# Selected Notes on the SCUM Lectures

**Student Colloquium for Undergraduates in Mathematics**

**Massachusetts Institute of Technology**

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These are notes I took during SCUM, held at MIT. It consists of one-hour talks every Wednesday, with free dinner, open to all. The website with SCUM information is

`http://web.mit.edu/vywang/www/scum_fa15.html`

As usual, any errors are due to the scribe, not the presenter.

# Contents

# 1 An Introduction to Wigner Matrices (Ravi Bajaj)

## 1.1 Setup

For a positive integer $n$ we consider a random symmetric matrix

$$M_n = (\xi_{ij})_{1 \le i,j \le n}$$

so that $\xi_{i,j} = \overline{\xi_{j,i}}$ for each $i, j$. We assume that we have

- $\xi_{ii}$ is identically distributed, with bounded variance and mean 0.
- $\xi_{ij}$ are identically distributed for $i \neq j$, with mean 0 and variance 1.

Since $M_n$ is Hermitian, it has real eigenvalues $\lambda_i(M_n)$ from $i = 1, \ldots, n$, sorted in ascending order. So this gives a measure

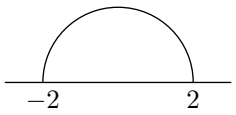
$$\mu_n \stackrel{\text{def}}{=} \mu_{\frac{1}{\sqrt{n}} M_n} \stackrel{\text{def}}{=} \frac{1}{n} \sum_i \delta_{\lambda_i(M_n)/\sqrt{n}}$$

i.e. we consider the measure with point masses at normalized eigenvalues (divided by $\sqrt{n}$).

Finally, define the semicircle measure by

$$\mu_{\text{SC}} \stackrel{\text{def}}{=} \frac{1}{2\pi} \sqrt{4 - |x|^2}\, dx.$$

which looks like a semicircle over $[-2, 2]$.




We are going to prove that

**Theorem 1.1.1**
As $n \to \infty$, $\mu_n$ tends almost surely to $\mu_{SC}$.

Our method of proof is the moment method.

## 1.2 The Moment Methods

The idea of the moment method is to show that for every integer $k$ we have

$$\mathbb{E}^{\mu_n}[x^k] \to \mathbb{E}^{\mu_{\text{SC}}}[x^k]$$

in the sense that for any $\varepsilon > 0$, the probability the two quantities above is less than $\varepsilon$ tends to zero.

**Example 1.2.1**

It turns out $\mathbb{E}^{\mu_{\text{SC}}}[X^k]$ is the $k/2$th Catalan number for $k$ even, and 0 when $k$ is odd. (Hint for proof: use $x = 2\sin\theta$.)

For the $k = 2$ case, observe that

$$\operatorname{Tr}(M^2) = \sum_i \lambda_i^2 = \sum_{i,j} |\xi_{ij}|^2$$

thus

$$\mathbb{E}(\operatorname{Tr}(M^2)) = n^2 - n + n\mathbb{E}[|\xi_{ij}|^2] = O(n^2)$$

according to the law of large numbers.

For a trickier argument, consider

$$\mathbb{E}\operatorname{Tr}(M^4) = \sum_{i_1,i_2,i_3,i_4} \mathbb{E}[\xi_{i_1,i_2}\xi_{i_2,i_3}\xi_{i_3,i_4}\xi_{i_4,i_1}] = O(n^3)$$

by some high-school level trick. To be precise, note by independence of variables, the only way this can not be zero is if it is of the form

$$\mathbb{E}[\xi_{ij}^2]\mathbb{E}[\xi_{ik}^2],$$

say; the first moments of each guy vanish. To be more precise, if we consider the graph on $n$ vertices, then $i_1 \to i_2 \to i_3 \to i_4 \to i_1$ is a walk, and if any edge is traversed exactly once then the entire product vanishes, since we can pull out a first moment.

More generally, if we extend this combinatorial argument one can compute

$$\mathbb{E}\left[\frac{1}{n}\operatorname{Tr}\left(\frac{1}{\sqrt{n}}M_n\right)^k\right] = \frac{1}{n^{k/2+1}}\sum_{i_1,\dots,i_k}\mathbb{E}\left[\prod_j X_{i_j,i_{j+1}}\right].$$

It turns out that by some combinatorial arguments, one can show this becomes the number of walks on $k$ guys with $k/2+1$ distinct vertices is $C_{k/2}$. This is the "main term" and everything else goes away (is $o(1)$).

# 2 Combinatorial Nullestellensatz and Graph Colorings (Evan Chen)

To be added; this was my own lecture, so I did not have someone to take notes for me! Until then, here is a loose outline. Relevant references are:

- `http://www.tau.ac.il/~nogaa/PDFS/null2.pdf`
- `http://www.tau.ac.il/~nogaa/PDFS/chrom3.pdf`
- `http://www.mit.edu/~evanchen/handouts/SPARC_Combo_Null_Slides/SPARC_Combo_Null_Slides.pdf`

## 2.1 Main Theorem

Let $F$ be a field.

**Fact 2.1.1.** Suppose a polynomial $P(x_1, \dots, x_n) \in F[x_1, \dots, x_n]$ has degree at most $t_i$ in each $x_i$, and we are given $|S_1| > t_1$, $|S_2| > t_2$, $\dots$, $|S_n| > t_n$. If $P$ vanishes on $S_1 \times \cdots \times S_n$ then it is the zero polynomial.

*Proof.* By induction on $n$. □

**Theorem 2.1.2** (Combinatorial Nullstellensatz)
Let $f \in F[x_1, \dots, x_n]$ be a polynomial of degree $t_1 + \cdots + t_n$. If $S_1, S_2, \dots, S_n \subseteq F$ satisfies $|S_i| > t_i$ for all $i$,

$$\exists s_i \in S_i : f(s_1, s_2, \dots, s_n) \neq 0$$

whenever the coefficient of $x_1^{t_1} x_2^{t_2} \dots x_n^{t_n}$ is nonzero.

*Proof.* Let $P_i(x) = \prod_{s \in S_i} (x - s)$. Then in $f$, repeatedly replace $x^{t_i+1}$ by the remainder mod $P_i(x)$. For example, if $S_1 = \{2, 3\}$ (hence $t_1 = 1$) repeatedly replace $x^2$ with $5x - 6$.

When the polynomial $f$ is completely reduced in this way, the $x_1^{t_1} x_2^{t_2} \dots x_n^{t_n}$ remains unharmed, but all other terms of degree $t_1 + \cdots + t_n$ have been reduced. So we can apply the previous fact. □

## 2.2 Contest Practice From 2007

**Example 2.2.1** (Russia 2007)
Two distinct numbers are written on each vertex of a convex 100-gon. Prove one can remove a number from each vertex so that the remaining numbers on any two adjacent vertices differ.

*Proof.* Define $P(x_1, \dots, x_{100})$ by

$$(x_1 - x_2)(x_2 - x_3)(x_3 - x_4)\dots(x_{99} - x_{100})(x_{100} - x_1).$$

The coefficient of $x_1x_2\dots x_{100}$ is 2. □

**Example 2.2.2** (IMO 2007/6)
Let $n$ be a positive integer. Consider

$$S = \{(x, y, z) \mid x, y, z \in \{0, 1, \dots, n\}, (x, y, z) \neq (0, 0, 0)\}$$

as a set of $(n+1)^3 - 1$ points in the three-dimensional space. Determine the smallest possible number of planes, the union of which contains $S$ but does not include $(0, 0, 0)$.

*Proof.* The answer is $3n$. For construction, use $x = 1, 2, \dots, n$, $y = 1, 2, \dots, n$ and $z = 1, 2, \dots, n$. Another possibly construction is $x + y + z = 1$, …, $x + y + z = 3n$.

Now uppose for contradiction we have $k < 3n$ planes. Let them be $a_ix+b_iy+c_iz+d_i = 0$. Construct

$$A(x, y, z) \stackrel{\text{def}}{=} \prod_{i=1}^{k} (a_ix + b_iy + c_iz + d_i)$$
$$B(x, y, z) \stackrel{\text{def}}{=} \prod_{i=1}^{n}(x - i) \prod_{i=1}^{n}(y - i) \prod_{i=1}^{n}(z - i)$$

The coefficient of $x^ny^nz^n$ in $A$ is 0. The coefficient of $x^ny^nz^n$ in $B$ is 1. Now define

$$P(x, y, z) \stackrel{\text{def}}{=} A(x, y, z) - \frac{A(0, 0, 0)}{B(0, 0, 0)} B(x, y, z).$$

Now $P(x, y, z) = 0$ for any $x, y, z \in \{0, 1, \dots, n\}^3$. But the coefficient of $x^ny^nz^n$ is $-\frac{A(0,0,0)}{B(0,0,0)}$. This is a contradiction of the nullstellensatz. □

## 2.3 List Coloring

Let $G$ be a simple graph with $n$ vertices.

**Definition 2.3.1.** A simple graph $G$ is **$k$-colorable** if it's possible to properly color its vertices with $k$ colors. The smallest such $k$ is the **chromatic number** $\chi(G)$.

**Definition 2.3.2.** A simple graph $G$ is **$k$-choosable** if its possible to properly color its vertices given a list of $k$ colors at each vertex. The smallest such $k$ is the **choice number** $\operatorname{ch}(G)$.

Obviously, $\operatorname{ch}(G) \geq \chi(G)$. We just saw that $\operatorname{ch}(C_{100}) = \chi(C_{100}) = 2$.

**Remark 2.3.3.** This is in general strict: $\operatorname{ch}(K_{q^q,q}) \geq q + 1$. (Can you see why?)

**Remark 2.3.4.** $\operatorname{ch}(G) \leq \Delta(G) + 1$.

**Theorem 2.3.5** (Known Results)

- (Eaton, Nancy, 2003) $\operatorname{ch}(G) \le \chi(G) \log n$.
- (Noel, Reed, Wu, 2012) If $n \le 2\chi(G) + 1$ then $\operatorname{ch}(G) = \chi(G)$.

A general approach via the nullstellensatz is to consider as before

$$f_G(x_1, \dots, x_n) = \prod_{\substack{(i,j) \in E(G) \\ i<j}} (x_i - x_j).$$

The coefficients of this correspond to "even" and "odd" acyclic orientations of $G$, depending on how many edges go from a smaller index to a larger index. Specifically, the coefficient of $x_1^{d_1} \dots x_n^{d_n}$ is

$$|DE(d_1, \dots, d_n)| - |DO(d_1, \dots, d_n)|$$

where $DE$, $DO$ are the set of even and odd orientations with outdegrees $d_1$, …, $d_n$.

Now fix a "base" orientation $D_0$ (either even or odd). Then the above is equal to $\pm(EE(D_0) - EO(D_0))$, where $EE$ and $EO$ are the number of even and odd Eulerian suborientations of $D_0$. This is by a "difference" trick. (Here an "even" suborientation is one with an even number of edges, and an "odd" suborientation is one with an odd number of edges).

In particular, according to the nullstellensatz, we obtain

**Theorem 2.3.6** (Alon)

Let $G$ be a graph with an orientation $D_0$ such that the maximum indegree of $D_0$ is $t$ and moreover the number of even and odd Eulerian suborientations of $D_0$ is not equal. Then, $G$ is $(t+1)$-choosable.

In fact, finding $D_0$ is done by the following:

**Lemma 2.3.7**

Let $L(G) \stackrel{\text{def}}{=} \max |E(H)|/|V(H)|$ where $H$ is a subgraph of $G$. Then $G$ has an orientation in which every outdegree is at most $\lceil L(G) \rceil$.

*Proof.* Standard application of Hall's Marriage Theorem. □

Thus, we derive

**Theorem 2.3.8** (Alon, Tarsi)

A bipartite graph $G$ is $\lceil L(G) \rceil + 1$ choosable. In particular, planar bipartite graphs are 3-choosable.

*Proof.* Pick any $D_0$ specified by the previous lemma. There are no odd Eulerian suborientations at all. The last remark follows from (subgraphs of) bipartite planar graphs having average degree at most 4, hence $L(G) \le 2$. □

The last bound is tight; $K_{4,2}$ is bipartite planar.

# 3 Finite Fourier Analysis (Victor Wang)

The $n$th roots of unity are the complex numbers $\exp(\frac{2\pi ik}{n})$ for $k = 0, \ldots, n-1$.

## 3.1 Roots of Unity Filters

Consider the classic problem of computing $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \ldots$. One way to do this is to just run an alternating sum

$$(1+1)^n = \binom{n}{0} + \binom{n}{1} + \ldots$$
$$(-1+1)^n = \binom{n}{0} - \binom{n}{1} + \ldots.$$

Thus, we deduce $\binom{n}{0} + \binom{n}{2} + \cdots = \frac{1}{2} \cdot (2^n + 0^n) = 2^{n-1}$.

We can do this in general: for example, we have

$$\sum_{k \geq 0} \binom{n}{3k} = (1+1)^n + (\omega+1)^n + (\omega^2+1)^n$$

where $\omega = \exp(\frac{2}{3}\pi i)$. So roots of unity are related to problems involving symmetry, in this way.

## 3.2 Solving the Cubic

Suppose we want to solve

$$x^3 - ax^2 + bx - c = 0.$$

We know *a priori* that this can be factored as $(x-r_0)(x-r_1)(x-r_2)$; by Vieta's formulas, we know $a = r_0 + r_1 + r_2$, $b = r_0r_1 + r_1r_2 + r_2r_0$, $c = r_0r_1r_2$. These expressions are invariant under three-cycles.

Now, we make the substitution

$$r_0 = u_0 + u_1 + u_2$$
$$r_1 = u_0 + u_1\omega + u_2\omega^2$$
$$r_2 = u_0 + u_1\omega^2 + u_2\omega^4$$

where again $\omega$ is a primitive cube root of unity. (Explicitly, $u_0 = \frac{1}{3}(r_0 + r_1 + r_2)$, $u_1 = \frac{1}{3}(r_0 + r_1\omega^{-1} + r_2\omega^{-2})$, $u_2 = \frac{1}{3}(r_0 + r_1\omega^{-2} + r_2\omega^{-4})$).

Conceptually,

$$\text{if } F(z) = u_0 + u_1z + u_2z^2 \text{ then } r_i = F(\omega^i).$$

The first of Vieta's relations now reads $a = F(1) + F(\omega) + F(\omega^2) = 3u_0$ which is a roots of unity filter on $F$. The second reads $b = F(1)F(\omega) + F(\omega)F(\omega^2) + F(\omega^2)F(1)$ which is a roots of unity filter on $F(z)F(\omega z) = (u_0 + u_1z + u_2z^2)(u_0 + u_1z\omega + u_2z^2\omega^2)$. Since in a filter, we only care about the cubic terms (here $u_0^2$ and $-u_1u_2z^3$), we find that

$b = 3u_0^2 - 3u_1u_2$. If we repeat the same thing for $c$ and compile everything together, we arrive at

$$\begin{aligned} a &= 3u_0 \\ b &= 3u_0^2 - 3u_1u_2 \\ c &= u_0^3 + u_1^3 + u_2^3 - 3u_0u_1u_2. \end{aligned}$$

The first equation gives us $u_0 = a/3$ for free, and we can then obtain $u_1u_2$ in terms of $a$, $b$, $c$. Finally, the last equation tells us what $u_1^3 + u_2^3$ are. So, we can compute the values of $u_1^3 + u_2^3$, $u_1^3 \cdot u_2^3$; this reduces us to a quadratic, which we can solve.

## 3.3 Ramsey Theorem

"I'm going to talk about a completely different problem".

There once was a sociologist [Szalai] who studied classrooms and found that in any group of around 20 children, there were either four mutual friends or four mutual strangers. Before doing any sociology, he consulted some prominent mathematicians; this became the start of Ramsey theory.

The philosophy of Ramsey theory is to find structure in large "unstructured" object. Here is an example of such a result.

**Theorem 3.3.1** (Roth, 1950)
Let $\rho$ be a fixed positive density. Then for sufficiently large $N$ (depending on $\rho$), arithmetic progressions of length 3 can be found in any subset $S \subset \{1, \dots, N\}$ of density $\geq \rho$.

The intuition of why Fourier analysis is helpful is that 3-arithmetic progressions feel "rotationally symmetric". Specifically, consider an arithmetic progression $a$, $a+d$, $\dots$, $a+nd$. Let $\omega$ be a primitive $d$Th root of unity. Then

$$\left|\omega^a + \omega^{a+d} + \cdots + \omega^{a+nd}\right| = n+1$$

since every guy equals $\omega^a$. More generally, given any $S$ the intuition is that if $\left|\sum_{s\in S} \omega^s\right|$ is large, then $S$ is "correlated" with an arithmetic progression of difference $d$.

Here are some of the main ideas in the proof. We want to count solutions to $a+b-2c = 0$ for $a, b, c \in S$, so we consider the generating function

$$F(z) = \sum_{a\in S} z^a.$$

Then, the number of solutions is the constant coefficient of

$$F(z)F(z)F(z^{-2}).$$

We have $|S|$ trivial solutions $a+b+c$, and we seek the other solutions.

To extract coefficients from a series in general, we want to extend the "finite" Fourier analysis we used before and use the entire circle rather than any finite polygon. For example,

$$\frac{1}{2\pi}\int_{\theta\in[0,2\pi]} \exp(i\theta) = 0.$$

Returning to the problem at hand, this gives us an analytic way to extract the constant term:

$$\frac{1}{2\pi}\int_{|z|=1} F(z)^2F(z^{-2}) = [z^0]F(z)^2F(z^{-2}).$$

This allows us to use some tricks, like `https://en.wikipedia.org/wiki/Parseval's_identity`, stating that

$$\frac{1}{2\pi}\int_{|z|=1} |f(z)|^2 = F(1) = |S|.$$

In any case, what we want is to find $\omega \neq 1$ such that $|F(\omega)|$ is big as mentioned before. ($F(1)$ is bad for obvious reasons). The key idea is to consider the weighted sum

$$G(z) = \left(\sum_{s\in S} z^s\right) - \frac{|S|}{N}\sum_{k=1}^{N} z^k.$$

This way $G(1) = 0$ which circumvents the issue earlier of $F(1)$.

Thus for $G(\omega)$ to be large, we must have a "correlation" and moreover $\omega$ cannot be too close to 1. In this way, it turns out that we can find a long arithmetic progression $J \subset \{1, \dots, n\}$ such that

$$\frac{|S\cap J|}{|J|} \geq \rho + \frac{\rho^3}{10}$$

i.e. when we restrict attention to $J$, we have a higher density. Then we repeat this argument.

Note that this is pretty local to 3-arithmetic progressions, since in that case we can capture the entire condition in one equation $a + b = 2c$. This is called the *density increment argument*.

# 4 A Categorical Proof of Orbit-Stabilizer (Peter Haine)

Overview:

1. Review of group actions.
2. Categorify this picture, and prove the orbit-stabilizer theorem
3. How these ideas relate to other things.

## 4.1 Group Actions

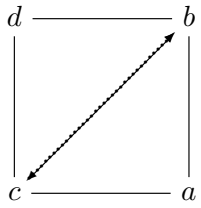
**Definition 4.1.1.** A group action of a group $G$ on a set $X$ is a map $\cdot : G \times X \to X$ which obeys the usual axioms

(1) $1 \cdot x = x$ and

(2) $(g' \cdot g \cdot x) = (g'g) \cdot x$.

For example,

- $\mathrm{GL}_n(\mathbb{R})$ acts on $\mathbb{R}^n$ by $A \cdot x \mapsto Ax$.
- The cyclic group $\mathbb{Z}/2$ acts on vertices of square $ABDC$ by reflecting across the diagonal $BC$.




**Definition 4.1.2.** The **orbit** of an $x \in X$, denoted $O_x$, is the set of images of $x \in X$ under the $G$saction. It partitions $X$ into equivalence classes.

For example, in the action of $ABCD$, the orbits are $\{a, d\}$, $\{b\}$, $\{c\}$.

**Definition 4.1.3.** The **stabilizer** of an $x \in X$ is the set of $g$ which fix $x \in X$; this is denoted $S_x$. Note that it's a subgroup of $G$.

Anyways, we have the usual

**Theorem 4.1.4** (Orbit-Stabilizer Theorem)
For any $x \in X$ if $G$ and $X$ are finite then

$$\#G = \#O_x \cdot \#S_x.$$

## 4.2 Categorification

Recall that a *monoid* is a "group without inverses. One can think of this as a 'one-object category": a monoid talks about the maps from an object to itself. For example, $\mathrm{GL}_n(\mathbb{R})$ is an isomorphisms of vector spaces $\mathbb{R}^n \to \mathbb{R}^n$.

A *category* is just a generalization of the idea of a monoid to *multiple* objects.

**Definition 4.2.1.** A category $\mathscr{C}$ consists of

(a) A class of *objects* $\mathrm{Ob}(\mathscr{C})$ (we'll use $c$, $c'$, ...).

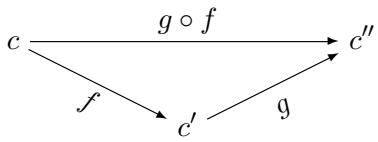
(b) A class of *morphisms* $\mathrm{Mor}(\mathscr{C})$ (we'll use $f$, $g$, ...).

such that

(a) Each $f \in \mathrm{Mor}(\mathscr{C})$ has a specified *source* and a specified *target*, i.e. we have a directed graph

(b) For all objects $c \in \mathrm{Obj}(C)$ there exists a specified identity morphism $\mathrm{id}_c : c \to c$.

(c) There is an *associative* composition of morphisms.




This must respect the identity $f \circ \mathrm{id}_c = f = \mathrm{id}_{c'} \circ f$ for any $c \xrightarrow{f} c'$.

Examples: category of sets, category of groups/rings/fields/..., $C^\infty\mathbb{R}$ (with single object $\mathbb{R}$ with morphisms of smooth functions), and $\mathbf{2}$, the two-object category.

**Definition 4.2.2.** An **isomorphism** is a category $C$ is a morphism $f : c \to c'$ such that there is an inverse $g : c' \to c$, meaning $g \circ f = \mathrm{id}_c$ and $f \circ g = \mathrm{id}_{c'}$.

The main point is that

**Lemma 4.2.3**

A group "is" a category with one object whose morphisms are all isomorphisms.

Thus we can generalize a group as follows.

**Definition 4.2.4.** A **groupoid** is a category where every morphism is an isomorphism.

For example, one can consider the category $\mathsf{GrpIso}$ whose objects are groups and whose morphisms are group isomorphisms.

**Definition 4.2.5.** A **functor** $F : \mathscr{C} \to \mathscr{D}$ given by assigning to each object $c \in \mathrm{Ob}(\mathscr{C})$ to an object $F(c) \in \mathrm{Ob}(\mathscr{D})$, and a map which induces morphisms

$$\begin{array}{ccc} c & & F(c) \\ \downarrow{\scriptstyle f} & \mapsto F(f) & \downarrow \\ c'' & & F(c') \end{array}$$

such that composition and identity are respected, meaning $F(g \circ f) = F(g) \circ F(f)$ and $F(\mathrm{id}_c) = \mathrm{id}_{F(c)}$.

Then the point is:

**Lemma 4.2.6**

A $G$-set $X$ "is" a functor $X : \{G\} \to \mathsf{Set}$, where $\{G\}$ is the one-object category corresponding to the group $G$.

In light of this, a morphism of $G$-sets $X \to Y$ can be defined as a natural transformations from $X$ to $Y$:

**Definition 4.2.7.** A **natural transformation** of functions $F, G : \mathscr{C} \to \mathscr{D}$ denoted $\eta : F \Rightarrow G$ is a collection of morphisms $\eta_c : F(c) \to G(c)$ for each $c \in \mathscr{C}$, such that for any $f : c \to c'$ the square

$$\begin{array}{ccc} F(c) & \xrightarrow{F(f)} & F(c') \\ \downarrow{\scriptstyle \eta_c} & & \downarrow{\scriptstyle \eta_{c'}} \\ G(c) & \xrightarrow{G(f)} & G(c') \end{array}$$

If each $\eta_c$ is an isomorphism it is called a **natural isomorphism**.

**Definition 4.2.8.** An **equivalence of categories** $\mathscr{C}$, $\mathscr{D}$, consists of functors $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ such that are naturally isomorphic to the identity functors.

**Theorem 4.2.9**

If $F : C \to D$ is part of an equivalence then it induces a bijection of isomorphism classes. Moreover the function

$$\operatorname{Mor}_{\mathscr{C}}(c, c') \to \operatorname{Mor}_{\mathscr{D}}(Fc, Fc') \qquad \text{by} \qquad f \mapsto F(f)$$

is a bijection.

**Definition 4.2.10.** A **skeletal category** is one in which every isomorphism class has exactly one object.

Every category is equivalent to a skeletal category.

### 4.2.1 Proof of Orbit-Stabilizer

Suppose we have a finite $G$-set $X$, viewed also as a functor $X : \{G\} \to \mathsf{Set}$. Define the **translation groupoid** $T_G X$ by

- Objects: the set $X$ itself
- Morphisms: there is a morphism $g : x \to y$ for each $g \in G$ such that $g \cdot x = y$.

Thus there are $\#X \cdot \#G$ morphisms. Pictorially, $T_G X$ looks like objects with arrows between them, and connected components correspond to orbits.

Now consider $x$ as a representative of its orbit $O_x$. Take a skeletal category $S$ equivalent to $T_G X$, which collapses each orbit to an object. By the theorem, there is a bijection

$$\operatorname{Mor}_S(O_x, O_x) \cong \operatorname{Mor}_{T_G X}(x, x) = \{g \in G \mid gx = x\} = S_X.$$

On the other hand, the morphisms in $T_G X$ which start at $x$ are in bijection with:

- The elements of $G$, and
- The disjoint union

$$\coprod_{y \in O_x} \operatorname{Mor}_{T_G}(x, y) \cong \coprod_{y \in O_x} \operatorname{Mor}_S(O_x, O_y) \cong \coprod_{y \in O_x} S_x.$$

This completes the proof.

# 5 Graph Magnitude and Homology (Yuzhou Gu)

The idea of "graph magnitude" is due to Leinster, and the associated homology to Willerton and Hopworth.

In this lecture, all graphs are finite and simple (hence undirected and unweighted). In some situations we will also make all the graphs connected.

## 5.1 Desired Properties of $\#G$

We want to find an algebraic invariant similar to the Euler characteristic; we denote it by $\#G$ and want it to satisfy the following properties:

(1) $\#(G \sqcup H) = \#G + \#H$.

(2) $\#(G \times H) = \#G \cdot \#H$ (Cartesian product of graphs).

(3) If $X = G \cup H$ then
$$\#X = \#G + \#H - \#(G \cap H)$$
holds "sometimes" (to be quantified later). Here, $G \cup H$ is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$; similarly for $G \cap H$.

Examples of functions that satisfy this:

- $\#G = |V(G)|$
- $\#G = |E(G)|$
- $\#G = |V(G)| + q\,|E(G)|$, viewed as an element of $\mathbb{Z}[q]/(q)^2$.

## 5.2 Three definitions of $\#G$

For a graph $G$ and vertices $x$, $y$ let $d(x, y)$ denote the distance between the two vertices (or $\infty$ if $x$, $y$ are not connected to each other).

Recall also that $\mathbb{Q}(q)$ is the set of *rational* functions in $\mathbb{Q}$, while $\mathbb{Z}[[q]]$ is the power series in $q$ with integer coefficients.

**Definition 5.2.1** (First Definition)**.** For a graph $G$ we define $Z(q)$ to be a $V(G) \times V(G)$ symmetric square matrix, whose $(x, y)$th entry is $q^{d(x,y)}$; by convention $q^\infty = 0$. We view the entries of $Z(q)$ as living in $\mathbb{Q}(q)$. Observe that $\det Z(0) = 1$, hence $Z(q)$ is invertible, so we may define the **graph magnitude**

$$\#G = \sum_{x,y \in V(G)} \left[Z^{-1}(q)\right]_{xy}.$$

In other words we sum all the $|V(G)|^2$ of the entries of $Z^{-1}(q)$.

**Remark 5.2.2.** This is actually a special case of a magnitude in an enriched categories. A graph is an $\mathcal{N}$-enriched category, where $\mathcal{N} = (\mathbb{N}, \geq 0, +)$.

**Definition 5.2.3** (Second Definition)**.** One can show there exists a unique function $w : V(G) \to \mathbb{Q}(q)$ such that

$$\forall x \in V(G) : \sum_y q^{d(x,y)} w(y) = 1.$$

Then, we define

$$\#G = \sum_x w(x).$$

This definition is the one easiest to use in proofs.

**Definition 5.2.4.** Let $\ell(x_0, \dots, x_k) = \sum_{0 \leq i < k} d(x_i, x_{i+1})$. Then

$$\#G = \sum_{k \geq 0} (-1)^k \sum_{(x_0,\dots,x_k)} q^{\ell(x_0,\dots,x_k)} \in \mathbb{Z}[[q]].$$

So in fact $\#G$ always lies in $\mathbb{Z}[[q]] \cap \mathbb{Q}(q)$. This definition gives a hint about how homology will be involved.

From this definition, one can check that the properties (1) and (2) hold, but (3) does not hold in general. One can ask when (3) holds.

**Example 5.2.5**

For $C_3$, we have $\#G$ is the sum of the entries of

$$\begin{pmatrix} 1 & q & q \\ q & 1 & q \\ q & q & 1 \end{pmatrix}^{-1}.$$

### 5.2.1 When Inclusion-Exclusion Holds

**Definition 5.2.6.** Let $U \subseteq G$ be a subgraph. We say $U$ is **convex** if for every $x, y \in vV(U)$ we have

$$d_U(x, y) = d_G(x, y).$$

**Definition 5.2.7.** If $U \subseteq G$ is convex, we say $G$ **projects** onto $U$ if there exists a map $\pi : V(G) \to V(U)$ such that

$$\forall g \in V(G), x \in U \text{ we have } d(g, x) = d(g, \pi(g)) + d(\pi(g), x).$$

So $\pi(g)$ is the "closest vertex to $g$".

**Theorem 5.2.8**

Let $X = G \cup H$ and suppose $G \cap H$ is convex in $X$, and moreover $H$ projects onto $G \cap H$. Then

$$\#X = \#G + \#H - \#(G \cap H).$$

*Proof.* Show $w_X = w_G + w_H - w_{G \cap H}$. □

**Corollary 5.2.9**
In a wedge sum $G \wedge H$, we have

$$\#(G \wedge H) = \#G + \#H - 1.$$

**Corollary 5.2.10**
If $X = G \cup H$ is a tree, then Inclusion-Exclusion holds.

**Corollary 5.2.11**
If $H$ is bipartite, and $G \cap H$ is the graph consisting of a single edge, then

$$\#X = \#G + \#H - \frac{2}{1+q}.$$

### 5.2.2 Whitney Twists

**Definition 5.2.12.** Let $G$ and $H$ be graphs, and define $g, g \in G$ and $h, h' \in H$. Define the graphs $X_1$ and $X_2$ by

- $X_1$ is $G \sqcup H$ modulo the equivalence $g \sim h$, $g' \sim h'$.
- $X_2$ is $G \sqcup H$ modulo the equivalence $g \sim h'$, $g' \sim h$.

In this case we say $X_1$ and $X_2$ are **Whitney twists**.

**Theorem 5.2.13**
When $(g, g') \in E(G)$ then $\#X_1 = \#X_2$.

*Proof.* Show that $\sum_{x_1} w_{X_1}(x_1) = \sum_{x_2} w_{X_2}(x_2)$. □

## 5.3 Homology

Digression on categorification. The idea is to replace sets with categories and functions with functors. This is **Baez categorification**. Examples:

- Singular homology is categorification of $X$
- Khovanov homology is categorification of Jones polynomial
- Burnside category is categorification of Burnside ring
- Category of representations of $S_n$ is categorification of ring of symmetric functions.

Now we define the homology as a categorification of #. We want to define a doubly graded homology theory

$$\mathrm{MH}_{**}(G).$$

To do this we have to define a chain complex. We do this as follows: define $\text{MC}_{k,\ell}(G)$ to be the free abelian group (with coefficients in $G$) of $(k+1)$-tuples $(x_0, \dots, x_k)$ of vertices, such that $\ell(x_0, \dots, x_k) = \ell$. Then, we can define the differential map

$$d : \text{MC}_{k,\ell} \to \text{MC}_{k-1,\ell}$$

by

$$d(x_0, \dots, x_k) = \sum_i \begin{cases} (-1)^i (x_0, \dots x_{i-1}, x_{i+1}, \dots, x_k) & \text{if the } \ell\text{-value is } \ell \\ 0 & \text{else.} \end{cases}$$

One can check that $d^2 = 0$, so $\text{MH}_{**}$ gives a homology of doubly graded abelian groups.

Now we want to make $\text{MH}_{**}$ into a functor. What does this mean? We first consider a category $\mathsf{Graph}$ whose objects are graphs, and whose morphisms $f : G \to H$ are given as follows: it's an $f : V(G) \to V(H)$ such that whenever $\{x, y\} \in E(G)$ either $f(x) = f(y)$ or $\{f(x), f(y)\} \in E(H)$.

Now we want to make $\text{MH}_{**}$ into a functor, we need to make a given $f : G \to H$ into

$$f_* : \text{MH}_{k\ell}(G) \to \text{MH}_{k\ell}(H).$$

We do this by declaring

$$f_*(x_0, \dots, x_k) = \begin{cases} (f(x_0), \dots, f(x_k)) & \text{if it has } \ell\text{-value } \ell \\ 0 & \text{otherwise.} \end{cases}$$

## 5.4 Properties of the Homology

Then it turns out that

$$\#G = \sum_{\ell \geq 0} \sum_{k \geq 0} (-1)^k \left( \text{rank}(\text{MH}_{k,\ell}(G)) q^\ell \right).$$

This satisfies the properties

1. $\text{MH}_{**}(G \sqcup H) = \text{MH}_{**}(G) \oplus \text{MH}_{**}(H)$.
2. Künneth formula: we have a natural exact sequence

$$0 \to \text{MH}_{**}(G) \otimes \text{MH}_{**}(H) \to \text{MH}_{**}(G \times H) \to \text{Tor}(\text{MH}_{*-1,*}(G), \text{MH}_{**}(H) \to 0.$$

Here, the left-hand term is

$$\bigoplus_{k_1+k_2=k, \ell_1+\ell_2=\ell} \left( \text{MH}_{k_1,\ell_1}(G) \otimes \text{MH}_{k_2,\ell_2}(H) \right).$$

We also have a *Mayer-Vietoris* sequence, which this time takes a short exact sequence

$$0 \to \text{MH}_{**}(G \cap H) \to \text{MH}_{**}(G) \oplus \text{MH}_{**}(H) \to \text{MH}_{**}(G \cup H) \to 0$$

given that $G \cap H$ is convex in $G \cup H$, and moreover $H$ projects onto $G \cap H$.

**Example 5.4.1**

Let $G$ be a single edge. Then $\text{MC}_{1,1}$ is generated by $(0,1)$ and $(1,0)$, and the other cycle groups vanish (except for $\text{MC}_{0,0} = \mathbb{Z}^2$). So $\text{MH}_{1,1} = \mathbb{Z}^2$ and $\text{MH}_{0,0} = \mathbb{Z}^2$ but all other homology groups vanish.

## 5.5 Open Problems

1. Is it possible that $\mathrm{MH}_{**}$ contain torsion?
2. Is $\mathrm{MH}_{**}$ isomorphic under connected Whitney twist?
3. How do we calculate the homology groups? For example, even $\mathrm{MH}_{**}(C_n)$ is unknown, though we have conjectures.
4. Can we put extra structure on the cohomology?

## 5.6 References

1. T. Leinster, The magnitude of a graph. arXiv: 1401.4623.
2. R. Hepworth and S. Willerton. Categorifying the magnitude of a graph. arXiv: 1505.04125.

# 6 Topology of Categories (Colin Aitken)

Reference for this lecture: "Higher Algebraic K-Theory I" (Quillen). There is no sequel, because he never got around to writing it.

## 6.1 Categorical Definitions

For this lecture, we will ignore set-theoretic issues.

**Definition 6.1.1.** A (small) **category** $C$ consists of a set of objects $\mathrm{ob}(C)$ and a set of morphisms $\mathrm{Hom}(x, y)$ for any $x, y \in \mathrm{ob}(C)$, plus the usual composition rules and distinguished $\mathrm{id}_x \in \mathrm{Hom}(x, x)$.

Examples are $\mathsf{Set}$, $\mathsf{Grp}$, $\mathsf{Vec}$, ....

**Definition 6.1.2.** A **functor** $F : C \to D$ is as usual.

Here are some of the standard non-concrete examples.

**Example 6.1.3** (Posetal Category)
The **posetal category** $C(P)$ associated to a poset $(P, \leq)$ has $\mathrm{ob}(P)$ the elements of $P$ with a unique map $x \to y$ iff $x \leq y$. Then, the functors from $C(P)$ to $C(P')$ correspond exactly with the order-preserving functions $P \to P'$.

**Example 6.1.4** (Group Category)
Given a group $G$, construct a category $C(G)$ with exactly one object $x$ such that $\mathrm{Hom}(x, x) \cong G$. Then, functors from $C(G)$ to $C(G')$ are the same as group homomorphisms from $G$ to $G'$.

"If something works in at least two cases, it's probably important."

## 6.2 Simplices

Let

$$\Delta_n = \{(x_0, \ldots, x_{n+1}) \mid 0 = x_0 \leq x_1 \leq \cdots \leq x_{n+1} = 1\} \subseteq \mathbb{R}^{n+2}.$$

This is equivalent to the "usual" $n$-simplices, just a little weirder.

For each $0 \leq i \leq n$, we define a **face map**

$$\delta_i : \Delta_{n-1} \hookrightarrow \Delta_n \qquad \text{by} \qquad (x_0, \ldots, x_n) \mapsto (x_0, \ldots, x_i, x_i, \ldots, x_n).$$

which essentially allows us to embed $\Delta_{n-1}$ as a face of $\Delta_n$. Also, for $0 \leq i \leq n$ we define

$$\delta_i : \Delta_{n-1} \hookrightarrow \Delta_n \qquad \text{by} \qquad (x_0, \ldots, x_n) \mapsto (x_0, \ldots, x_i, x_{i+2}, \ldots, x_n)$$

which is the inverse operation.

## 6.3 Nerves

**Definition 6.3.1.** Given a category $C$ we define the **nerve** $N(C)$ to be the graded set whose $i$th component is

$$N(C)_i = \{\text{diagrams } x_0 \to x_1 \cdots \to x_i\}.$$

These look quite similar to the previous thing, so we'll define "dual" operators as follows. Let

$$d_i : N(C)_n \to N(C)_{n-1} \quad \text{by} \quad (x_0 \to \cdots \to x_n) \mapsto (x_0 \to \cdots \to x_{i-1} \to x_{i+1} \to \ldots. \to x_n)$$

where $x_{i-1} \to x_{i+1}$ is the composition $x_{i-1} \to x_i \to x_{i+1}$, and

$$\sigma_i : N(C)_n \to N(C)_{n+1} \quad \text{by} \quad (x_0 \to \cdots \to x_n) \mapsto (x_0 \to \cdots \to x_i \xrightarrow{\text{id}} x_i \to \ldots. \to x_n)$$

Finally, we define a complex

$$BC = \coprod_i N(C)_i \times \Delta_i / \sim$$

where $(d_i k_n, \ell_{n-1}) \sim (k_n, \delta_i, \ell_{n-1})$ and $(\sigma_i k_n, \ell_{n+1} \sim (k_n, s_i \ell_{n+1})$.

The letter $B$ is what we usually use for classifying spaces. For example, $B(C(G)) = BG$.

## 6.4 Examples

**Definition 6.4.1.** The **psedoucircle** $\mathbb{S}^1$ has four points $a$, $b$, $c$, $d$ and the topology is

$$\tau(\mathbb{S}^1) = \{\varnothing, a, b, ab, abc, abd, abcd\}.$$

We now claim there is a bijection

$$\mathsf{FinTop} \iff \text{Transitive, reflexive relations} \iff \text{Categories } C \text{ with } |\operatorname{Hom}(x,y)| \le 1.$$

In one direction, given a finite topological space $T$ we put a partial ordering on $T$ by $x \le y$ if $y$ is the smallest open set containing $x$.

**Theorem 6.4.2** (McCord)

Let $T$ be a finite topological space. There is a weak homotopy equivalence $\pi : BT \to T$.

*Proof.* Depends on the following lemma.

**Lemma**

Given an open basis $\mathscr{U}$ of $Y$, and $f : X \to Y$ such that $f$ is a weak homotopy equivalence when restricted to $f^{-1}(U)$ (for every $U \in \mathscr{U}$), then $f$ is a weak homotopy equivalence.

Let $\mathscr{U} = \{U_x : x \in T\}$ now. □

## 6.5 Fun Facts

Let $F, G : C \to C'$ functors of categories and let $\theta$ be a natural transformation. Then $\theta$ can be thought of as a morphism

$$C \times \mathbf{2} \xrightarrow{(F,G)} C'$$

where $\mathbf{2}$ is the 2-object category.

Then, applying $B$, we can get

$$BC \times [0,1] \xrightarrow{H} BC'$$

where $H(-,0) = BF$ and $H(-,1) = BG$. So in fact, we have a homotopy from $BF$ to $BG$. This is a little weird since we think of natural transformations as *morphisms* $F$ to $G$, but the induced map $H$ is an isomorphism-like thing (it's a homotopy).

We actually also have

**Theorem 6.5.1**

If $F$ is an adjoint, then $BF$ is a homotopy equivalence.

**Corollary 6.5.2**

If $C$ has initial object, then $BC$ is contractible.

Sadly, this means that every category we care about in real life (like $\mathsf{Set}$, $\mathsf{Grp}$, $\mathsf{Top}$, $\mathsf{Vect}$) carries no topological information.

Here is the second fun fact. Let $C$ be a category; for simplicity we assume its connected, so $BC$ is connected. We will compute $\pi_1(BC)$. Actually, we claim that

**Theorem 6.5.3**

Coverings $p : E \to BC$ are in bijection which functors $F : C \to \mathsf{Set}^{\text{iso}}$.

Here $\mathsf{Set}^{\text{iso}}$ is the category of sets, but whose only arrows are isomorphisms.

*Proof.* Given a covering, define $F(c) = p^{-1}(c)$. Then interpret $F(c \xrightarrow{f} d)$ by treating $f$ as a path.

For the other direction, given $F$ and $C$, let $FC$ be the category whose objects are $(X, x) \in C \times F(X)$ and whose arrows $v : (X, x) \to (Y, x)$ are maps $v : X \to Y$ with $F(v)x = y$. □

Now let $G$ be the category $C$ with formal inverses, and for $x \in C$ and let $G_x$ be the subcategory of $C$ with $\text{ob}(G_x) = x$. Then $G_x$ is a group $G$, and functors from $C$ to $\text{Set}^{\text{iso}}$ correspond to functors $G \to \mathsf{Set}$ or to $G_x$ sets. Hence

$$\pi_1(BC) = G_x.$$

**Example 6.5.4**

(a) $\pi(\text{Set}^{\text{iso}}, x) = S_{|x|}$.

(b) $\pi_1(M)$ where $M$ is a monoid (interpreted as a category) is the group completion of $M$.

(c) $\pi_1(P)$ for a poset $P$ turns out to be walking along the edges of $P$ which make a cycle. (??)

# 7 Chip Firing (Ziv Scully)

Reference: `http://arxiv.org/abs/1211.6786`

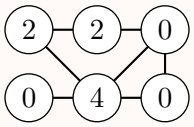
## 7.1 The Game

In the **parallel chip-firing game**, we take a connected undirected graph $G$. Then

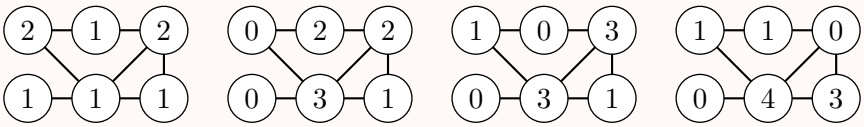
- Put a nonnegative integer number of chips on each vertex.
- Every turn, if a vertex can give a chip to *each* of its neighbor, it "fires" and does so. Otherwise, it "waits".

**Example 7.1.1** (An Example Game)

Start with




Then, the next states are




After the fifth game above, we start going in a loop, back to the second configuration.

## 7.2 Basic Properties

**Proposition 7.2.1** (Periodicity)

Chip-firing is eventually periodic.

*Proof.* We observe that this game has a finite state space. Since it's deterministic, we now see that it is eventually periodic. □

**Proposition 7.2.2**

Every vertex fires the same number of times per period.

This is surprisingly most of the low-hanging fruit.

## 7.3 Known Results

One question is what the possible periods are for a graph $G$. Here is what we know.

**Theorem 7.3.1** (Possible Cycles)
For the following classes of graphs, the possible periods are:

- Trees: period 1 or 2.
- $n$-cycles: period 2 or divisor of $n$.
- $K_n$: $1, \dots, n$
- $K_{a,b}$: $1, \dots, \min(a,b)$, $2, \dots, 2\min(a,b)$.

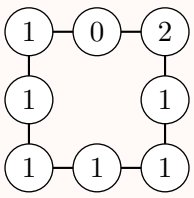
Despite these small numbers, we will soon see that given a graph $G$ the period can be exponentially large (but is bounded for any given graph $G$). In fact, we know very little about this game, but we did manage to prove:

**Theorem 7.3.2** (Chip-firing is hard)
Chip-firing is Turing complete.

So this problem really is hard.

Here are some examples:

**Example 7.3.3** (Cyclic Graph: "Gliders"; Period $n$)
This construction gives for general $n$ a period $n$. We can obtain divisors of this by then putting "gliders" periodically.




**Example 7.3.4** (Period 2)
Let $G$ be a biparite graph, colored red and blue. Then

- Place 0 chips on every blue vertex.
- On each red vertex $r$, place $\deg r$ chips.

This gets period 2. (It turns out one can do this for a general graph $G$ with more than two vertices by taking a spanning tree.)

## 7.4 Proof of Results for Trees and $n$-cycles

We will now prove these are the only possible cycle lengths, i.e. we will complete the proof of our earlier theorem for trees and $n$-cycles.

**Definition 7.4.1.** We define the following notation:

- $N(v)$ is the set of neighbors of $v$.
- $d(v)$ is the degree of $v$.
- $\sigma_t(v)$ is the number of chips of $v$ at time $t$.
- $F_t(v)$ is 1 if $v$ fires at time $t$, and 0 otherwise.
- $\Phi_t(v) = \sum_{w \in N(v)} F_t(w)$, the number of chips $v$ gets at time $t$.

Thus the transition rule for this game is

$$\sigma_{t+1}(v) = \sigma_t(v) + \underbrace{\Phi_t(v)}_{\text{Chips gained}} \underbrace{- d(v)F_t(v)}_{\text{If fired}}.$$

**Lemma 7.4.2** (Duality)
One can consider a dual game by

$$\sigma_t(v) \mapsto 2d(v) - 1 - \sigma_t(v)$$

provided the outputs are all nonnegative. This dual game swaps firing and waiting.

**Lemma 7.4.3**
The number of vertices $v$ such that $\sigma_t(v) \geq 2d(v)$ is nonincreasing with $t$.

*Proof.* If we have a vertex $v$ with $\sigma_t(v) \geq d(v)$, then

$$\sigma_{t+1}(v) = \sigma_t(v) + \Phi_t(v) - d(v) \leq \sigma_t(v)$$

i.e. as long as we're firing we can't gain chips. But if we don't fire (meaning $\sigma_t(v) \leq d(v)-1$) then $\sigma_{t+1}(v) \leq \sigma_t(v) + d(v) \leq 2d(v) - 1$. □

From now on assume $\sigma_t(v) \leq 2d(v) - 1$ always holds.

**Lemma 7.4.4** (Clumping)
If $v$ waits (resp. fires) for all times in some interval $[a, b]$ then it has a neighbor $u \in N(v)$ that also waits (resp. fires) for all times in the interval $[a-1, b-1]$.

*Proof.* We only do the waiting case. We have

$$0 \leq \sigma_a(v) \leq \sigma_{a+1}(v) \leq \cdots \leq \sigma_b(v) \leq d(v) - 1.$$

In other words we gained at most $d(v) - 1$ chips over time. Since we have $d(v)$ neighbors, this implies by Pigeonhole that some $u$ never gave us any chips in $[a-1, b-1]$. (This proof is a little sloppy; there are some edge cases at the very beginning.) □

**Definition 7.4.5.** An **$f$-clump for a vertex $v$** is an interval of length at least two during which $v$ always either waits (for $f = 0$) or fires (for $f = 1$).

Let $T$ be a tree with a distinguished vertex $e$, called a **motor**, which we can think of as an "external vertex": it is connected to several "virtual" vertices and fires in a fixed pattern, without regard to the usual rules of the game. In particular, it can have a negative number of chips

**Theorem 7.4.6**

Let $T$ be a tree with motor $e$, $v$ a vertex of $T$ If $[a, b]$ is a $f$-clump of $v$ in a tree, then $[a - D, b - D]$ is an $f$-clump of $e$, where $D$ is the distance from $e$ to $v$.

The proof is essentially to use the clumping lemma repeatedly, looking at maximal clumps to avoid "backtracking". In particular

**Corollary 7.4.7**

In a tree there are no clumps.

**Corollary 7.4.8**

If nobody fires twice in a row, the tree exactly copies $e$ (with time shift).

# 8 Li+e/near Algebra (Janson Ng)

## 8.1 Definitions

Let $k$ be algebraically closed of characteristic zero.

**Definition 8.1.1.** A **Lie algebra** $\mathfrak{g}$ is a $k$-vector space $\mathfrak{g}$ equipped with a skew-symemtric bracket $[,]$ which satisfies the Jacobi identity.

**Example 8.1.2**

(a) Any one-dimensional space is a Lie algebra with zero bracket.

(b) $\mathfrak{gl}(n)$, with the commutator bracket $[A, B] = AB - BA$.

(c) $\mathfrak{sl}(n)$, the traceless $n \times n$ matrices.

Today's main interest is representations of

$$\mathfrak{sl}(2) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mid a, b, c \in k \right\}$$

**Definition 8.1.3.** A **representation** of $\mathfrak{sl}(2)$ is a linear transformation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ which preserves the Lie bracket.

**Example 8.1.4**

Let $\mathfrak{g} = \mathfrak{gl}(2)$, then $V = k^{\oplus 2}$ is a representation with action $\rho(A)(v) = Av$.

**Definition 8.1.5.** A **subrepresentation** of $V$ is a $\mathfrak{g}$-invariant vector space $W \subseteq V$. We say $V$ is an **irrep** if the only subrepresentation of $V$ is $V$ itself.

### 8.1.1 $\mathfrak{sl}(2)$

Note that $\mathfrak{sl}(2)$ has a basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

These satisfy $[h, e] = 2e$, $[h, f] = -2f$, and $[e, f] = h$.

Let $V$ be an irrep of $\mathfrak{sl}(2)$, and $\lambda \in k$. We consider

$$V_\lambda = \{v \in V \mid h \cdot v = \lambda v\}.$$

**Claim 8.1.6.** If $v \in V_\lambda$, then $f \cdot v \in V_{\lambda-2}$ and $e \cdot v \in V_{\lambda+2}$.

*Proof.* Use $h \cdot (f \cdot v) - f \cdot (h \cdot v) = [h, f] \cdot v$. □

# 9 Quantum Groups (Charles Fu)

This is actually a generalization of Lie algebras, not groups.

## 9.1 Review of Lie algebras

Recall $\mathfrak{sl}(2)$:

- $e$, $f$, $h$ as usual
- Triangular decomposition $\mathfrak{sl}_2 = n^+ \oplus h \oplus n^-$.
- $U(\mathfrak{sl}(2))$ is the universal enveloping algebra.
- $U(\mathfrak{sl}(2)) = U(n^+) \oplus U(h) \oplus U(n^-)$.

We can make

$$U(\mathfrak{sl}(2)) = T(\mathfrak{sl}(2))/(a \otimes b - b \otimes a - [a, b])$$

into a Hopf algebra, as follows:

- The multiplication $\nabla$ is the multiplication.
- The unit $\eta$ is the obvious one.
- $\Delta(x) = 1 \otimes x + x \otimes 1$.
- $\varepsilon$ is the zero map.
- $S(x) = -x$.

Let $P(a \otimes b) = b \otimes a$. We say a Hopf algebra $B$ is

- commutative if $\nabla \circ P = \nabla$.
- cocommutative if $P \circ \Delta = \Delta$.

Note that our example $U(\mathfrak{sl}_2)$ is not commutative (nor is $U(\mathfrak{g})$ for any reasonable $\mathfrak{g}$) but $U(\mathfrak{sl}_2)$ is cocommutative (or $U\mathfrak{g}$ for any reasonable $\mathfrak{g}$). In quantum groups, we are going to break the cocommutative things.

## 9.2 Quantum group

Let $q \in \mathbb{C}$, $q \neq 0$ and $q^2 \neq \pm 1$. Then we define

$$U_q(\mathfrak{sl}(2))$$

by generators $E$, $F$, $K$, $K^{-1}$ satisfying relations

- $KK^{-1} = K^{-1}K = 1$.
- $KEK^{-1} = q^2E$

- $KFK^{-1} = q^{-2}F$.
- $EF = FE = \frac{K-K^{-1}}{q-q^{-1}}$.

We make it into a Hopf algebra by putting

$$\begin{aligned}
\Delta(E) &= E \otimes 1 + K \otimes E \\
\Delta(F) &= F \otimes K^{-1} + 1 \otimes F \\
\Delta(K) &= K \otimes K \\
0 &= \varepsilon(E) = \varepsilon(F) \\
1 &= \varepsilon(K) \\
S(E) &= -K^{-1}E \\
S(F) &= -FK \\
S(K) &= K^{-1}
\end{aligned}$$

**Remark 9.2.1.** This is not a unique structure. In fact, every author has their own taste about which structure to use.

Define

$$[a] = \frac{q^a - q^{-a}}{q - q^{-1}}.$$

**Theorem 9.2.2**

Assume $q$ is not a root of unity. Then representations of $U_q(\mathfrak{sl}(2))$ are completely reducible. Moreover, the irreducible representations are of the following form:

- $V_{n+1}^+ = \{m_0, \dots, m_n\}$ where $Km_i = q^{n-2i}m_i$, $Fm_i = m_{i+1}$ and $Em_i = [i][n+1-i]m_{i-1}$.
- $V_{n+1}^- = \{m_0, \dots, m_n\}$, with a similar action (omitted).