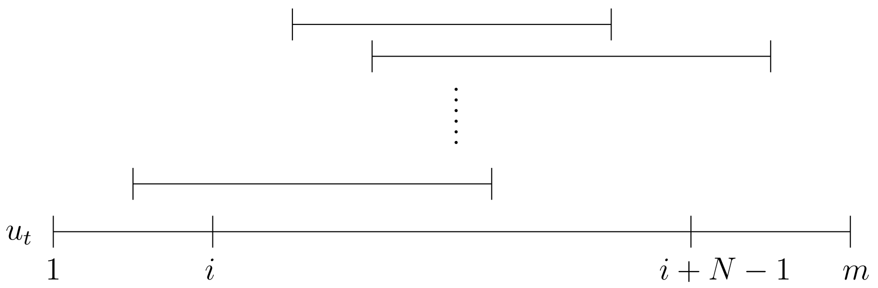
Proof.

First observe that it suffices to prove that the set of completely covered words is recognizable. Indeed, if we denote this set by $\texttt{Covered}(W)$ then we have $\texttt{Collage}(W) = \Box^*(\texttt{Covered}(W)\Box^+)^*\texttt{Covered}(W)\Box^*$.

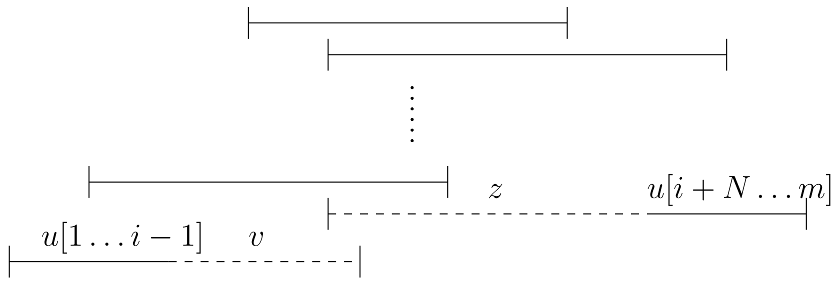
The crux of the proof is that the pile defining a given collage may be assumed of bounded height.

Lemma 2 *Let N be the number of states of an automaton recognizing W. For each $w \in \texttt{Covered}(W)$ there exists a pile of height at most $2N$ whose associated collage is w.*

Proof. There is no loss of generality to assume that, given $w \in \texttt{Collage}(W)$, the associated pile $\texttt{Pile}^n_P$ where $P = (x_t, u_t)_{1 \le t \le r}$, satisfies the condition that (x_t, u_t) is not obscured by (x_{t+1}, u_{t+1}) for $0 \le t \le r-1$, since otherwise we can eliminate (x_t, u_t) to start with. We can further assume that no factor of length N of a word u_t is obscured by the set of words $\{u_{t+1}, u_{t+2}, \ldots, u_r\}$. Indeed, consider the factor $u_t[i \ldots i+N-1]$ and assume that for all $i \le j \le i+N-1$, the position $x_t + j - 1$ is covered by some u_s where $t < s \le r$.



Let q (resp. p) be the state of the automaton after reading $u[1\ldots i-1]$ (resp. $u[1\ldots i+N-1]$) starting from the initial state. Let v (resp. z) be a word of length less than N taking q to some final state (resp. the initial state to p). Replacing the pair (x_t, u_t) by the pair $(x_t, u_t[1\ldots i-1]v)$ followed by the pair $(x_t+i+N-|z|, zu_t[i+N\ldots m])$ where $m=|u_t|$, results in the same collage.



Now assume a pile satisfying the preliminary claim has height greater than or equal to $2N$ at a position $1\leq i\leq n$. Let $1\leq s_1<\ldots<s_k\leq r$ be the maximal increasing sequence of indices such that $u_{s_1}, u_{s_2}, \ldots, u_{s_k}$ cover position i and for $t=1,\ldots,k$ let $[\lambda_{s_t}, \rho_{s_t}]$ be the interval covered by the sequence $u_{s_t}, u_{s_{t+1}}, \ldots, u_{s_k}$ with $k\geq 2N$ by hypothesis. The sequence

$$[\lambda_{s_1}, \rho_{s_1}]\supset[\lambda_{s_2}, \rho_{s_2}]\supset\ldots\supset[\lambda_{s_k}, \rho_{s_k}]$$

is strictly decreasing by the preliminary remark, and each element contains the position i. Then either $i-\lambda_{s_1}-1$ or $\rho_{s_k}-i-1$ is greater than N, a contradiction.

■

We now turn to the proof of our theorem. We call a language over a given alphabet Σ *marked local* whenever it is possible to partition the alphabet $\Sigma=I\cup H\cup F$ in such a way that a word belongs to the language if and only if its initial letter belongs to I, its final letter belongs to F, the remaining letters belong to H and the transitions between consecutive letters belong to a subset $V\subseteq\Sigma\times\Sigma$. This is a strengthening of the standard notion of local languages. It is clear that the recognizable language W is the image of some marked local language U in a letter-to-letter morphism f. It is also clear that the collage of W is the image of the collage of U in the morphism f. Consequently, we may assume without loss of generality that W itself is marked local. Because of Lemma 2, it suffices to consider piles of words of length bounded by some integer h in order to generate all words in the collage of W. If we prove that the set of piles, viewed as words over the alphabet $\Delta = \bigcup_{0 \leq i \leq h} \Sigma^i$, is recognizable, then the collage itself is recognizable. This is achieved by showing that the language $\mathtt{Pile}(W)$ is marked local. Indeed, the possible initial (resp. final) letters (over the alphabet Δ) are of the form $a_1 \cdots a_k$ with $0 < k \leq h$ and $a_i \in I$ (resp. $a_i \in F$) for $i = 1, \ldots, k$. The allowed transitions are the pairs (A, B) where $A = a_1 \cdots a_k$ and $B = b_1 \cdots b_\ell$ satisfy the following condition. Let $a_{i_1}, a_{i_2}, \ldots, a_{i_r}$ with $1 \leq i_1 < i_2 \ldots < i_r$ be the sequence of the letters of the alphabet $I \cup H$ in A and $b_{j_1}, b_{j_2}, \ldots, b_{j_s}$ with $1 \leq j_1 < j_2 \ldots < j_s$ be the sequence of the letters of the alphabet $F \cup H$ in B. Then $r = s$ and the pairs (a_{i_t}, b_{j_t}) belong to V for $t = 1, \ldots, r$.

We illustrate this last construction with example of Table 1, without distinguishing explicitly between the three subalphabets I, F and H. Consider the transition between column 3 and 4. Then $A = b_1 a_4$ and $B = a_1 b_2 a_4$. For A the subsequence of letters in $I \cup H$ is b_1, a_4, for B the subsequence of letters in $F \cup H$ is a_1, a_4. Similarly, consider the transition between column 4 and 5. Then $A = a_1 b_2 a_4$ and $B = b_2 b_4$. For A the subsequence of letters in $I \cup H$ is b_2, a_4 and for B the subsequence of letters in $F \cup H$ is b_2, a_4.

■

3 Preliminaries on picture languages

Here we borrow the terminology to the Chapter of the Handbook of Formal Languages written by D. Giammaresi and A. Restivo, [4]. The reader is also referred to [6]. We restrict ourselves to the results which are necessary for a self-contained exposition of our work.

The definitions for the free monoid extend to two-dimensional strings in a rather natural way. A *two-dimensional string* (or *picture*) is a two-dimensional rectangular array of elements in Σ. The *size* of a picture p is the pair $(r(p), c(p))$ of its number of rows and columns, also denoted by (r, c) when the picture p is understood. The element at position (i, j) with $1 \leq i \leq r$, $1 \leq j \leq c$, also called *pixel*, is denoted by $p[i, j]$. As for usual arrays, the indices grow from top to bottom for the rows and from left to right for the columns.

The set of all pictures over Σ is denoted by $\Sigma^{*\times*}$. The subset of all pictures with n columns (resp. with p rows, with n columns and p rows) is denoted by $\Sigma^{*\times n}$ (resp. $\Sigma^{p\times *}$, $\Sigma^{p\times n}$). A *two-dimensional language* over Σ is a subset of $\Sigma^{*\times *}$.

3.1 Different characterizations

The first attempt at defining some procedure for recognizing pictures is creditted to Blum and Hewitt in the seventies, [3]. Their model is an extension of the ordinary two-way one-tape automata by allowing the read head to move in all cardinal directions. It was however superseded by the more powerful and robust class of recognizable languages.

There are different and equivalent definitions for recognizable picture languages, see [4, Theorem 8.7]. In particular the notions of tiling systems and of (some type of) regular expressions lead to the same family. The notion of tiling systems is the most suitable for our purpose and we recall it now.

3.1.1 Tiling systems

Before running a procedure on the pictures, we border them with occurrences of a symbol $\sharp \notin \Sigma$, e. g.,

$$\begin{array}{|cccccc|}\hline \sharp & \sharp & \sharp & \sharp & \sharp & \sharp \\ \sharp & 2 & 1 & 3 & 1 & \sharp \\ \sharp & 2 & 2 & 3 & 3 & \sharp \\ \sharp & 1 & 3 & 3 & 1 & \sharp \\ \sharp & 3 & 2 & 3 & 2 & \sharp \\ \sharp & 1 & 2 & 3 & 2 & \sharp \\ \sharp & \sharp & \sharp & \sharp & \sharp & \sharp \\ \hline \end{array}$$

We first define a *local language* as a language consisting of all pictures whose 2×2-subpictures belong to a fixed subset of $\{\Sigma \cup \sharp\}^{2\times 2}$. E. g., the following 10 subpictures define all rectangular chessboards of odd number of rows and columns.

$$\begin{array}{|c|c|}\hline \sharp & \sharp \\ \hline \sharp & 0 \\ \hline \end{array}, \begin{array}{|c|c|}\hline \sharp & 0 \\ \hline \sharp & \sharp \\ \hline \end{array}, \begin{array}{|c|c|}\hline \sharp & \sharp \\ \hline 0 & \sharp \\ \hline \end{array}, \begin{array}{|c|c|}\hline 0 & \sharp \\ \hline \sharp & \sharp \\ \hline \end{array}, \begin{array}{|c|c|}\hline \sharp & \sharp \\ \hline 1 & 0 \\ \hline \end{array}, \begin{array}{|c|c|}\hline \sharp & \sharp \\ \hline 0 & 1 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 1 & 0 \\ \hline \sharp & \sharp \\ \hline \end{array}, \begin{array}{|c|c|}\hline 0 & 1 \\ \hline \sharp & \sharp \\ \hline \end{array}, \begin{array}{|c|c|}\hline 0 & 1 \\ \hline 1 & 0 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 1 & 0 \\ \hline 0 & 1 \\ \hline \end{array},$$

Formally, we have

Definition 3 *A* local system *is a pair* (Σ, Θ) *where* Σ *is a finite alphabet and* Θ *a subset of* $\Sigma^{2\times 2}$. *The* language defined by the system *is the set of all pictures whose* 2×2 *subpictures belong to* Θ. ■

The definition of the more general family of recognizable picture languages requires the notion of *projection* which is a mapping h from an alphabet Δ into some other alphabet Σ which extends to pictures by substituting the color $h(a) \in \Sigma$ for color $a \in \Delta$ for each pixel, resulting in a picture of the same size. Formally, we have

Definition 4 *A* tiling system *is a quadruple* $(\Sigma, \Delta, \Theta, h)$ *where* Σ *and* Δ *are finite alphabets,* Θ *a subset of* $\Delta^{2\times 2}$ *and* $h : \Delta \to \Sigma$ *a projection.*

The language recognized *by the tiling system is the projection by h of the local language recognized by the local system* (Δ, Θ)*.*

A language is tiling recognizable *if it is recognized by some tiling system.*

■

With the previous example, identifying 0 and 1 defines the collection of all pictures of uniform contents and odd number of rows and columns. Using this characterization, it can be seen that the collection of all squares is tiling recognizable but not local.

3.1.2 Regular expressions

The allowed operations are the union, intersection (not complementation), row- and column-concatenation which are partial operations and row- and column-Kleene closure. By row-concatenation of two picture languages $P, Q \subseteq \Sigma^{*\times *}$ is meant the language, denoted $P \ominus Q$, of all pictures obtained by taking two arbitrary pictures $p \in P$ and $q \in Q$ with the same number of columns and by putting p on top of q. The Kleene row-concatenation closure of a language $P \subseteq \Sigma^{*\times *}$ is the set of all pictures obtained by taking a finite sequence of pictures with the same number of columns $p_1, \ldots, p_n \in P$ and by putting p_1 on top of p_2, …,p_{n-1} on top of p_n. The notions of column-concatenation and column-Kleene closure are defined dually by concatenating from left to right. The column-concatenation of two picture languages $P, Q \subseteq \Sigma^{*\times *}$ is denoted by $P \oplus Q$.

The fundamental result of this theory is that the collection of languages recognized by some tiling system is identical to the smallest family of languages of pictures comprising all finite languages and closed under union, intersection, row- and column-concatenation, under Kleene row-and column concatenation closure and projection. Henceforth, this collection of pictures is called the family of *recognizable* picture languages.

Example 2

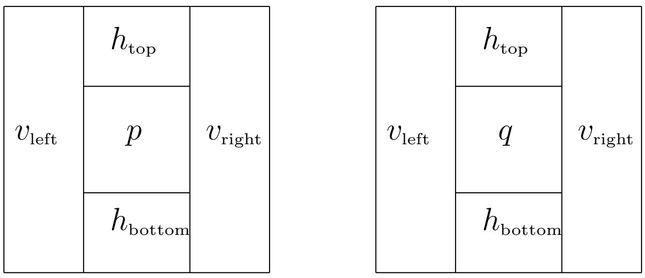
$$p = \begin{array}{|c|c|c|}\hline 1&0&2\\\hline 0&1&0\\\hline\end{array}\,, q = \begin{array}{|c|c|c|}\hline 2&2&1\\\hline 0&0&0\\\hline 1&1&1\\\hline\end{array}\,, p \ominus q = \begin{array}{|c|c|c|}\hline 1&0&2\\\hline 0&1&0\\\hline 2&2&1\\\hline 0&0&0\\\hline 1&1&1\\\hline\end{array}$$

$$p = \begin{array}{|c|c|}\hline 1&0\\\hline 0&1\\\hline 2&2\\\hline\end{array}\,, q = \begin{array}{|c|c|c|}\hline 2&2&1\\\hline 0&0&0\\\hline 1&1&1\\\hline\end{array}\,, p \oplus q = \begin{array}{|c|c|c|c|c|}\hline 1&0&2&2&1\\\hline 0&1&0&0&0\\\hline 2&2&1&1&1\\\hline\end{array}$$

3.2 A necessary condition

Recognizable languages over strings are characterized by the finiteness of the number of different right (or left) contexts. For picture languages there exists some weaker version. Indeed, it can be shown that for such a language to be recognizable, the number of non-equivalent pictures of a given size may not grow too quickly relative to the size of the picture. More precisely, given two pictures p and q with r rows and c columns respectively and a picture language X, we say that $p, q \in X$ are equivalent relative to X when for all pictures $h_{\text{top}}, h_{\text{bottom}}, v_{\text{left}}, v_{\text{right}}$ of suitable size, we have

$$v_{\text{left}} \oplus (h_{\text{top}} \ominus p \ominus h_{\text{bottom}}) \oplus v_{\text{right}} \in X$$
$$\Leftrightarrow$$
$$v_{\text{left}} \oplus (h_{\text{top}} \ominus q \ominus h_{\text{bottom}}) \oplus v_{\text{right}} \in X$$



Given a picture language X and two integers r, c, we denote by $f(r, c)$ the number of non-equivalent pictures relative to X. Then we have a weak form of syntactic characterization.

Proposition 3 *If the language X is recognizable then there exists an integer k such that for all pairs (r, c), the number of non-equivalent pictures relative to X is less than k^{r+c}.*

Because of the fundamental theorem on recognizable picture languages, this family is closed under projection. It can be proven that it is not closed under complementation, [4, Theorem 7.5]. The following property transforms the contents of the pictures, not their size. It is inspired from bit blitting operations used in computer graphics.

Let Σ_1, Σ_2 and Σ_3 be three alphabets and let $f : \Sigma_1 \times \Sigma_2 \to \Sigma_3$ be a function. Given two pictures $p_1 \in \Sigma_1^{r \times c}$ and $p_2 \in \Sigma_2^{r \times c}$ define $F(p_1, p_2)$ as the picture $p_3 \in \Sigma_3^{r \times c}$ where $p_3[i, j] = f(p_1[i, j], p_2[i, j])$. This operation extends naturally to pairs of picture languages. For example, on pictures over the binary alphabet $\{0, 1\}$, if we take f to be logical disjunction and if $X, Y \subseteq \{0, 1\}^{* \times *}$ then $F(X, Y)$ will be the set of pictures obtained by combining one picture of X and one picture of Y (these two pictures having the same dimension) with a logical OR operation. The following Proposition shows that the resulting language is recognizable if X and Y are.

Proposition 4 *If $X \subseteq \Sigma_1^{* \times *}$ and $Y \subseteq \Sigma_2^{* \times *}$ are recognizable languages then $F(X, Y)$ is recognizable.*

Proof. Let $(\Sigma_1, \Delta_1, \Theta_1, h_1)$ and $(\Sigma_2, \Delta_2, \Theta_2, h_2)$ be tiling systems recognizing X and Y. Define $\Delta_3 = \Delta_1 \times \Delta_2$, $\Theta_3 = \{t \mid \pi_1(t) \in \Theta_1 \wedge \pi_2(t) \in \Theta_2\}$ and $h_3(x, y) = f(h_1(x), h_2(y))$ where $\pi_1 : \Delta_1 \times \Delta_2 \to \Delta_1$ and $\pi_2 : \Delta_1 \times \Delta_2 \to \Delta_2$ are projections. The system $(\Sigma_3, \Delta_3, \Theta_3, h_3)$ recognizes $F(X, Y)$. ∎

4 Collage of pictures

Here, we extend to the two-dimensional case the notions introduced in section 2. It consists of "piling up" pictures belonging to a given collection one on top of the other, above a horizontal surface filled with a blank symbol. The result is the picture seen from above, the top symbol at each position obscuring all symbols under it. We directly define the collage of a picture instead of proceeding as in the previous one-dimensional case with the intermediate notion of pile.

Definition 5 *Let $P \subseteq \Sigma^{* \times *}$ be a collection of pictures, called* patches*, let (r, c) be a pair of integers and let S be a finite sequence of triples $(x, y, p) \in \mathbb{N}^2 \times P$, called a* stack*. The collage $\texttt{Collage}_S^{(r,c)}$ is the $r \times c$-array of symbols in $\Sigma \cup \{\square\}$ defined by induction on the number of elements in S as follows.*

If $S = \emptyset$ *then* $\texttt{Collage}_S^{(r,c)}$ *is the* $r \times c$*-array whose entries are all equal to the letter* $\square$. *Otherwise let* S' *be the sequence* S *deprived of its last triple* (x, y, p).

$$\texttt{Collage}_S^{(r,c)}[i,j] = \begin{cases} p[i-x+1, j-y+1] & \text{if } i \in [x, x+r(p)-1] \\ & \text{and } j \in [y, y+c(p)-1] \\ \texttt{Collage}_{S'}^{(r,c)}[i,j] & \text{otherwise} \end{cases}$$

■

Example 3 The sequence of triples $S = \{(1,1,p_1), (4,2,p_2), (4,2,p_3), (2,2,p_4), (4,1,p_5)\}$ with the following patches

$$p_1 = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 1 & 1 \\ \hline 1 & 1 \\ \hline 1 & 1 \\ \hline \end{array} \quad p_2 = \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 2 & 2 \\ \hline 2 & 2 \\ \hline 2 & 2 \\ \hline 2 & 2 \\ \hline \end{array} \quad p_3 = \begin{array}{|c|c|}\hline 3 & 3 \\ \hline 3 & 3 \\ \hline 3 & 3 \\ \hline 3 & 3 \\ \hline \end{array} \quad p_4 = \begin{array}{|c|c|}\hline 4 & 4 \\ \hline 4 & 4 \\ \hline 4 & 4 \\ \hline \end{array} \quad p_5 = \begin{array}{|c|c|}\hline 5 & 5 \\ \hline 5 & 5 \\ \hline 5 & 5 \\ \hline 5 & 5 \\ \hline \end{array}$$

defines the collage

$$p = \begin{array}{r|c|c|c|}\cline{2-4} 1 & 1 & 1 & \square \\ \cline{2-4} 2 & 1 & 4 & 4 \\ \cline{2-4} 3 & 1 & 4 & 4 \\ \cline{2-4} 4 & 5 & 5 & 4 \\ \cline{2-4} 5 & 5 & 5 & 3 \\ \cline{2-4} 6 & 5 & 5 & 3 \\ \cline{2-4} 7 & 5 & 5 & 3 \\ \cline{2-4} 8 & \square & 2 & 2 \\ \cline{2-4} \end{array}$$

■

We are interested in studying the languages of pictures obtained by applying the collage operation from a recognizable picture language. This requires to extend the operation to subsets of pictures.

Definition 6 *Given a set of patches* $P \subseteq \Sigma^{*\times*}$, *its* collage closure *is the set*

$$\texttt{Collage}(P) = \{p \in (\Sigma \cup \square)^{*\times*} \mid p = \texttt{Collage}_S^{(r,c)} \text{ for some } r, c \text{ and } S\} \quad (2)$$

■

When the resulting picture p does not contain the symbol $\square$, it is said to be *completely covered* by P.

The two-dimensional case does not enjoy as nice closure properties as the one-dimensional, as far as the collage operation is concerned. Indeed, the collage of a set of recognizable patches is not recognizable in general. However, there is a number of hypotheses under which this property still holds. As an immediate consequence of section 2, this is the case when the patches all have a unique column (or all have a unique row). Another type of restriction is when the stacks associated with a collage have bounded height. Unary alphabets are special in the sense that closure by collage is recognizable, regardless of the set of patches we start with (it may even be non-recursive). We finally give an example of a set of two patches whose collage is non-recognizable.

Proposition 5 *Let $P \subseteq \Sigma^{*\times 1}$ (resp. $P \subseteq \Sigma^{1\times *}$) be a recognizable language of patches. Then the picture language* $\texttt{Collage}(P) \subseteq (\Sigma \cup \{\sharp\})^{*\times *}$ *is again recognizable.*

Proof. Indeed, a picture in $\texttt{Collage}(P)$ is a row- (column-) concatenation of unidimensional collages. We may conclude by Theorem 1 and by the definition of recognizable picture languages. ∎

We may also bound the height of the stack, which is defined as follows. Let $S = (x_1, y_1, p_1), \ldots, (x_k, y_k, p_k)$ be a stack of k elements. Intuitively, we may consider the elements as falling on the ground and being prevented from hitting it only by previously fallen other elements occupying a position overlapping their own position. The ℓ-th element of the stack is placed at a particular integer altitude $z_\ell \geq 0$ such that two elements at the same altitude do not overlap while minimizing the maximum altitude, which by definition is the height of the stack. Observe that the number of patches covering a particular position may be bounded even if the height is not, think of a staircase for example.

Formally given a collage $\texttt{Collage}_S^{(r,c)}$ we define the height $h(i,j)$ of each pixel $1 \leq i \leq r$, $1 \leq j \leq c$ by induction on the cardinality k of the stack S. If $k = 0$ then $h(i,j) = 0$. Otherwise, let S' be the stack S deprived of its last triple (x, y, p) and let $h'(i,j)$ be the height of the pixel (i,j) in this collage. Then we have by setting $I = [x, x + r(p) - 1] \times [y, y + c(p) - 1]$

$$h(i,j) = \begin{cases} 1 + \max\{h'(k,\ell) \mid (k,\ell) \in I\} & \text{if } (i,j) \in I \\ h'(i,j) & \text{otherwise} \end{cases}$$

The *height* of the stack is the maximum value of $h(i,j)$ when (i,j) runs over the picture.

Proposition 6 *Let $P \subseteq \Sigma^{*\times*}$ be a recognizable language of patches and let k be an integer. The set $\texttt{Collage}_{h\leq k}(P)$ of collages of P which can be obtained by stacks of height k or less is recognizable.*

Proof.

In the case where $k = 1$, the proposition is a consequence of [7, Proposition 5.1.] asserting that tilings of recognizable picture language are recognizable. Indeed, a tiling is a collage of patches such that no two patches overlap and that the whole picture is covered by some patch. Then the tiling by the recognizable picture language $P \cup \{\square\}^{*\times*}$ is precisely a collage of height 1. Let P' be this language. Let $f : (\Sigma \cup \{\square\})^2 \rightarrow \Sigma \cup \{\square\}$ be defined by $f(x, y) = x$ for $x \neq \square$ and $f(\square, y) = y$ otherwise. This function allows us to combine layers of tilings of P by treating $\square$ as a transparent color. We then have, with the notations of Proposition 4

$$\texttt{Collage}_h(P) = \underbrace{F(P', F(P', \ldots F(P', P') \ldots))}_{h-1 \text{ times}}$$

Using the closure under union allows us to complete the proof. ∎

In the one-letter case, the resulting pictures are binary with e.g., 1 standing for the letter and 0 for the symbol $\square$.

Theorem 7 *If $P \subseteq \{a\}^{*\times*}$ is an arbitrary picture language then the picture language $\texttt{Collage}(P)$ is recognizable.*

Proof. Let us start with some elementary observations. If p belongs to P, then each occurrence of a rectangle q is contained in some occurrence of a rectangle p satisfying $r(p) \leq r(q)$ and $c(p) \leq c(q)$. Thus, $\texttt{Collage}(P)$ equals $\texttt{Collage}(Q)$ where Q is the set of minimal patches in P where minimal is meant componentwise. By Dickson's Lemma asserting in particular that all subsets of $\mathbb{N}^2$ have finitely many minimal elements, the subset Q is finite. In the unary case, collage corresponds to taking the logical OR on pixels, thus by Proposition 4 where the function f achieves the logical disjunction of the pixels, we see that it suffices to consider the case where Q is reduced to a unique element.

Let $P = \{a\}^{r\times c}$ be a singleton. Consider an element $p \in \texttt{Collage}(P)$. Every pixel (i, j) of p can belong to a number of occurrences. Since there is only one non-blank letter, the order in which these patches are laid is irrelevant. The number of times a given patch is laid on a given position is also irrelevant. Position of the patches however is relevant. We may therefore consider the set $B_{i,j}$ of pairs (k, ℓ) such that the rectangle $a^{r\times c}$ can be placed in p with its top left corner at position $(i-k+1, j-\ell+1)$, that is if the subpicture $[i-k+1 \ldots i-k+r] \times [j-\ell+1 \ldots j-\ell+c]$ of p is made of all a's. Let Δ be the power set of $\{1, \ldots, r\} \times \{1, \ldots, c\}$. A tiling system $(\Sigma, \Delta, \Theta, h)$ recognizing $\texttt{Collage}(P)$ is specified as follows. The primary alphabet is $\Sigma = \{a, \Box\}$ and the auxiliary alphabet is Δ. The projection h maps $\emptyset$ to $\Box$ and every other element to a. Consider the following four subsets of $(\Delta \cup \{\#\})^{2\times 2}$:

$$\Theta_1 = \left\{ \begin{array}{|c|c|}\hline x & y \\ \hline z & t \\ \hline \end{array} \mid \forall k \forall \ell \;\; (k,\ell) \in x \wedge \ell < c \Rightarrow (k, \ell+1) \in y \right\}$$
$$\Theta_2 = \left\{ \begin{array}{|c|c|}\hline x & y \\ \hline z & t \\ \hline \end{array} \mid \forall k \forall \ell \;\; (k,\ell) \in y \wedge \ell > 1 \Rightarrow (k, \ell-1) \in x \right\}$$
$$\Theta_3 = \left\{ \begin{array}{|c|c|}\hline x & y \\ \hline z & t \\ \hline \end{array} \mid \forall k \forall \ell \;\; (k,\ell) \in x \wedge k < r \Rightarrow (k+1, \ell) \in z \right\}$$
$$\Theta_4 = \left\{ \begin{array}{|c|c|}\hline x & y \\ \hline z & t \\ \hline \end{array} \mid \forall k \forall \ell \;\; (k,\ell) \in z \wedge k > 1 \Rightarrow (k-1, \ell) \in x \right\}$$

The set Θ_1 (resp. Θ_2, Θ_3, Θ_4) enforces coherent propagation of the hypotheses towards the right (resp. leftwards, downwards and upwards). We set $\Theta = \bigcap_{1 \le i \le 4} \Theta_i$. We make the somewhat untidy convention that a condition of the form $(k,\ell) \in x$ is false whenever $x = \#$. It is clear that all pictures in $\texttt{Collage}(P)$ are recognized by the tiling system. Conversely, assume a picture is recognized by the system. Then it suffices to observe that if (k,ℓ) is an element of a subset of Δ which labels the pixel at position (i,j), then all pixels at positions $(i+\alpha, j+\beta)$ satisfying $i-k+1 \le i+\alpha \le i-k+r$ and $j-\ell+1 \le j+\beta \le j-\ell+c$ are labeled by a subset containing the element $(k+\alpha, \ell+\beta)$, proving thus that the picture is a union of occurrences of the rectangle.

■

4.2 The general case

We show in this paragraph that even if the language P of patches is finite, $\texttt{Collage}(P)$ might no longer be recognizable. Actually we prove it with a set P consisting of two patches of dimension 1×3 and 3×1 respectively.

Indeed, consider the language of pictures over the alphabet $\{a, b, \mathrm{e}\}$ (b for suggesting the beginning and e the end) consisting of the *horizontal patch*

$$\begin{array}{|c|c|c|}\hline b & a & \mathrm{e} \\ \hline \end{array} \qquad (3)$$

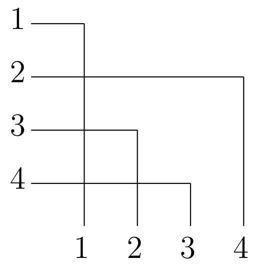


Fig. 1. The permutation $\sigma(1) = 1$, $\sigma(2) = 4$, $\sigma(3) = 2$, $\sigma(4) = 3$

and of the *vertical patch*

$$\begin{array}{|c|}\hline b \\ \hline a \\ \hline \mathrm{e} \\ \hline \end{array} \tag{4}$$

Theorem 8 *The language* `Collage`(P) *where* P *consists of the two horizontal and vertical patches 3 and 4 is not recognizable.*

Proof.

Given a permutation σ on the set of integers $\{1, \ldots, p\}$ we construct a picture with $3p - 1$ rows and $3p + 1$ columns. This picture is based on a structure composed of p different paths in the discrete plane, each path being itself composed of a horizontal line followed by a vertical, see figure 1. The coordinates of the end points of these two lines are respectively

$(3i - 2, 1)$ and $(3i - 2, 3\sigma(i) + 1)$

$(3i - 2, 3\sigma(i))$ and $(3p - 1, 3\sigma(i))$

Now we view each path of this picture as obtained by pasting one on top of the previous one, from left to right and from top to bottom, occurrences of the horizontal and then of the vertical patch. The horizontal line starts at position $(3i - 2, 1)$, has length $3\sigma(i) + 1$ and is covered by occurrences of the horizontal patch with periodic shift resulting in the sequence of labels $b, b, a, b, b, a, \ldots, b, b, a$ followed by the final sequence b, b, b, e, see figure 2. The vertical path starts at position $(3i - 2, 3\sigma(i))$ has length $3(n - i) + 2$ and is covered by occurrences of the vertical patch starting with the sequence of labels b, b and followed by a periodic sequence $b, a, b, b, a, \ldots, b, b, a, b$.

Consequently, the picture consists of p different strips built on the previous p paths, which are covered by piling up occurrences of the two patches in such a way that the collage of a patch is done on top of the previous patch. Furthermore, the order of achieving the collage of two strips is irrelevant as they intersect on an element labeled by a letter belonging to both patches.

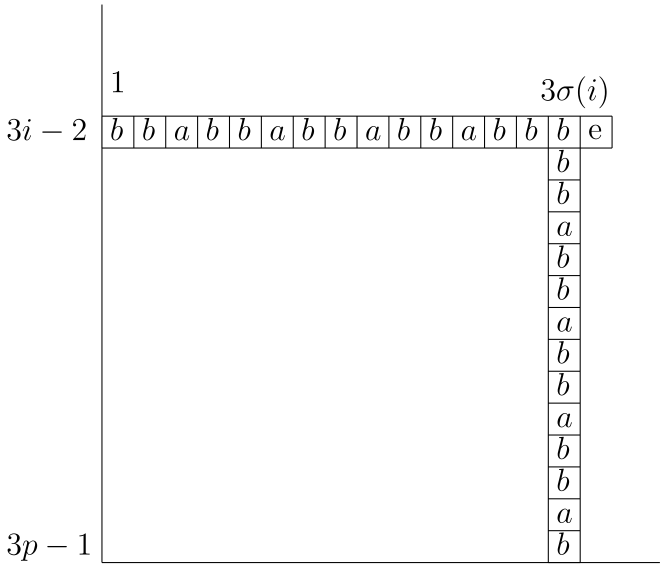


Fig. 2. A path associated with the pair $(i, \sigma(i))$

1	b	b	b	e									
			b										
			b										
2	b	b	a	b	b	a	b	b	a	b	b	b	e
			b									b	
			b									b	
3	b	b	a	b	b	b	e					a	
			b			b						b	
			b			b						b	
4	b	b	a	b	b	a	b	b	b	e		a	
			b			b			b			b	
			1			2			3			4	

Fig. 3. The 4 paths associated with the permutation of Figure 1

Consider two different permutations σ and τ and assume $\sigma(i) \neq \tau(i)$ for some $1 \leq i \leq p$. Then it is not difficult to design a context which, as the figure 4 suggests intuitively, connects $\sigma(i)$ back to i

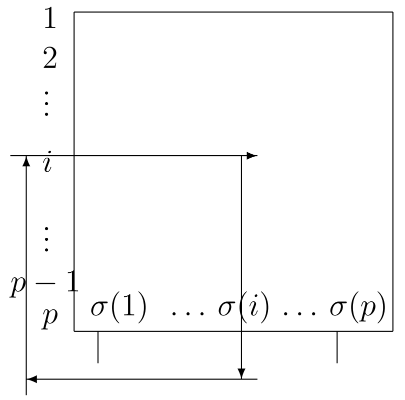


Fig. 4. A general view with a context creating a loop by closing the path connecting i and $\sigma(i)$.

and adds the minimum information so that all paths associated with the integer $j \neq i$ represent a legal collage of patches. This latter is done by simply appending $\begin{array}{|c|}\hline a \\ \hline \mathrm{e} \\ \hline \end{array}$ below positions $(3p-1, 3\sigma(j))$ for all $j \neq i$.

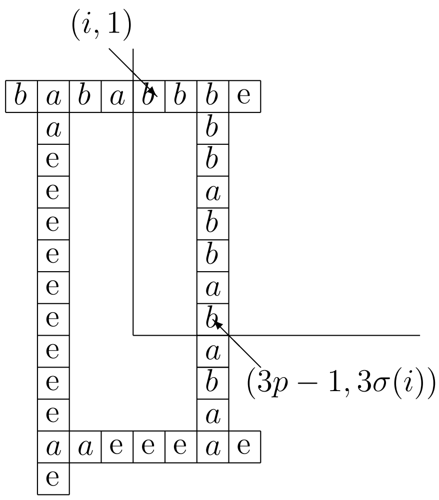


Fig. 5. A closer view at the loop.

Since all permutations on $\{1, \ldots, p\}$ define a picture having context discriminating them among all other permutations, there exist $\Omega(p!)$ non-equivalent pictures whose number of rows and columns is in $O(p)$, contradicting thus Proposition 3 and completing the proof.

			b	b	b	e									
					b										
					b										
b	a	b	b	b	a	b	b	a	b	b	a	b	b	b	e
	a				b									b	
	e				b									b	
	e		b	b	a	b	b	b	e					a	
	e				b			b						b	
	e				b			b						b	
	e		b	b	a	b	b	a	b	b	b	e		a	
	e				b			b			b			b	
	e				a			a			a			a	
	e				e			e			e			b	
	e													a	
	a	a	e	e	e	e	e	e	e	e	e	e	e	a	e
	e														

Fig. 6. The sub-picture associated with the current permutation σ surrounded by a context creating a loop

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References

[1] A. Apostolico and A. Ehrenfeucht. Efficient detection of quasiperiodicities in strings. *Theoretical Computer Science*, 119(1):247–265, 1993.

[2] M. Barnsley. *Fractals Everywhere.* Academic Press, Boston, 1988.

[3] M. Blum and C. Hewitt. Automata on two-dimensional tape. In *IEEE Symposium on Switching and Automata Theory*, pages 155–160, 1967.

[4] D. Giammarresi and A. Restivo. Two-dimensional languages. In G. Rozenberg and A. Salomaa, editors, *Handbook of formal languages: beyond words*, volume 3, pages 215–267. Springer Verlag, 1997.

[5] A. Habel and H.-J. Kreowski. Collage grammars. In H. Ehrig, H.-J. Kreowski, and G. Rozenberg, editors, *Proc. Fourth Intl. Workshop on Graph Grammars and Their Appliations to Computer Science*, number 532 in Lecture Notes in Computer Science, pages 411–429, 1991.

[6] K. Inoue and I. Takanami. A survey of two-dimensional automata theory. In *Proc. 5th Int. Meeting of Young Computer Scientists*, number 381 in Lecture Notes in Computer Science, pages 72–91, 1990.

[7] D. Simplot. A characterization of recognizable picture languages by tilings by finite sets. *Theoretical Computer Science*, 218(2):297–323, 1999.